Electronic Journal of Linear Algebra

Volume 30 *Volume 30 (2015)* Article 45

2015

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Recommended Citation

Colakoglu, Nurhan and Lancaster, Peter. (2015), "Spectral Properties of Finite-dimensional Waveguide Systems", *Electronic Journal of Linear Algebra*, Volume 30, pp. 670-692.
DOI: https://doi.org/10.13001/1081-3810.2855

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

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http://math.technion.ac.il/iic/ela

SPECTRAL PROPERTIES OF FINITE-DIMENSIONAL WAVEGUIDE SYSTEMS*

NURHAN ÇOLAKOĞLU† AND PETER LANCASTER‡

Abstract. This is a largely expository paper in which a finite dimensional model for gyroscopic/waveguiding systems is studied. Properties of the spectrum that play an important role when computing with such models are studied. The notion of "waveguide-type" is explored in this context. The main theorem provides a form of the central result (due to Abramov) concerning the existence of real spectrum for such systems. The roles of semisimple/defective eigenvalues are discussed, as well as the roles played by eigenvalue "types" (or "Krein signatures"). The theory is illustrated with examples.

Key words. Matrix polynomial, Waveguide, Eigenfunctions.

AMS subject classifications. 15A22, 47A56, 78M10.

1. Introduction. We make some preliminary definitions concerning matrix-valued functions of a complex variable. They are consistent with the usage of references [6]–[8], for example. See also [10]. Let $L_1,\ L_2 \in \mathbb{C}^{n\times n}$ and consider monic, quadratic matrix polynomials:

$$\mathbb{L}(\lambda) = I\lambda^2 + L_1\lambda + L_2, \quad \lambda \in \mathbb{C}. \tag{1.1}$$

Definition 1.1. (a) $\mathbb{L}$ is said to be *selfadjoint* if L_1 and L_2 are Hermitian. (b) $\mathbb{L}$ is said to be *gyroscopic* if it is selfadjoint and $L_1 = iG$, where $G^T = -G \in \mathbb{R}^{n\times n}$. Thus, for $\lambda \in \mathbb{C}$,

$$\mathbb{L}(\lambda) = I\lambda^2 + iG\lambda + L_2, \quad G^T = -G \in \mathbb{R}^{n\times n}, \quad L_2^* = L_2 \in \mathbb{C}^{n\times n}. \tag{1.2}$$

More generally, the leading coefficient could be a positive definite matrix $A \in \mathbb{R}^{n\times n}$, but we assume that the usual reduction to the identity has been applied maintaining the symmetries of the definitions (i.e., $\mathbb{L}(\lambda) \to A^{-1/2}\mathbb{L}(\lambda)A^{-1/2}$). Given the

*Received by the editors on January 14, 2015. Accepted for publication on October 2, 2015. Handling Editor: Panayiotis Psarrakos.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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skew-symmetry of G, systems of the form (1.2) may be described as *gyroscopic*, but it is important to note that the coefficient L_2 will generally be non-real and indefinite. Indeed, L_2 is generally parameter dependent with the form $C - \omega^2 R$, $\omega \in \mathbb{R}$. See Section 2.6.1 of the comprehensive work by Silbergleit and Kopilevich [18], where the problems are first formulated in a Hilbert space context. In that work, and many others, *propagating waves* are studied - in contrast to *evanescent* (fading) waves, and this implies a special interest in *real* eigenvalues for $\mathbb{L}(\lambda)$. Indeed, a major result of the following theory is the fact that a waveguide system always has at least two real eigenvalues (see Theorem 3.1). For a general treatment of waveguides in the context of electromagnetism see [17].

We adapt the earlier treatment of Abramov [1] to our finite-dimensional setting (see also [4] and [11]). In particular, gyroscopic and waveguide systems as defined here (see Definition 2.2 and Proposition 2.3) are obtained when using finite-dimensional approximations (finite-element or finite-difference methods) for continuous systems. Truncation errors involved in such an approximation process are important, but are not our concern in this paper.

The theory is further developed in an infinite-dimensional context in [5] and [11] where special attention is paid to the distribution of the real spectrum (see Figure 4 of [5]) and, physically, to the presence of specific energy-transporting waveforms. It is our objective to study these phenomena in the context of finite dimensional problems - and hence linear algebra - which is generally necessary before computation is possible. This is the context of papers of Chugunova and Pelinovsky [3], Nicolet and Geuzaine [16], and Treysséde and Laguerre [19], for example. As in the more general theoretical context adopted by Abramov, and by Silbergleit and Kopilevich, stability depends on the presence of *real* spectrum (see Section 19.1 of [18]). Eigenvalue problems similar to (1.2) appear in the theory of "photonic crystal fibres". However, in that case, the leading coefficient is singular (see Section 9.4 of Zolla et al. [20].)

Our discussion depends heavily on the four following notions, which are generally useful in the spectral analysis of selfadjoint matrix polynomials.

1.1. The spectrum. The spectrum of $\mathbb{L}$ is defined by

$$\sigma(\mathbb{L}) = \{\lambda \in \mathbb{C} : \det \mathbb{L}(\lambda) = 0\},$$

and members of $\sigma(\mathbb{L})$ are known as *eigenvalues* of $\mathbb{L}$. An eigenvalue is said to be semisimple if its algebraic and geometric multiplicities are equal. Because the leading coefficient of $\mathbb{L}$ is invertible the spectrum (the set af all eigenvalues) is bounded. The selfadjoint property of Definition 1.1(a) ensures that $\sigma(\mathbb{L})$ is symmetric with respect to the real axis. In the special case that $L_2 \in \mathbb{R}^{n\times n}$, $\sigma(\mathbb{L})$ has Hamiltonian symmetry,

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but this is not generally the case if $L_2 \notin \mathbb{R}^{n\times n}$.

