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CONVERGENCE ON GAUSS-SEIDEL ITERATIVE METHODS FOR LINEAR SYSTEMS WITH GENERAL $H-$MATRICES*

CHENG-YI ZHANG†, DAN YE‡, CONG-LEI ZHONG§, AND SHUANGHUA LUO¶

Abstract. It is well known that as a famous type of iterative methods in numerical linear algebra, Gauss-Seidel iterative methods are convergent for linear systems with strictly or irreducibly diagonally dominant matrices, invertible $H-$matrices (generalized strictly diagonally dominant matrices) and Hermitian positive definite matrices. But, the same is not necessarily true for linear systems with nonstrictly diagonally dominant matrices and general $H-$matrices. This paper firstly proposes some necessary and sufficient conditions for convergence on Gauss-Seidel iterative methods to establish several new theoretical results on linear systems with nonstrictly diagonally dominant matrices and general $H-$matrices. Then, the convergence results on preconditioned Gauss-Seidel (PGS) iterative methods for general $H-$matrices are presented. Finally, some numerical examples are given to demonstrate the results obtained in this paper.

Key words. Gauss-Seidel iterative methods, Convergence, Nonstrictly diagonally dominant matrices, General $H-$matrices.

AMS subject classifications. 15A15, 15F10.

1. Introduction. In this paper, we consider the solution methods for the system of n linear equations

$$Ax = b, \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and is nonsingular, $b, x \in \mathbb{C}^n$ and x unknown. Let us recall the standard decomposition of the coefficient matrix $A \in \mathbb{C}^{n\times n}$,

$$A = D_A - L_A - U_A,$$

where $D_A = diag(a_{11}, a_{22}, \ldots, a_{nn})$ is a diagonal matrix, L_A and U_A are strictly lower and strictly upper triangular matrices, respectively. If $a_{ii} \neq 0$ for all $i \in \langle n\rangle =$

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$\{1, 2, \ldots, n\}$, the Jacobi iteration matrix associated with the coefficient matrix A is

$$H_J = D_A^{-1}(L_A + U_A);$$

the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iteration matrices associated with the coefficient matrix A are

$$H_{FGS} = (D_A - L_A)^{-1}U_A, \tag{1.2}$$

$$H_{BGS} = (D_A - U_A)^{-1}L_A, \tag{1.3}$$

and

$$\begin{aligned} H_{SGS} &= H_{BGS}H_{FGS} \\ &= (D_A - U_A)^{-1}L_A(D_A - L_A)^{-1}U_A, \end{aligned} \tag{1.4}$$

respectively. Then, the Jacobi, FGS, BGS and SGS iterative method can be denoted the following iterative scheme:

$$x^{(i+1)} = Hx^{(i)} + f, \quad i = 0, 1, 2, \ldots \tag{1.5}$$

where H denotes iteration matrices H_J, H_{FGS}, H_{BGS} and H_{SGS}, respectively, correspondingly, f is equal to $D_A^{-1}b$, $(D_A - L_A)^{-1}b$, $(D_A - U_A)^{-1}b$ and $(D_A - U_A)^{-1}D_A(D_A - L_A)^{-1}b$, respectively. It is well-known that (1.5) converges for any given $x^{(0)}$ if and only if $\rho(H) < 1$ (see [11]), where $\rho(H)$ denotes the spectral radius of the iteration matrix H. Thus, to establish the convergence results of iterative scheme (1.5), we mainly study the spectral radius of the iteration matrix in the iterative scheme (1.5).

As is well known in some classical textbooks and monographs, see [11], Jacobi and Gauss-Seidel iterative methods for linear systems with Hermitian positive definite matrices, strictly or irreducibly diagonally dominant matrices and invertible $H-$matrices (generalized strictly diagonally dominant matrices) are convergent. Recently, the class of strictly or irreducibly diagonally dominant matrices and invertible $H-$matrices has been extended to encompass a wider set, known as the set of *general $H-$matrices*. In a recent paper, Ref. [2, 3, 4], a partition of the $n \times n$ general $H-$matrix set, H_n, into three mutually exclusive classes was obtained: The Invertible class, H_n^I, where the comparison matrices of all general $H-$matrices are nonsingular, the Singular class, H_n^S, formed only by singular $H-$matrices, and the Mixed class, H_n^M, in which singular and nonsingular $H-$matrices coexist. Lately, Zhang in [16] proposed some necessary and sufficient conditions for convergence on Jacobi iterative methods for linear systems with general $H-$matrices.

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A problem has to be proposed, i.e., whether Gauss-Seidel iterative methods for linear systems with nonstrictly diagonally dominant matrices and general $H-$matrices are convergent or not. Let us investigate the following examples.

Example 1.1. Assume that either A or B is the coefficient matrix of linear system (1.1), where $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$. It is verified that both A and B are nonstrictly diagonally dominant and nonsingular. Direct computations yield that $\rho(H^A_{FGS}) = \rho(H^B_{BGS}) = 1$, while $\rho(H^A_{BGS}) = \rho(H^B_{FGS}) = 0.3536 < 1$ and $\rho(H^A_{SGS}) = \rho(H^B_{SGS}) = 0.5797 < 1$. This shows that BGS and SGS iterative methods for the matrix A are convergent, while the same is not FGS iterative method for A. However, FGS and SGS iterative methods for the matrix B are convergent, while the same is not BGS iterative method for B.

Example 1.2. Assume that either A or B is the coefficient matrix of linear system (1.1), where $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$. It is verified that A is nonstrictly diagonally dominant matrix and B is a mixed $H-$matrix. Further, they are nonsingular. By direct computations, it is easy to get that $\rho(H^A_{FGS}) = 0.4215 < 1$, $\rho(H^A_{BGS}) = 0.3536 < 1$ and $\rho(H^A_{SGS}) = 0.3608 < 1$, while $\rho(H^B_{FGS}) = \rho(H^B_{BGS}) = \rho(H^B_{SGS}) = 1$. This shows that FGS, BGS and SGS iterative methods converge for the matrix A, while they fail to converge for the matrix B.

In fact, the matrices A and B in Example 1.1 and Example 1.2, respectively, are all general $H-$matrices, but are not invertible $H-$matrices. Gauss-Seidel iterative methods for these matrices sometime may converge for some given general $H-$matrices, but may fail to converge for other given general $H-$matrices. Thus, an important question is how one can obtain the convergence on Gauss-Seidel iterative methods for the class of general $H-$matrices without a direct computation of the spectral radius?

Aimed at the problem above, some necessary and sufficient conditions for convergence on Gauss-Seidel iterative methods are first proposed to establish some new results on nonstrictly diagonally dominant matrices and general $H-$matrices. In particular, the convergence results on preconditioned Gauss-Seidel (PGS) iterative methods for general $H-$matrices are presented. Furthermore, some numerical examples are given to demonstrate the results obtained in this paper.

The paper is organized as follows. Some notations and preliminary results about special matrices are given in Sections 2 and 3. Some special matrices will be defined, based on which some necessary and sufficient conditions for convergence on Gauss-Seidel iterative methods are firstly proposed in Section 4. Some convergence results on preconditioned Gauss-Seidel iterative methods for general $H-$matrices are

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then presented in Section 5. In Section 6, some numerical examples are given to demonstrate the results obtained in this paper. Conclusions are given in Section 7.

2. Preliminaries. In this section, we give some notions and preliminary results about special matrices that are used in this paper.

$\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) will be used to denote the set of all $m\times n$ complex (real) matrices. $\mathbb{Z}$ denotes the set of all integers. Let $\alpha \subseteq \langle n\rangle = \{1,2,\ldots,n\} \subset \mathbb{Z}$. For nonempty index sets $\alpha,\beta \subseteq \langle n\rangle$, $A(\alpha,\beta)$ is the submatrix of $A\in\mathbb{C}^{n\times n}$ with row indices in α and column indices in β. The submatrix $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$. Let $A\in\mathbb{C}^{n\times n}$, $\alpha\subset\langle n\rangle$ and $\alpha' = \langle n\rangle - \alpha$. If $A(\alpha)$ is nonsingular, the matrix

$$A/\alpha = A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha')$$

is called the *Schur complement* with respect to $A(\alpha)$, indices in both α and α' are arranged with increasing order. We shall confine ourselves to the nonsingular $A(\alpha)$ as far as A/α is concerned.

Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{C}^{m\times n}$, $A\circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{m\times n}$ denotes *the Hadamard product* of the matrices A and B. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *nonnegative* if $a_{ij} \geq 0$ for all $i,j\in\langle n\rangle$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *$Z-$matrix* if $a_{ij}\leq 0$ for all $i\neq j$. We will use Z_n to denote the set of all $n\times n$ $Z-$matrices. A matrix $A = (a_{ij}) \in Z_n$ is called an *$M-$matrix* if A can be expressed in the form $A = sI - B$, where $B \geq 0$, and $s \geq \rho(B)$, the spectral radius of B. If $s > \rho(B)$, A is called a *nonsingular $M-$matrix*; if $s = \rho(B)$, A is called a *singular $M-$matrix*. M_n, $M_n^\bullet$ and M_n^0 will be used to denote the set of all $n\times n$ $M-$matrices, the set of all $n\times n$ nonsingular $M-$matrices and the set of all $n\times n$ singular $M-$matrices, respectively. It is easy to see that

$$M_n = M_n^\bullet \cup M_n^0 \quad \text{and} \quad M_n^\bullet \cap M_n^0 = \emptyset. \tag{2.1}$$

The *comparison matrix* of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i\neq j. \end{cases} \tag{2.2}$$

It is clear that $\mu(A)\in Z_n$ for a matrix $A\in\mathbb{C}^{n\times n}$. The set of *equimodular matrices* associated with A, denoted by $\omega(A) = \{B\in\mathbb{C}^{n\times n} : \ \mu(B) = \mu(A)\}$. Note that both A and $\mu(A)$ are in $\omega(A)$. A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called a *general $H-$matrix* if $\mu(A)\in M_n$ (see [1]). If $\mu(A)\in M_n^\bullet$, A is called an *invertible $H-$matrix*; if $\mu(A)\in M_n^0$ with $a_{ii} = 0$ for at least one $i\in\langle n\rangle$, A is called a *singular $H-$matrix*; if $\mu(A)\in M_n^0$ with $a_{ii}\neq 0$ for all $i\in\langle n\rangle$, A is called a *mixed $H-$matrix*. H_n, H_n^I, H_n^S and H_n^M

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ELA
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will denote the set of all $n \times n$ general $H-$matrices, the set of all $n \times n$ invertible $H-$matrices, the set of all $n \times n$ singular $H-$matrices and the set of all $n \times n$ mixed $H-$matrices, respectively (see [2]). Similar to equalities (2.1), we have

$$H_n = H_n^I \cup H_n^S \cup H_n^M \quad \text{and} \quad H_n^I \cap H_n^S \cap H_n^M = \emptyset. \tag{2.3}$$

For $n \geq 2$, an $n \times n$ complex matrix A is *reducible* if there exists an $n \times n$ permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is called *irreducible.* If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Definition 2.1. A matrix $A \in \mathbb{C}^{n\times n}$ is called *diagonally dominant by row* if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}| \tag{2.4}$$

holds for all $i \in \langle n \rangle$. If inequality in (2.4) holds strictly for all $i \in \langle n \rangle$, A is called *strictly diagonally dominant by row.* If A is irreducible and the inequality in (2.4) holds strictly for at least one $i \in \langle n \rangle$, A is called *irreducibly diagonally dominant by row.* If (2.4) holds with equality for all $i \in \langle n \rangle$, A is called *diagonally equipotent by row.*

$D_n(SD_n,\ ID_n)$ and DE_n will be used to denote the sets of all $n \times n$ (strictly, irreducibly) diagonally dominant matrices and the set of all $n \times n$ diagonally equipotent matrices, respectively.

Definition 2.2. A matrix $A \in \mathbb{C}^{n\times n}$ is called *generalized diagonally dominant* if there exist positive constants α_i, $\ i \in \langle n \rangle$, such that

$$\alpha_i |a_{ii}| \geq \sum_{j=1, j\neq i}^{n} \alpha_j |a_{ij}| \tag{2.5}$$

holds for all $i \in \langle n \rangle$. If inequality in (2.5) holds strictly for all $i \in \langle n \rangle$, A is called *generalized strictly diagonally dominant.* If (2.5) holds with equality for all $i \in \langle n \rangle$, A is called *generalized diagonally equipotent.*

We denote the sets of all $n \times n$ generalized (strictly) diagonally dominant matrices and the set of all $n \times n$ generalized diagonally equipotent matrices by $GD_n(GSD_n)$

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ELA
http://math.technion.ac.il/iic/ela

and GDE_n, respectively.

Definition 2.3. A matrix A is called *nonstrictly diagonally dominant*, if either (2.4) or (2.5) holds with equality for at least one $i \in \langle n\rangle$.

Remark 2.4. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be nonstrictly diagonally dominant and $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of A, then the following hold:

- $A(\alpha, \alpha') = 0$, which shows that A is reducible;
- $A(i_1) = (a_{i_1 i_1})$ being (generalized) diagonally equipotent implies $a_{i_1 i_1} = 0$.

Remark 2.5. Definition 2.2 and Definition 2.3 show that

$$D_n \subset GD_n \text{ and } GSD_n \subset GD_n.$$

The following will introduce the relationship of (generalized) diagonally dominant matrices and general $H-$matrices and some properties of general $H-$matrices that will be used in the rest of the paper.

Lemma 2.6. (See [12, 13, 15, 14]) *Let $A \in D_n(GD_n)$. Then $A \in H_n^I$ if and only if A has no (generalized) diagonally equipotent principal submatrices. Furthermore, if $A \in D_n \cap Z_n(GD_n \cap Z_n)$, then $A \in M_n^\bullet$ if and only if A has no (generalized) diagonally equipotent principal submatrices.*

Lemma 2.7. (See [1]) $SD_n \cup ID_n \subset H_n^I = GSD_n$.

Lemma 2.8. (See [2]) $GD_n \subset H_n$.

It is interesting whether $H_n \subseteq GD_n$ is true or not. The answer is "NOT". Some counterexamples are given in [2] to show that $H_n \subseteq GD_n$ is not true. But, under the condition of "irreducibility", the following conclusion holds.

Lemma 2.9. (See [2]) *Let $A \in \mathbb{C}^{n\times n}$ be irreducible. Then $A \in H_n$ if and only if $A \in GD_n$.*

More importantly, under the condition of "reducibility", we have the following conclusion.

Lemma 2.10. *Let $A \in \mathbb{C}^{n\times n}$ be reducible. Then $A \in H_n$ if and only if in the Frobenius normal form of A*

$$PAP^T = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1s} \\ 0 & R_{22} & \cdots & R_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{ss} \end{bmatrix}, \tag{2.6}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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each irreducible diagonal square block R_{ii} is generalized diagonally dominant, where P is a permutation matrix, $R_{ii} = A(\alpha_i)$ is either 1×1 zero matrices or irreducible square matrices, $R_{ij} = A(\alpha_i, \alpha_j),\ i \neq j,\ i, j = 1, 2, \ldots, s$, further, $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{s} \alpha_i = \langle n \rangle$.

The proof of this lemma follows from Theorem 5 in [2] and Lemma 2.9.

LEMMA 2.11. *A matrix $A \in H_n^M \cup H_n^S$ if and only if in the Frobenius normal from (2.6) of A, each irreducible diagonal square block R_{ii} is generalized diagonally dominant and has at least one generalized diagonally equipotent principal submatrix.*

Proof. It follows from (2.3), Lemma 2.6 and Lemma 2.10 that the conclusion of this lemma is obtained immediately. ◻

3. Some special matrices and their properties. In order to investigate convergence on Gauss-Seidel iterative methods, some definitions of special matrices will be defined and their properties will be proposed to be used in this paper.

DEFINITION 3.1. Let $E^{i\theta} = (e^{i\theta_{rs}}) \in \mathbb{C}^{n \times n}$, where $e^{i\theta_{rs}} = \cos\theta_{rs} + i \sin\theta_{rs}$, $i = \sqrt{-1}$ and $\theta_{rs} \in \mathbb{R}$ for all $r, s \in \langle n \rangle$.

1. The matrix $E^{i\theta} = (e^{i\theta_{rs}}) \in \mathbb{C}^{n \times n}$ is called a $\pi-$ray pattern matrix if
   (a) $\theta_{rs} + \theta_{sr} = 2k\pi$ holds for all $r, s \in \langle n \rangle,\ r \neq s$, where $k \in \mathbb{Z}$;
   (b) $\theta_{rs} - \theta_{rt} = \theta_{ts} + (2k+1)\pi$ holds for all $r, s, t \in \langle n \rangle$ and $r \neq s,\ r \neq t,\ t \neq s$, where $k \in \mathbb{Z}$;
   (c) $\theta_{rr} = 0$ for all $r \in \langle n \rangle$.
2. The matrix $E^{i\theta} = (e^{i\hat{\theta}_{rs}}) \in \mathbb{C}^{n \times n}$ is called a forward $\theta-$ray pattern matrix if
   (a) $\widehat{\theta}_{rs} = \theta_{rs} + \theta$ if $r > s$ and $r, s \in \langle n \rangle$; $\widehat{\theta}_{rs} = \theta_{rs}$, otherwise;
   (b) for all $r, s, t \in \langle n \rangle$, θ_{rs}, θ_{sr}, θ_{rt} and θ_{rr} satisfy the conditions (a), (b) and (c) of 1.
3. The matrix $E^{i\theta} = (e^{i\tilde{\theta}_{rs}}) \in \mathbb{C}^{n \times n}$ is called a backward $\theta-$ray pattern matrix if
   (a) $\tilde{\theta}_{rs} = \theta_{rs} + \theta$ if $r < s$ and $r, s \in \langle n \rangle$; $\tilde{\theta}_{rs} = \theta_{rs}$, otherwise;
   (b) for all $r, s, t \in \langle n \rangle$, θ_{rs}, θ_{sr}, θ_{rt} and θ_{rr} satisfy the conditions (a), (b) and (c) of 1.

It is easy to see that a $\pi-$ray pattern matrix is a $\theta-$ray pattern matrix defined in Definition 3.4 in [16]. In addition, forward $\theta-$ray pattern matrix and backward $\theta-$ray pattern matrix are both $\pi-$ray pattern matrices when $\theta = 0$.

EXAMPLE 3.2. $R = \begin{bmatrix} 1 & e^{i7\pi/4} & e^{i5\pi/3} & e^{i11\pi/6} \\ e^{i\pi/4} & 1 & e^{i11\pi/12} & e^{i13\pi/12} \\ e^{i\pi/3} & e^{i13\pi/12} & 1 & e^{i7\pi/6} \\ e^{i\pi/6} & e^{i11\pi/12} & e^{i5\pi/6} & 1 \end{bmatrix}$,

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ELA
http://math.technion.ac.il/iic/ela

$$R_\theta = \begin{bmatrix} 1 & e^{i7\pi/4} & e^{i5\pi/3} & e^{i11\pi/6} \\ e^{i(\pi/4+\theta)} & 1 & e^{i11\pi/12} & e^{i13\pi/12} \\ e^{i(\pi/3+\theta)} & e^{i(13\pi/12+\theta)} & 1 & e^{i7\pi/6} \\ e^{i(\pi/6+\theta)} & e^{i(11\pi/12+\theta)} & e^{i(5\pi/6+\theta)} & 1 \end{bmatrix} \text{ and}$$

$$R^\theta = \begin{bmatrix} 1 & e^{i(7\pi/4+\theta)} & e^{i(5\pi/3+\theta)} & e^{i(11\pi/6+\theta)} \\ e^{i\pi/4} & 1 & e^{i(11\pi/12+\theta)} & e^{i(13\pi/12+\theta)} \\ e^{i\pi/3} & e^{i13\pi/12} & 1 & e^{i(7\pi/6+\theta)} \\ e^{i\pi/6} & e^{i11\pi/12} & e^{i5\pi/6} & 1 \end{bmatrix}$$

are $\pi-$ray pattern matrix, forward $\theta-$ray pattern matrix and backward $\theta-$ray pattern matrix, respectively.

In fact, any complex matrix $A = (a_{rs}) \in \mathbb{C}^{n\times n}$ has the following form:

$$A = e^{i\eta} \cdot |A| \circ E^{i\theta} = (e^{i\eta} \cdot |a_{rs}| e^{i\theta_{rs}}) \in \mathbb{C}^{n\times n}, \tag{3.1}$$

where $\eta \in \mathbb{R}$, $|A| = (|a_{rs}|) \in \mathbb{R}^{n\times n}$ and $E^{i\theta} = (e^{i\theta_{rs}}) \in \mathbb{C}^{n\times n}$ with $\theta_{rs} \in \mathbb{R}$ and $\theta_{rr} = 0$ for $r, s \in \langle n\rangle$. The matrix $E^{i\theta}$ is called a *ray pattern matrix* of the matrix A.

Definition 3.3. If the ray pattern matrix $E^{i\theta}$ of the matrix A given in (3.1) is a $\pi-$ray pattern matrix, then A is called a $\pi-$ray matrix; if the ray pattern matrix $E^{i\theta}$ is a forward $\theta-$ray pattern matrix, then A is called a forward $\theta-$ray matrix; and if the ray pattern matrix $E^{i\theta}$ is a backward $\theta-$ray pattern matrix, then A is called a backward $\theta-$ray matrix.

$\mathscr{R}_n^\pi$, $\mathscr{L}_n^\theta$ and $\mathscr{U}_n^\theta$ denote the set of all $n \times n$ $\pi-$ray matrices, the set of all $n \times n$ forward $\theta-$ray matrices and the set of all $n\times n$ backward $\theta-$ray matrices, respectively. Obviously, if a matrix $A \in \mathscr{R}_n^\pi$, then $\xi \cdot A \in \mathscr{R}_n^\pi$ for all $\xi \in \mathbb{C}$, the same is the matrices in $\mathscr{L}_n^\theta$ and $\mathscr{U}_n^\theta$, respectively.

Example 3.4. Let $\eta = \pi/4$, $|A| = \begin{bmatrix} 5 & 2 & 9 & 3 \\ 0 & 3 & 1 & 1 \\ 5 & 3 & 5 & 2 \\ 2 & 4 & 2 & 1 \end{bmatrix}$. Then $B = e^{i\eta} \cdot |A| \circ R$, $C = e^{i\eta} \cdot |A| \circ R_\theta$ and $D = e^{i\eta} \cdot |A| \circ R^\theta$ are $\pi-$ray matrix, forward $\theta-$ray matrix and backward $\theta-$ray matrix, respectively.

Theorem 3.5. *Let a matrix $A = D_A - L_A - U_A = (a_{rs}) \in \mathbb{C}^{n\times n}$ with $D_A = diag(a_{11}, a_{22}, \dots, a_{nn})$. Then $A \in \mathscr{R}_n^\pi$ if and only if there exists an $n \times n$ unitary diagonal matrix D such that $D^{-1}AD = e^{i\eta} \cdot (|D_A| - |L_A| - |U_A|)$ for $\eta \in \mathbb{R}$.*

Proof. According to Definition 3.3, $A = e^{i\eta} \cdot |A| \otimes E^{i\theta} = (e^{i\eta} \cdot |a_{rs}| e^{i\theta_{rs}})$. Define a diagonal matrix $D_\phi = diag(e^{i\phi_1}, e^{i\phi_2}, \dots, e^{i\phi_n})$ with $\phi_r = \theta_{1r} + \phi_1 + (2k+1)\pi$ for $\phi_1 \in \mathbb{R}$, $r = 2, 3, \dots, n$, and $k \in \mathbb{Z}$. By Definition 3.1, $D^{-1}AD = e^{i\eta} \cdot (|D_A| - |L_A| - |U_A|)$, which shows that the necessity is true.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

The following will prove the sufficiency. Assume that there exists an $n\times n$ unitary diagonal matrix $D_\phi = diag(e^{i\phi_1}, \ldots, e^{i\phi_n})$ such that $D_\phi^{-1}AD_\phi = e^{i\eta}\cdot(|D_A| - |L_A| - |U_A|)$. Then

$$\theta_{rs} = \phi_r - \phi_s + (2k_{rs}+1)\pi \tag{3.2}$$

holds for all $r, s \in \langle n\rangle$ and $r \neq s$, where $k_{rs} \in \mathbb{Z}$. In (3.2), $\theta_{rs}+\theta_{sr} = 2(k_{rs}+k_{sr}+1)\pi = 2k\pi$ with $k = k_{rs} + k_{sr} + 1 \in \mathbb{Z}$ and for all $r, s \in \langle n\rangle,\ r \neq s$. Following (3.2), $\theta_{ts} = \phi_s - \phi_t + (2k_{ts} + 1)\pi$. Hence, $\phi_s - \phi_t = \theta_{ts} - (2k_{ts} + 1)\pi$. Consequently, $\theta_{rs}-\theta_{rt} = \phi_s-\phi_t+2(k_{rs}-k_{rt})\pi = \theta_{ts}+[2(k_{rs}-k_{rt}-k_{ts}-1)+1]\pi = \theta_{ts}+(2k'+1)\pi$ for all $r, s, t \in \langle n\rangle$ and $r \neq s,\ r \neq t,\ t \neq s$, where $k' = k_{rs} - k_{rt} - k_{ts} - 1 \in \mathbb{Z}$. In the same method, we can prove that $\theta_{sr} - \theta_{tr} = \theta_{st} + (2\hat{k} + 1)\pi$ hold for all $r, s, t \in \langle n\rangle$ and $r \neq s,\ r \neq t,\ t \neq s$, where $\hat{k} \in \mathbb{Z}$. Furthermore, it is obvious that $\theta_{rr} = \theta$ for all $r \in \langle n\rangle$. This completes the sufficiency. $\square$

In the same method of proof as Theorem 3.5, the following conclusions will be established.

THEOREM 3.6. *Let a matrix $A = D_A - L_A - U_A = (a_{rs}) \in \mathbb{C}^{n\times n}$ with $D_A = diag(a_{11}, a_{22}, \ldots, a_{nn})$. Then $A \in \mathscr{L}_n^\theta$ if and only if there exists an $n \times n$ unitary diagonal matrix D such that $D^{-1}AD = e^{i\eta}\cdot[(|D_A| - |U_A|]) - e^{i\theta}|L_A|$ for $\eta \in \mathbb{R}$.*

THEOREM 3.7. *Let a matrix $A = D_A - L_A - U_A = (a_{rs}) \in \mathbb{C}^{n\times n}$ with $D_A = diag(a_{11}, a_{22}, \ldots, a_{nn})$. Then $A \in \mathscr{U}_n^\theta$ if and only if there exists an $n \times n$ unitary diagonal matrix D such that $D^{-1}AD = e^{i\eta}\cdot[(|D_A| - |L_A|) - e^{i\theta}|U_A|]$ for $\eta \in \mathbb{R}$.*

COROLLARY 3.8. $\mathscr{R}_n^\pi = \mathscr{L}_n^0 = \mathscr{U}_n^0 = \mathscr{L}_n^\theta \cap \mathscr{U}_n^\theta$.

Proof. By Theorem 3.5, Theorem 3.6 and Theorem 3.7, the proof is obtained immediately. $\square$

4. Convergence on Gauss-Seidel iterative methods. In numerical linear algebra, the Gauss-Seidel iterative method, also known as the Liebmann method or the method of successive displacement, is an iterative method used to solve a linear system of equations. It is named after the German mathematicians Carl Friedrich Gauss(1777-1855) and Philipp Ludwig von Seidel(1821-1896), and is similar to the Jacobi method. Later, this iterative method was developed as three iterative methods, i.e., the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iterative methods. Though these iterative methods can be applied to any matrix with non-zero elements on the diagonals, convergence is only guaranteed if the matrix is strictly or irreducibly diagonally dominant matrix, Hermitian positive definite matrix and invertible $H-$matrix. Some classic results on convergence on Gauss-Seidel

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ELA
http://math.technion.ac.il/iic/ela

iterative methods as follows:

Theorem 4.1. (See [8, 10, 11]) *Let $A \in SD_n \cup ID_n$. Then $\rho(H_{FGS}) < 1$, $\rho(H_{BGS}) < 1$ and $\rho(H_{SGS}) < 1$, where H_{FGS}, H_{BGS} and H_{SGS} are defined in (1.2), (1.3) and (1.4), respectively, and therefore, the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. The proof of convergence on FGS-scheme is seen in the proof of Theorem 3.4 in [11]. In the same method as the proof on convergence of FGS-scheme in Theorem 3.4 of [11], we can prove the convergence of BGS-scheme. In addition, according to Lemma 2.7 and Theorem 5.22 in [8] for $\omega = 1$, SGS-scheme converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. $\square$

Theorem 4.2. (See [8, 10, 11]) *Let $A \in H_n^I$. Then the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. The proof of convergence on FGS-scheme is seen in the proof of Theorem 5.12 in [8]. Similar to the proof of Theorem 5.12 in [8], we can prove the convergence of BGS-scheme. Further, according to Theorem 5.22 in [8] for $\omega = 1$, the proof of convergence on SGS-scheme is obtained immediately. $\square$

Theorem 4.3. (See [8, 10, 11]) *Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix. Then the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. The proof of convergence on FGS-scheme is seen in the proof of Corollary 1 of Theorem 3.6 in [11]. Similar to the proof of this corollary, the convergence of BGS-scheme can be proved. Furthermore, when $\omega = 1$, Theorem 5.23 in [8] implies that SGS-scheme converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. $\square$

In this section, we consider convergence on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices and general $H-$matrices. Above all, we investigate the case of nonstrictly diagonally dominant matrices.

Theorem 4.4. *Let $A \in D_n(GD_n)$. If A has no (generalized) diagonally equipotent principal submatrices, then the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. Since $A \in D_n(GD_n)$ and has no (generalized) diagonally equipotent principal submatrices, it follows from Lemma 2.6 that $A \in H_n^I$. Then Theorem 4.2 shows

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

that the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. □

Theorem 4.4 indicates that if we study convergence on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices we only investigate the case of (generalized) diagonally equipotent matrices.

4.1. Convergence on forward Gauss-Seidel iterative methods.

In this subsection, we mainly establish some convergence results on FGS-iterative method for linear systems with (generalized) diagonally equipotent matrices and then generalize these results to nonstrictly diagonally dominant matrices and general $H-$matrices. Finally, an analogous way presents convergence on BGS-iterative method for nonstrictly diagonally dominant matrices and general $H-$matrices. Let us consider firstly the case of 2×2 (generalized) diagonally equipotent matrices.

Theorem 4.5. *Let an irreducible matrix $A=(a_{ij})\in GDE_2$. Then $\rho(H_{FGS})=\rho(H_{BGS})=\rho(H_{SGS})=1$, where H_{FGS}, H_{BGS} and H_{SGS} are defined in (1.2), (1.3) and (1.4), respectively, and therefore, the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, doesn't converge to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. Assume $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\in GDE_2$. By Definition 2.2, $\alpha_1|a_{11}|=\alpha_2|a_{12}|$ and $\alpha_2|a_{22}|=\alpha_1|a_{21}|$ with $a_{ij}\neq 0$ and $\alpha_i>0$ for all $i,j=1,2$. Consequently, $A\in GDE_2$ if and only if $|a_{12}a_{21}|/|a_{11}a_{22}|=1$. Direct computations give that $\rho(H_{FGS})=\rho(H_{BGS})=\rho(H_{SGS})=|a_{12}a_{21}|/|a_{11}a_{22}|=1$ and consequently, the sequence $\{x^{(i)}\}$ generated by FGS-, BGS- and SGS-scheme (1.5), respectively, doesn't converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. □

In what follows, we consider the convergence of FGS-scheme for linear systems with $n\times n$ ($n\geq 3$) (generalized) diagonally equipotent matrices. Continuing in this direction, some lemmas will be introduced firstly to be used in this section.

Lemma 4.6. (See [7, 13]) *Let an irreducible matrix $A\in D_n(GD_n)$. Then A is singular if and only if $D_A^{-1}A\in DE_n(GDE_n)\cap\mathscr{R}_n^{\pi}$, where $D_A=diag(a_{11},\ldots,a_{nn})$.*

Lemma 4.7. *Let $A=(a_{ij})\in DE_n$ ($n\geq 3$) be irreducible. Then $e^{i\theta}$ is an eigenvalue of H_{FGS} if and only if $D_A^{-1}A\in\mathscr{U}_n^{\theta}$, where $D_A=diag(a_{11},a_{22},\ldots,a_{nn})$ and $\theta\in\mathbb{R}$.*

Proof. We prove the sufficiency firstly. Since $A=(a_{ij})\in DE_n$ is irreducible, $a_{ii}\neq 0$ for all $i\in\langle n\rangle$ and $D_A-L_A\in ID_n$ is nonsingular. Consequently, $(D_A-L_A)^{-1}$ and H_{FGS} exist, where $D_A=diag(a_{11},a_{22},\ldots,a_{nn})$. Assume $D_A^{-1}A\in\mathscr{U}_n^{\theta}$. Theorem

Electronic Journal of Linear Algebra ISSN 1081-3810
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3.6 shows that there exists an unitary diagonal matrix D such that

$$D^{-1}(D_A^{-1}A)D = [(I - |D_A^{-1}L_A|) - e^{i\theta}|D_A^{-1}U_A|]$$

for $\theta \in \mathbb{R}$. Hence,

$$D_A^{-1}A = D(I - |D_A^{-1}L_A|)D^{-1} - e^{i\theta}D|D_A^{-1}U_A|D^{-1}$$

and

$$\begin{aligned} H_{FGS} &= (D_A - L_A)^{-1}U_A = (I - D_A^{-1}L_A)^{-1}D_A^{-1}U_A \\ &= [D(I - |D_A^{-1}L_A|)D^{-1}]^{-1}(e^{i\theta}D|D_A^{-1}U_A|D^{-1}) \\ &= e^{i\theta}D[(I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|]D^{-1}. \end{aligned} \tag{4.1}$$

Using (4.1),

$$\begin{aligned} \det(e^{i\theta}I - H_{FGS}) &= \det[e^{i\theta}I - e^{i\theta}D((I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|)D^{-1}] \\ &= e^{i\theta} \cdot \det(I - (I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|) \\ &= \frac{e^{i\theta} \cdot \det(I - |D_A^{-1}L_A| - |D_A^{-1}U_A|)}{\det(I - |D_A^{-1}L_A|)} \\ &= \frac{e^{i\theta} \cdot \det \mu(D_A^{-1}A)}{\det(I - |D_A^{-1}L_A|)}. \end{aligned} \tag{4.2}$$

Since $A \in DE_n$ is irreducible, so is $\mu(D_A^{-1}A) \in DE_n$. Furthermore, according to the definition (2.2) and Theorem 3.5, we have $\mu(D_A^{-1}A) \in \mathscr{R}_n^{\pi}$. Thus, $\mu(D_A^{-1}A) \in DE_n \cap \mathscr{R}_n^{\pi}$ is irreducible. Then it follows from Lemma 4.6 that $\mu(D_A^{-1}A)$ is singular. As a result, (4.2) gives $\det(e^{i\theta}I - H_{FGS}) = 0$ to reveal that $e^{i\theta}$ is an eigenvalue of H_{FGS}. This completes the sufficiency.

The following prove the necessity. Let $e^{i\theta}$ is an eigenvalue of H_{FGS}. Then

$$\begin{aligned} \det(e^{i\theta}I - H_{FGS}) &= \det(e^{i\theta}I - (D_A - L_A)^{-1}U_A) \\ &= \frac{\det[e^{i\theta}(D_A - L_A) - U_A]}{\det(D_A - L_A)} \\ &= 0. \end{aligned}$$

Thus, $\det(e^{i\theta}(D_A - L_A) - U_A) = 0$ which shows that $e^{i\theta}(D_A - L_A) - U_A$ is singular. Since $e^{i\theta}(D_A - L_A) - U_A \in DE_n$ is irreducible due to irreducibility of A, Lemma 4.6 implies that $I - D_A^{-1}L_A - e^{-i\theta}D_A^{-1}U_A \in \mathscr{R}_n^{\pi}$. Thus, according to Theorem 3.5, there exists a unitary diagonal matrix D such that

$$D^{-1}(I - D_A^{-1}L_A - e^{-i\theta}D_A^{-1}U_A)D = I - |D_A^{-1}L_A| - |D_A^{-1}U_A|. \tag{4.3}$$

But, equality (4.3) shows

$$D^{-1}(D_A^{-1}L_A)D = |D_A^{-1}L_A| \text{ and } D^{-1}(D_A^{-1}U_A)D = e^{i\theta}|D_A^{-1}U_A|.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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Then,

$$\begin{aligned} D^{-1}(D_A^{-1}A)D &= I - D^{-1}(D_A^{-1}L_A)D - D^{-1}(D_A^{-1}U_A)D \\ &= I - |D_A^{-1}L_A| - e^{i\theta}|D_A^{-1}U_A|, \end{aligned}$$

and there exists an unitary diagonal matrix D such that

$$D^{-1}(D_A^{-1}A)D^{-1} = I - |D_A^{-1}L_A| - e^{i\theta}|D_A^{-1}U_A|).$$

Theorem 3.6 shows that $D_A^{-1}A \in \mathscr{U}_n^\theta$. So, we finish the necessity. □

Theorem 4.8. *Let $A \in DE_n$ $(n \geq 3)$ be irreducible. Then $\rho(H_{FGS}) < 1$, where H_{FGS} is defined in (1.2), i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{U}_n^\theta$.*

Proof. The sufficiency can be proved by contradiction. We assume that there exists an eigenvalue λ of H_{FGS} such that $|\lambda| \geq 1$. Then

$$det(\lambda I - H_{FGS}) = 0. \tag{4.4}$$

If $|\lambda| > 1$, then $\lambda I - H_{FGS} = (D_A - L_A)^{-1}(\lambda D_A - \lambda L_A - U_A)$. Obviously, $\lambda I - \lambda L - U \in ID_n$ and is nonsingular (see Theorem 1.21 in [11]). As a result, $det(\lambda I - H_{FGS}) \neq 0$, which contradicts (4.4). Thus, $|\lambda| = 1$. Set $\lambda = e^{i\theta}$, where $\theta \in R$. Then Lemma 4.7 shows that $D_A^{-1}A \in \mathscr{U}_n^\theta$, which contradicts the assumption $A \notin \mathscr{U}_n^\theta$. Therefore, $\rho(H_{FGS}) < 1$. The sufficiency is finished.

Let us prove the necessity by contradiction. Assume that $D_A^{-1}A \in \mathscr{U}_n^\theta$. It then follows from Lemma 4.7 that $\rho(H_{FGS}) = 1$ which contradicts $\rho(H_{FGS}) < 1$. A contradiction arises to demonstrate that the necessity is true. Thus, we complete the proof. □

Following, the conclusion of Theorem 4.8 will be extended to irreducibly generalized diagonally equipotent matrices and irreducibly mixed $H-$matrices.

Theorem 4.9. *Let $A = (a_{ij}) \in GDE_n$ $(n \geq 3)$ be irreducible. Then the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{U}_n^\theta$.*

Proof. According to Definition 2.2, the exists a diagonal matrix $E = diag(e_1, e_2, \ldots, e_n)$ with $e_k > 0$ for all $k \in \langle n \rangle$, such that $AE = (a_{ij}e_j) \in DE_n$. Let $AE = F = (f_{ij})$ with $f_{ij} = a_{ij}e_j$ for all $i, j \in \langle n \rangle$. Then $H_{FGS}^F = E^{-1}H_{FGS}E$ and $D_F^{-1}F = E^{-1}(D_A^{-1}A)E$ with $D_F = D_AE$. Theorem 4.8 yields that $\rho(H_{FGS}^F) < 1$ if and only if $D_F^{-1}F \notin \mathscr{U}_n^\theta$. Since $\rho(H_{FGS}^F) = \rho(H_{FGS}$ and $D_A^{-1}A \notin \mathscr{U}_n^\theta$ for $D_F^{-1}F = E^{-1}(D_A^{-1}A)E \notin \mathscr{U}_n^\theta$ and $E = diag(e_1, e_2, \ldots, e_n)$ with $e_k > 0$ for all

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

$k \in \langle n \rangle$, $\rho(H_{FGS}) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{U}_n^{\theta}$., i.e., $\rho(H_{FGS}) < 1$ if and only if $D_A^{-1}A \notin \mathscr{U}_n^{\theta}$. ◻

THEOREM 4.10. *Let $A = (a_{ij}) \in H_n^M$ $(n \geq 3)$ be irreducible. Then the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{U}_n^{\theta}$.*

Proof. Since $A \in H_n^M$ $(n \geq 3)$ be irreducible, it follows from Lemma 2.9 and Lemma 2.11 that $A \in GDE_n$ $(n \geq 3)$ be irreducible. Therefore, Theorem 4.9 shows that the conclusion of this theorem holds. ◻

It follows that some convergence results on forward Gauss-Seidel iterative method are established for nonstrictly diagonally dominant matrices and general $H-$matrices.

THEOREM 4.11. *Let $A = (a_{ij}) \in D_n(GD_n)$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \notin \mathscr{U}_k^{\theta} \cap DE_k(\mathscr{U}_k^{\theta} \cap GDE_k)$, where $D_{A_k} = diag(a_{i_1i_1}, a_{i_2i_2}, \ldots, a_{i_ki_k})$.*

Proof. The proof is obtained immediately by Theorem 4.4, Theorem 4.5, Theorem 4.8 and Theorem 4.9. ◻

THEOREM 4.12. *Let $A = (a_{ij}) \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \notin \mathscr{U}_k^{\theta} \cap GDE_k$.*

Proof. If $A \in H_n$ is irreducible, it follows from Theorem 4.5 and Theorem 4.10 that the conclusion of this theorem is true. If $A \in H_n$ is reducible, since $A \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$, Theorem 2.10 shows that each diagonal square block R_{ii} in the Frobenius normal from (2.6) of A is irreducible and generalized diagonally dominant for $i = 1, 2, \ldots, s$. Let $H_{FGS}^{R_{ii}}$ denote the Gauss-Seidel iteration matrix associated with diagonal square block R_{ii}. Direct computations give

$$\rho(H_{FGS}) = \max_{1 \leq i \leq s} \rho(H_{FGS}^{R_{ii}}).$$

Since R_{ii} is irreducible and generalized diagonally dominant, Theorem 4.4, Theorem 4.5, Theorem 4.8, Theorem 4.9 and Theorem 4.11 show that $\rho(H_{FGS}) = \max_{1 \leq i \leq s} \rho(H_{FGS}^{R_{ii}}) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.5) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 843-870, December 2015

ELA
http://math.technion.ac.il/iic/ela

only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \dots, i_k)$, $3 \le k \le n$, such that $D_{A_k}^{-1} A_k \notin \mathscr{U}_k^{\theta} \cap GDE_k$. $\square$

As has been shown in Theorem 4.8, Theorem 4.9, Theorem 4.10, Theorem 4.11 and Theorem 4.12, convergence on FGS-scheme is established for (generalized) diagonally equipotent matrices, nonstrictly diagonally dominant matrices and general $H-$matrices. The following will list the same convergence results on the BGS-scheme. The proofs can be derived in an analogous way to the ones that are given.

Theorem 4.13. *Let $A \in DE_n$ $(n \ge 3)$ be irreducible. Then the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1} A \notin \mathscr{L}_n^{\theta}$.*

Theorem 4.14. *Let $A = (a_{ij}) \in GDE_n$ $(n \ge 3)$ be irreducible. Then the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1} A \notin \mathscr{L}_n^{\theta}$.*

Theorem 4.15. *Let $A = (a_{ij}) \in H_n^M$ $(n \ge 3)$ be irreducible. Then the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1} A \notin \mathscr{L}_n^{\theta}$.*

Theorem 4.16. *Let $A = (a_{ij}) \in D_n(GD_n)$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \dots, i_k)$, $3 \le k \le n$, such that $D_{A_k}^{-1} A_k \notin \mathscr{L}_k^{\theta} \cap DE_k(\mathscr{L}_k^{\theta} \cap GDE_k)$, where $D_{A_k} = diga(a_{i_1 i_1}, a_{i_2 i_2}, \dots, a_{i_k i_k})$.*

Theorem 4.17. *Let $A = (a_{ij}) \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \dots, i_k)$, $3 \le k \le n$, such that $D_{A_k}^{-1} A_k \notin \mathscr{L}_k^{\theta} \cap GDE_k$.*

4.2. Convergence on symmetric Gauss-Seidel iterative method.

In this subsection, convergence on SGS-iterative method will be established for nonstrictly diagonally dominant matrices and general $H-$matrices.

Above all, the case of (generalized) diagonally equipotent matrices will be studied. The following lemma will be used in this section.

Lemma 4.18. (See Lemma 3.13 in [15]) *Let $A = \begin{bmatrix} E & U \\ L & F \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$, where*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 843-870, December 2015
ELA
http://math.technion.ac.il/iic/ela

$E, F, L, U \in C^{n\times n}$ and E is nonsingular. Then the Schur complement of A with respect to E, i.e., $A/E = F - LE^{-1}U$ is nonsingular if and only if A is nonsingular.

THEOREM 4.19. *Let $A \in DE_n$ $(n \geq 3)$ be irreducible. Then $\rho(H_{SGS}) < 1$, where H_{SGS} is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\pi}$.*

Proof. The sufficiency can be proved by contradiction. We assume that there exists an eigenvalue λ of H_{SGS} such that $|\lambda| \geq 1$. According to equality (1.4),

$$\begin{aligned}\det(\lambda I - H_{SGS}) &= \det(\lambda I - (D_A - U_A)^{-1}L_A(D_A - L_A)^{-1}U_A)\\ &= \det[(D_A - U_A)^{-1}]\det[\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A]\\ &= \frac{\det[\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A]}{\det(D_A - U_A)}\\ &= 0.\end{aligned} \tag{4.5}$$

Equality (4.5) gives

$$\det B = \det[\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A]] = 0, \tag{4.6}$$

i.e., $B := \lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A$ is singular. Let $E = D_A - L_A$, $F = \lambda(D_A - U_A)$ and

$$C = \begin{bmatrix} E & -U_A \\ -L_A & F \end{bmatrix} = \begin{bmatrix} D_A - L_A & -U_A \\ -L_A & \lambda(D_A - U_A) \end{bmatrix}. \tag{4.7}$$

Then $B = F - L_AE^{-1}U_A$ is the Schur complement of C with respect to the principal submatrix E. Now, we investigate the matrix C. Since A is irreducible, both $L_A \neq 0$ and $U_A \neq 0$. As a result, C is also irreducible. If $|\lambda| > 1$, then (4.7) indicates $C \in ID_{2n}$. Consequently, C is nonsingular, so is $B = \lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A$ coming from Lemma 4.18, i.e., $\det B \neq 0$, which contradicts (4.6). Therefore, $|\lambda| = 1$. Let $\lambda = e^{i\theta}$ with $\theta \in R$. (4.6) and Lemma 4.18 yield that $C = \begin{bmatrix} D_A - L_A & -U_A \\ -L_A & e^{i\theta}(D_A - U_A) \end{bmatrix}$, and hence, $C_1 = \begin{bmatrix} D_A - L_A & -U_A \\ -e^{-i\theta}L_A & D_A - U_A \end{bmatrix}$ are singular. Since $A = I - L - U \in DE_n$ and is irreducible, both C and C_1 are irreducible diagonally equipotent. The singularity of C_1 and Lemma 4.6 yield that $D_{C_1}^{-1}C_1 \in \mathscr{R}_{2n}^{\pi}$, where $D_{C_1} = diag(D_A, D_A)$, i.e., there exists an $n \times n$ unitary diagonal matrix D such that $\widetilde{D} = diag(D, D)$ and

$$\begin{aligned}\widetilde{D}^{-1}(D_{C_1}^{-1}C_1)\widetilde{D} &= \begin{bmatrix} I - D^{-1}(D_A^{-1}L_A)D & -D^{-1}(D_A^{-1}U_A)D \\ -e^{-i\theta}D^{-1}(D_A^{-1}L_A)D_A^{-1}L_AD & I - D^{-1}(D_A^{-1}U_A)D \end{bmatrix}\\ &= \begin{bmatrix} I - |D_A^{-1}L_A| & -|D_A^{-1}U_A| \\ -|D_A^{-1}L_A| & I - |D_A^{-1}U_A| \end{bmatrix}.\end{aligned} \tag{4.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 843-870, December 2015

(4.8) indicates that $\theta = 2k\pi$, where k is an integer. Thus, $\lambda = e^{i2k\pi} = 1$, and there exists an $n \times n$ unitary diagonal matrix D such that $D^{-1}(D_A^{-1}A)D = I - |D_A^{-1}L_A| - |D_A^{-1}U_A|$, i.e., $D_A^{-1}A \in \mathscr{R}_n^{\pi}$. However, this contradicts $D_A^{-1}A \notin \mathscr{R}_n^0$. Thus, $|\lambda| \neq 1$. According to the proof above, we have that $|\lambda| \geq 1$ is not true. Therefore, $\rho(H_{SGS}) < 1$, i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

The following will prove the necessity by contradiction. Assume that $D_A^{-1}A \in \mathscr{R}_n^{\pi}$. Then there exists an $n\times n$ unitary diagonal matrix D such that $D_A^{-1}A = I - D_A^{-1}L_A - D_A^{-1}U_A = I - D|D_A^{-1}L_A|D^{-1} - D|D_A^{-1}U_A|D^{-1}$ and

$$\begin{aligned} H_{SGS} &= (D_A - U_A)^{-1}L_A(D_A - L_A)^{-1}U_A \\ &= [I - (D_A^{-1}U_A)]^{-1}(D_A^{-1}L_A)[I - (D_A^{-1}L_A)]^{-1}(D_A^{-1}U_A) \\ &= D[(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|]D^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &\det(I - H_{SGS}) \\ &= \det\{I - D[(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|]D^{-1}\} \\ &= \det[I - (I - |D_A^{-1}U_A|)^{-1}|D_A^{-1}L_A|(I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|] \\ &= \frac{\det[(I - |D_A^{-1}U_A|) - |D_A^{-1}L_A|(I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|]}{\det(I - |D_A^{-1}U_A|)}. \end{aligned} \tag{4.9}$$

Let $V = \begin{bmatrix} I - |D_A^{-1}L_A| & -|D_A^{-1}U_A| \\ -|D_A^{-1}L_A| & I - |D_A^{-1}U_A| \end{bmatrix}$ and $W = (I - |D_A^{-1}U_A|) - |D_A^{-1}L_A|(I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|$. Then W is the Schur complement of V with respect to $I - |D_A^{-1}L_A|$. Since $A = I - L - U \in DE_n$ is irreducible, $D_A^{-1}A = I - D_A^{-1}L_A - D_A^{-1}U_A \in DE_n$ is irreducible. Therefore, $V \in DE_{2n} \cap \mathscr{R}_{2n}^{\pi}$ and is irreducible. Lemma 4.6 shows that V is singular, and hence,

$$\det V = \det[(I - |D_A^{-1}U_A|) - |D_A^{-1}L_A|(I - |D_A^{-1}L_A|)^{-1}|D_A^{-1}U_A|] = 0.$$

Therefore, (4.9) yields $\det(I - H_{SGS}) = 0$, which shows that 1 is an eigenvalue of H_{SGS}. Thus, $\rho(H_{SGS}) \geq 1$, i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) doesn't converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. This is a contradiction which shows that the assumption is incorrect. Therefore, $A \notin \mathscr{R}_n^{\pi}$. $\square$

Lemma 4.6 shows that the following corollary holds.

Corollary 4.20. *Let $A \in DE_n$ $(n \geq 3)$ be irreducible and nonsingular. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 843-870, December 2015

THEOREM 4.21. *Let $A \in H_n^M$ (GDE_n) be irreducible for $n \geq 3$. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathscr{R}_n^{\pi}$.*

Proof. According to Lemma 2.9 and Lemma 2.11, under the condition of irreducibility, $H_n^M = GDE_n$. Then, similar to the proof of Theorem 4.9, we can obtain the proof by Definition 2.2 and Theorem 4.19. ☐

COROLLARY 4.22. *Let $A \in H_n^M$ $(n \geq 3)$ be irreducible and nonsingular. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. It follows from Lemma 4.6 that the proof of this corollary is obtained immediately. ☐

In what follows, some convergence results on symmetric Gauss-Seidel iterative methods are established for nonstrictly diagonally dominant matrices.

THEOREM 4.23. *Let $A = (a_{ij}) \in D_n(GD_n)$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \notin \mathscr{R}_k^{\pi} \cap DE_k(\mathscr{R}_k^{\pi} \cap GDE_k)$.*

Proof. It follows from Theorem 4.4, Theorem 4.5, Theorem 4.19 and Theorem 4.21 that the proof of this theorem is obtained immediately. ☐

THEOREM 4.24. *Let $A \in GD_n$ be nonsingular. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has no 2×2 irreducibly generalized diagonally equipotent principal submatrices.*

Proof. Since $A \in GD_n$ is nonsingular, it follows from Theorem 3.11 in [16] that A hasn't any irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \in \mathscr{R}_k^{\pi}$, and hence, $D_{A_k}^{-1}A_k \notin \mathscr{R}_k^{\pi} \cap GDE_k$. Then the conclusion of this theorem follows Theorem 4.23. ☐

In the rest of this section, the convergence results on symmetric Gauss-Seidel iterative method for nonstrictly diagonally dominant matrices will be extended to general $H-$matrices.

THEOREM 4.25. *Let $A = (a_{ij}) \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$ if and only if A has neither 2×2 irre-*

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ELA
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ducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \ldots, i_k)$, $3 \le k \le n$, *such that* $D_{A_k}^{-1} A_k \notin \mathscr{R}_k^\pi \cap GDE_k$.

Proof. Similar to the proof of Theorem 4.12, we can obtain the proof immediately by Theorem 2.10 and Theorem 4.23. ◻

THEOREM 4.26. *Let* $A \in H_n$ *be nonsingular. Then the sequence* $\{x^{(i)}\}$ *generated by SGS-scheme (1.5) converges to the unique solution of* (1.1) *for any choice of the initial guess* $x^{(0)}$ *if and only if* A *has no* 2×2 *irreducibly generalized diagonally equipotent principal submatrices.*

Proof. The proof is similar to the proof of Theorem 4.24. ◻

4.3. Conclusions and remarks.

In the end, the convergence results established in Subsection 1 and 2 are summarized into a table (see Table 4.1) which shows that the comparison results on convergence of FGS, BGS and SGS method for different class of general $H-$matrices. In Table 4.1, DD, $GDE(DE)$, PSM, Y, N, C, D and $\times$ denote diagonal dominance, generalized diagonally equipotent (diagonally equipotent), principal submatrices (submatrix), yes, no, convergence, divergence and unapplicable, respectively.

The research in this section shows that the FGS iterative method associated with the irreducible matrix $A \in H_n^M \cap \mathscr{U}_n^\theta$ fails to converge, the same does for the BGS iterative method associated with the irreducible matrix $A \in H_n^M \cap \mathscr{L}_n^\theta$ and the SGS iterative method associated with the irreducible matrix $A \in H_n^M \cap \mathscr{R}_n^\pi$. It is natural to consider convergence on preconditioned Gauss-Seidel iterative methods for nonsingular general $H-$matrices.

5. Convergence on preconditioned Gauss-Seidel iterative methods. In this section, Gauss-type preconditioning techniques for linear systems with nonsingular general $H-$matrices are chosen such that the coefficient matrices are invertible $H-$matrices. Then based on structure heredity of the Schur complements for general $H-$matrices in [16], convergence on preconditioned Gauss-Seidel iterative methods will be studied and some results will be established.

Many researchers have considered the left Gauss-type preconditioner applied to linear system (1.1) such that the associated Jacobi and Gauss-Seidel methods converge faster than the original ones. Milaszewicz [9] considered the preconditioner

$$P_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix}. \tag{5.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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Table 4.1. The comparison results on convergence of FGS, BGS and SGS method for different class of general $H-$matrices

items	classes	a_{ii}	DD	PSM	order	ray matrices	singular	FGS	BGS	SGS
H_n	H_n^I	$a_{ii} \neq 0$ $\forall i \in \langle n \rangle$	GSD_n $(SD_n \cup ID_n)$	no GDE (DE) PSM			N	C	C	C
	H_n^M	$a_{ii} \neq 0$ $\forall i \in \langle n \rangle$	Irreducible GD_n	GDE_n (DE_n)	$n = 2$		Y or N	D	D	D
					$n \geq 3$	$\mathscr{U}_n^\theta / \mathscr{R}_n^\pi$	N	D	C	C
						$\mathscr{L}_n^\theta / \mathscr{R}_n^\pi$	N	C	D	C
						$\mathscr{R}_n^\pi$	Y	D	D	D
						otherwise	N	C	C	C
			Reducible GD_n and other	$k \times k$ irreducible $GDE(DE)$ PSM A_k exist	$k = 2$		Y or N	D	D	D
					$k \geq 3$ and $k < n$	$D_{A_k}^{-1} A_k \in \mathscr{U}_k^\theta / \mathscr{R}_k^\pi$	N	D	C	C
						$D_{A_k}^{-1} A_k \in \mathscr{L}_k^\theta / \mathscr{R}_k^\pi$	N	C	D	C
						$D_{A_k}^{-1} A_k \in \mathscr{R}_k^\pi$	Y	D	D	D
						otherwise	N	C	C	C
	H_n^S	$a_{ii} = 0$ $\exists i \in \langle n \rangle$	Reducible GD_n and other	$GDE(DE)$ PSM exist			Y	$\times$	$\times$	$\times$

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ELA
http://math.technion.ac.il/iic/ela

Later, Hadjidimos et al. [6] generalized Milaszewicz's preconditioning technique and presented the preconditioner

$$P_1(\alpha) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n1} & 0 & \cdots & 1 \end{bmatrix}. \tag{5.2}$$

Recently, Zhang et al. [17] proposed the left Gauss type preconditioning techniques which utilizes the Gauss transformation [5] matrices as the base of the Gauss type preconditioner based on Hadjidimos et al. [6], Milaszewicz [9] and LU factorization method [5]. The construction of Gauss transformation matrices is as follows:

$$M_k = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\tau_{k+1} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & -\tau_n & 0 & \cdots & 1 \end{bmatrix}, \tag{5.3}$$

where $\tau_i = a_{ik}/a_{kk},\ i = k+1, \ldots, n$ and $k = 1, 2, \ldots, n-1$. Zhang et al. [17] consider the following left preconditioners:

$$\mathscr{P}_1 = M_1,\ \mathscr{P}_2 = M_2 M_1,\ \cdots,\ \mathscr{P}_{n-1} = M_{n-1} M_{n-2} \cdots M_2 M_1.$$

Let $\mathscr{H}_n = \{A \in H_n :\ A \textit{ is nonsingular}\}$. Then $H_n^I \subset \mathscr{H}_n$ while $H_n^I \neq \mathscr{H}_n$. Again, let $\hat{H}_n^M = \{A \in H_n^M :\ A \textit{ is nonsingular}\}$. In fact, $\mathscr{H}_n = H_n^I \cup \hat{H}_n^M$. Thus, nonsingular general $H-$matrices that the matrices in $\mathscr{H}_n$ differ from invertible $H-$matrices. In this section, we will propose some Gauss-type preconditioning techniques for linear systems with the coefficient matrices belong to $\mathscr{H}_n$ and establish some convergence results on preconditioned Gauss-Seidel iterative methods.

Firstly, we consider the case that the coefficient matrix $A \in \mathscr{H}_n$ is irreducible. Then let us generalize the preconditioner of (5.1),(5.2) and (5.3) as follows:

$$\mathscr{P}_k = \begin{bmatrix} 1 & \cdots & 0 & -\tau_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & -\tau_{k-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\tau_{k+1} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\tau_n & 0 & \cdots & 1 \end{bmatrix}, \tag{5.4}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

where $\tau_i = a_{ik}/a_{kk},\ i = 1, \ldots, n;\ i \neq k$ and $k \in \langle n\rangle$. Assume that $\tilde{A}_k = \mathscr{P}_k A$ for $k \in \langle n\rangle$, $H_J^A,\ H_{FGS}^A,\ H_{BGS}^A$ and H_{SGS}^A denote the Jacobi and the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iteration matrices associated with the coefficient matrix A, respectively.

THEOREM 5.1. *Let $A \in \mathscr{H}_n$ be irreducible. Then $\tilde{A}_k = \mathscr{P}_k A \in H_n^I$ for all $k \in \langle n\rangle$, where $\mathscr{P}_k$ is defined in (5.4). Furthermore, the following conclusions hold:*

1. $\rho(H_J^{\tilde{A}_k}) \leq \rho(H_J^{\mu(A/k)}) < 1$ *for all* $k \in \langle n\rangle$*, where* $A/k = A/\alpha$ *with* $\alpha = \{k\}$*;*
2. $\rho(H_{FGS}^{\tilde{A}_k}) \leq \rho(H_{FGS}^{\mu(A/k)}) < 1$ *for all* $k \in \langle n\rangle$*;*
3. $\rho(H_{BGS}^{\tilde{A}_k}) \leq \rho(H_{BGS}^{\mu(A/k)}) < 1$ *for all* $k \in \langle n\rangle$*;*
4. $\rho(H_{SGS}^{\tilde{A}_k}) \leq \rho(H_{SGS}^{\mu(A/k)}) < 1$ *for all* $k \in \langle n\rangle$*,*

i.e., the sequence $\{x^{(i)}\}$ generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.5) converge to the unique solution of (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. Since $A \in H_n^M$ is irreducible and nonsingular for $A \in \mathscr{H}_n$ is irreducible, it follows from Theorem 5.9 in [16] that A/α is an invertible $H-$matrix, where $\alpha = \{k\}$. For the preconditioner $\mathscr{P}_k$, there exists a permutation matrix P_k such that $P_k\mathscr{P}_kP_k^T = \begin{bmatrix} 1 & 0 \\ -\tau & I_{n-1} \end{bmatrix}$, where $\tau = (\tau_1, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_n)^T$. As a consequence,

$$P_k(\mathscr{P}_kA)P_k^T = P_k\mathscr{P}_kP_k^TP_kAP_k^T = \begin{bmatrix} a_{kk} & \alpha_k \\ 0 & A/\alpha \end{bmatrix}$$

is an invertible $H-$matrix, so is $\mathscr{P}_kA$. Following, Theorem 4.1 in [16] and Theorem 4.2 show that the four conclusions hold. $\square$

On the other hand, if an irreducible matrix $A \in \mathscr{H}_n$ has a principal submatrix $A(\alpha)$ which is easy to get its inverse matrix or is a (block)triangular matrix, there exists a permutation matrix P_α such that

$$P_\alpha AP_\alpha^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha', \alpha') \end{bmatrix}, \tag{5.5}$$

where $\alpha' = \langle n\rangle - \alpha$. Let

$$M = \begin{bmatrix} I_{|\alpha|} & 0) \\ -[A(\alpha)]^{-1}A(\alpha', \alpha) & I \end{bmatrix}. \tag{5.6}$$

Then

$$MP_\alpha AP_\alpha^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ 0 & A/\alpha \end{bmatrix},$$

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ELA
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where $A(\alpha)$ and A/α are both invertible $H-$matrices, so is $MP_\alpha AP_\alpha^T$. As a result, $P_\alpha^T MP_\alpha A = P^T(MP_\alpha AP_\alpha^T)P$ is an invertible $H-$matrix. Therefore, we consider the following preconditioner

$$\mathscr{P}_\alpha = P_\alpha^T MP_\alpha, \tag{5.7}$$

where P_α and M are defined by (5.5) and (5.6), respectively.

THEOREM 5.2. *Let $A \in \mathscr{H}_n$ be irreducible. Then $\tilde{A}_\alpha = \mathscr{P}_\alpha A \in H_n^I$ for all $\alpha \subset \langle n\rangle$, $\alpha \neq \emptyset$, where $\mathscr{P}_\alpha$ is defined in (5.7). Furthermore, the following conclusions hold:*

1. $\rho(H_J^{\tilde{A}_\alpha}) \leq \max\{\rho(H_J^{\mu(A(\alpha))}), \rho(H_J^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
2. $\rho(H_{FGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{FGS}^{\mu(A(\alpha))}), \rho(H_{FGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
3. $\rho(H_{BGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{BGS}^{\mu(A(\alpha))}), \rho(H_{BGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
4. $\rho(H_{SGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{SGS}^{\mu(A(\alpha))}), \rho(H_{SGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$,

i.e., the sequence $\{x^{(i)}\}$ generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.5) converge to the unique solution of (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. The proof is similar to the proof of Theorem 5.1. □

Following, we consider the case that the coefficient matrix $A \in \mathscr{H}_n$ is reducible. If there exists a proper $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$ such that $A(\alpha)$ and $A(\alpha')$ are both invertible $H-$matrices, we consider the preconditioner (5.7) and have the following conclusion.

THEOREM 5.3. *Let $A \in \mathscr{H}_n$ and a proper $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$, $\alpha \neq \emptyset$, such that $A(\alpha)$ and $A(\alpha')$ are both invertible $H-$matrices. Then $\tilde{A}_\alpha = \mathscr{P}_\alpha A \in H_n^I$, where $\mathscr{P}_\alpha$ is defined in (5.7). Furthermore, the following conclusions hold:*

1. $\rho(H_J^{\tilde{A}_\alpha}) \leq \max\{\rho(H_J^{\mu(A(\alpha))}), \rho(H_J^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
2. $\rho(H_{FGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{FGS}^{\mu(A(\alpha))}), \rho(H_{FGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
3. $\rho(H_{BGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{BGS}^{\mu(A(\alpha))}), \rho(H_{BGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$;
4. $\rho(H_{SGS}^{\tilde{A}_\alpha}) \leq \max\{\rho(H_{SGS}^{\mu(A(\alpha))}), \rho(H_{SGS}^{\mu(A/\alpha)})\} < 1$ *for all* $\alpha \in \langle n\rangle$,

i.e., the sequence $\{x^{(i)}\}$ generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.5) converge to the unique solution of (1.1) *for any choice of the initial guess $x^{(0)}$.*

Proof. It is obvious that here exists a permutation matrix P_α such that (5.5) holds. Further,

$$\mathscr{P}_\alpha A = P_\alpha^T MP_\alpha A = P^T(MP_\alpha AP_\alpha^T)P = P^T \begin{bmatrix} A(\alpha) & A(\alpha,\alpha') \\ 0 & A/\alpha \end{bmatrix} P. \tag{5.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Since $A \in \mathscr{H}_n$, $A \in H_n^I \cup H_n^M$ is nonsingular. Again, $A(\alpha)$ and $A(\alpha')$ are both invertible $H-$matrices, it follows from Theorem 5.2 and Theorem 5.11 in [16] that A/α is an invertible $H-$matrix. Therefore, $\tilde{A}_\alpha = \mathscr{P}_\alpha A \in H_n^I$ coming from (5.8). Following, Theorem 4.1 in [16] and Theorem 4.2 yield that the four conclusions hold, which completes the proof. $\square$

It is noted that the preconditioner $\mathscr{P}_\alpha$ has at least two shortcomings when the coefficient matrix $A \in \mathscr{H}_n$ is reducible. One is choice of α. For a large scale reducible matrix $A \in \mathscr{H}_n \cap H_n^M$, we are not easy to choose α such that $A(\alpha)$ and $A(\alpha')$ are both invertible $H-$matrices. The other is the computation of $[A(\alpha)]^{-1}$. Although $A(\alpha)$ is an invertible $H-$matrices, it is difficult to obtain its inverse matrix for large $A(\alpha)$. These shortcomings above are our further research topics.

6. Numerical examples. In this section, some examples are given to illustrate the results obtained in Section 4 and Section 5.

Example 6.1. Let the coefficient matrix A of linear system (1.1) be given by the following $n \times n$ matrix

$$A_n = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}. \tag{6.1}$$

It is easy to see that $A_n \in DE_n \subset H_n$ is irreducible and $A_n \notin H_n^I$, but Lemma 4.3 in [13] shows that A_n is nonsingular. Thus, $A_n \in \mathscr{H}_n$ is irreducible. Since

$$D_n^{-1} A_n D_n = |D_{A_n}| - |L_{A_n}| - e^{i\pi}|U_{A_n}|,$$

where

$$D_n = diag[1, -1, \ldots, (-1)^{k-1}, \ldots, (-1)^{n-1}],$$

it follows from Theorem 3.6 that $A_n \in \mathscr{U}_n^\pi$. In addition, it is obvious that $A_n \in \mathscr{L}_n^\pi$. Therefore, Theorem 4.9 and Theorem 4.10 show that

$$\rho(H_{FGS}(A_{100})) = \rho(H_{BGS}(A_{100})) = 1.$$

Further, Theorem 4.16 shows that $\rho(H_{SGS}(A_{100})) < 1$. In fact, direct computations also get $\rho(H_{FGS}(A_{100})) = \rho(H_{BGS}(A_{100})) = 1$ and $\rho(H_{SGS}(A_{100})) = 0.3497 < 1$

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ELA
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which demonstrates that the conclusions of Theorem 4.9, Theorem 4.10 and Theorem 4.16 in Section 4 are correct and effective.

The discussion above shows that FGS and BGS iterative schemes fail to converge to the unique solution of linear system (1.1) with the coefficient matrix (6.1) of for any choice of the initial guess $x^{(0)}$, but SGS iterative schemes does. Now we consider preconditioned Gauss-Seidel iterative methods for linear system (1.1) with the coefficient matrix (6.1).

Choose two set $\alpha = \{1\} \in \langle n\rangle$ and $\beta = \{1, n\} \in \langle n\rangle$ and partition A_n into

$$A_n = \begin{bmatrix} 1 & -a^T \\ a & A_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -b^T & 0 \\ b & A_{n-2} & -c^T \\ 0 & c & 1 \end{bmatrix},$$

where $a = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n-1}$, $b = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n-2}$, $c^T = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n-2}$ and $A_{n-2} = tri[1, 2, -1] \in \mathbb{R}^{(n-2)\times(n-2)}$, we get two preconditioners

$$\mathscr{P}_1 = \begin{bmatrix} 1 & 0 \\ -a & I_{n-1} \end{bmatrix} \text{ and } \mathscr{P}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ -b & I_{n-2} & c^T \\ 0 & c & 1 \end{bmatrix},$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Then Theorem 5.9 in [16] shows that $\tilde{A}_1 = \mathscr{P}_1 A_n = \begin{bmatrix} 1 & -a^T \\ 0 & A_n/\alpha \end{bmatrix}$ and $\tilde{A}_\beta = \mathscr{P}_\beta A_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_n/\beta & 0 \\ 0 & c & 1 \end{bmatrix}$ are both invertible $H-$matrices. According to Theorem 5.1 and Theorem 5.2, for these two preconditioners, the preconditioned FGS, BGS and SGS iterative schemes converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

In fact, by direct computations, Table 6.1 in the following is obtained to show that $\rho(H_{FGS}^{\tilde{A}_1}) = \rho[H_{FGS}^{\mu(\tilde{A}_1)}] = 0.9970 < 1$, $\rho(H_{BGS}^{\tilde{A}_\beta}) = \rho[H_{BGS}^{\mu(\tilde{A}_\beta)}] = 0.9970 < 1$ and $\rho(H_{SGS}^{\tilde{A}_1}) = 0.3333 < \rho[H_{SGS}^{\mu(\tilde{A}_1)}] = 0.9950 < 1$, $\rho(H_{SGS}^{\tilde{A}_\beta}) = 0.3158 < \rho[H_{SGS}^{\mu(\tilde{A}_\beta)}] = 0.9979 < 1$, which illustrate specifically that Theorem 5.1 and Theorem 5.2 are both valid.

X	$\rho(H_X^{\tilde{A}_1})$	$\rho[H_X^{\mu(\tilde{A}_1)}]$	$\rho(H_X^{\tilde{A}_\beta})$	$\rho[H_X^{\mu(\tilde{A}_\beta)}]$
FGS	0.9970	0.9970	0.9900	0.9900
BFS	0.9970	0.9970	0.9900	0.9900
SGS	0.3333	0.9950	0.3158	0.9979

Table 6.1
The comparison result of spectral radii of PGS iterative matrices.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

Example 6.2. Let the coefficient matrix A of linear system (1.1) be given by the following 6×6 matrix

$$A = \begin{bmatrix} 5 & -1 & 1 & 1 & 1 & -1 \\ 1 & 5 & -1 & 1 & 1 & 1 \\ 1 & 1 & 5 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Although $A \in DE_6$ are reducible but there is not any principal submatrix A_k $(k < 6)$ in A such that $D_{A_k}^{-1} A_k \in \mathscr{R}_k^0$, Theorem 3.16 in [16] shows that A is nonsingular. Thus, $A_n \in \mathscr{H}_6$ is reducible. Furthermore, there is not any principal submatrix A_k in A such that $D_{A_k}^{-1} A_k \in \mathscr{U}_k^{\psi}$ and $D_{A_k}^{-1} A \in \mathscr{L}_k^{\phi}$. It follows from Theorem 4.20, Theorem 4.21 and Theorem 4.22 that FGS, BGS and SGS iterative schemes converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

From the first column in Table 6.2, one has $\rho(H_{FGS}) = \rho(H_{BGS}) = 0.3536 < 1$ and $\rho(H_{SGS}) = 0.2500 < 1$. This naturally verifies the results of Theorem 4.20, Theorem 4.21 and Theorem 4.22.

X	$\rho(H_X)$	$\rho(H_X^{\tilde{A}_\alpha})$	$\rho[H_X^{\mu(\tilde{A}_\alpha)}]$
FGS	0.3536	0.6000	0.6000
BFS	0.3536	0.6000	0.6000
SGS	0.2500	0.6000	0.6000

Table 6.2
The comparison result of spectral radii of GS and PGS iterative matrices.

Now, we consider convergence on preconditioned Gauss-Seidel iterative methods. Set $\alpha = \{3,4\} \subset \langle 6 \rangle = \{1,2,3,4,5,6\}$, and set $\beta = \{3,4\} \subset \langle 6 \rangle$ and $\gamma = \{3,4\} \subset \langle 6 \rangle$. Since $A(\beta \cup \gamma) \in H_4^I$, it follows from Theorem 4.3 in [15] that $A/\alpha \in H_4^I$. Thus, we choose a preconditioner

$$\mathscr{P}_\alpha = \begin{bmatrix} I_2 & -A(\beta,\alpha)[A(\alpha)]^{-1} & 0 \\ 0 & I_2 & c^T \\ 0 & -A(\gamma,\alpha)[A(\alpha)]^{-1} & I_2 \end{bmatrix}$$

such that $\tilde{A}_\alpha = \mathscr{P}_\alpha A \in H_6^I$. From Theorem 5.3, it is obvious to see that the preconditioned FGS, BGS and SGS iterative schemes (1.5) converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 843-870, December 2015

ELA
http://math.technion.ac.il/iic/ela

As is shown in Table 6.2, $\rho(H_{FGS}^{\tilde{A}_\alpha}) = \rho[H_{FGS}^{\mu(\tilde{A}_\alpha)}] = 0.6000 < 1$, $\rho(H_{BGS}^{\tilde{A}_\alpha}) = \rho[H_{BGS}^{\mu(\tilde{A}_\alpha)}] = 0.6000 < 1$ and $\rho(H_{SGS}^{\tilde{A}_\alpha}) = \rho[H_{SGS}^{\mu(\tilde{A}_\alpha)}] = 0.6000 < 1$, which directly verifies the results of Theorem 5.3.

7. Conclusions. This paper studies convergence on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices and general $H-$matrices. The definitions of some special matrices are firstly proposed to establish some new results on convergence of Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices and general $H-$matrices. Following, convergence of Gauss-Seidel iterative methods for preconditioned linear systems with general $H-$matrices is established. Finally, some numerical examples are given to demonstrate the results obtained in this paper.

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