DEFINITION 1.2. The system will be said to be *unstable* if either or both of the following conditions hold: (a) There is a real eigenvalue which is not semisimple. (b) There is at least one non-real eigenvalue (and hence a conjugate pair).

Or, what is equivalent: The system is *stable* if and only if all eigenvalues are real and semisimple. Note that, when $L_2 \in \mathbb{R}^{n\times n}$ the spectrum has Hamiltonian symmetry and a nonzero real eigenvalue μ is always accompanied by the real eigenvalue $-\mu$. (In the theory of waveguides *real* eigenvalues are of special interest and are associated with "running waves".)

1.2. The eigencurves. The eigencurves are defined on $\mathbb{R}$ as

$$\{\mu \in \mathbb{R} : \ \mu \in \sigma\{\mathbb{L}(\lambda_0)\} \ \text{ for some } \ \lambda_0 \in \mathbb{R}\}.$$

Notice that, because $\mathbb{L}(\lambda_0)$ is Hermitian when $\lambda_0 \in \mathbb{R}$ then, for a fixed λ_0, $\mathbb{L}(\lambda_0)$ has n *real* eigenvalues, μ, (counting multiplicities). Indeed, there are real *analytic eigenfunctions* $\mu_1(\lambda), \dots, \mu_n(\lambda)$ defined on $\mathbb{R}$, whose zeros are the eigenvalues of $\mathbb{L}(\lambda)$ (see Section 12.4 of [6]). Their graphs generate n smooth *eigencurves*. Thus, the points at which these curves meet the real axis are the eigenvalues of $\mathbb{L}(\lambda)$. Notice that, for every eigencurve, $\mu \to \infty$ as $\lambda_0 \to \infty$ or $\lambda_0 \to -\infty$.

When $L_2^T = L_2 \in \mathbb{R}^{n\times n}$ these curves are symmetric about the origin, but this symmetry is lost if $L_2^* = L_2 \notin \mathbb{R}^{n\times n}$ (compare Figures 3.1 and 3.2 below). See Sections 12.4, 12.5 of [8] and [10] for details, and compare with Sections 14 and 19 of [18].

The eigencurves admit direct analysis of the system without recourse to the technique of *linearization* in which (1.2) is transformed to a linear eigenvalue problem on $\mathbb{C}^{2n}$.

1.3. Real eigenvalue types.

DEFINITION 1.3. Let λ_0 be a *real eigenvalue* (in the sense of item 1.1 above) and suppose that there are exactly k eigenfunctions $\mu_1(\lambda), \dots, \mu_k(\lambda)$ which vanish at λ_0. Then:

(a) λ_0 has *positive (resp., negative) type* if $\mu_j^{(1)}(\lambda_0) > 0$, (resp., < 0) for $j = 1, 2, \dots, k$.
(b) λ_0 has *neutral type* if $\mu_j^{(1)}(\lambda_0) = 0$ for $j = 1, 2, \dots, k$.
(c) If there are eigenfunctions $\mu_r(\lambda)$, $\mu_s(\lambda)$, $r \neq s$, for which $\mu_r(\lambda_0) = \mu_s(\lambda_0) = 0$, $\mu_r^{(1)}(\lambda_0)$ and $\mu_s^{(1)}(\lambda_0)$ are not both zero, and $\mu_r^{(1)}(\lambda_0)\,\mu_s^{(1)}(\lambda_0) \le 0$ then λ_0 is said to have *mixed type*.

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(d) The *sign-characteristic* of the real eigenvalue λ_0 is the set of integers consisting of +1's, -1's, or 0's defined by:

$$\{\text{sgn}\,\mu_j^{(1)}(\lambda_0)\}_{j=1}^k.$$

(The derivative $\mu_j^{(1)}(\lambda_0)$ is associated with the notion of *group velocity* in Chapter 5 of [18]. The positive and negative "types" can be associated with a direction of motion (p. 230 of [9]). See also the discussion of [2].)

1.4. The numerical range. The numerical range of $\mathbb{L}$ is defined by

$$NR(\mathbb{L}) := \{\lambda \in \mathbb{C} : \ x^*\mathbb{L}(\lambda)x = 0 \ \text{ for some nonzero } \ x \in \mathbb{C}^n\}. \tag{1.3}$$

See Section 10.6 of [6] for the following facts:

1. $\sigma(\mathbb{L}) \subset NR(\mathbb{L})$.
2. Every real frontier point of $NR(\mathbb{L})$ is an eigenvalue.
3. There are no real eigenvalues if and only if there are no real numbers in $NR(\mathbb{L})$.
4. $NR(\mathbb{L})$ is bounded.

(Notice the significance of item 3 in the context of waveguide systems where, as noted above, stable systems have only real eigenvalues.)

2. Systems of waveguide-type. We first summarize the approach of Abramov [1] in the context of (1.2), i.e., finite-dimensional gyroscopic systems. Since we are particularly interested in *real* eigenvalues, an important role is played by the real-valued "discriminant functional" associated with the scalar quadratic equation, $x^*\mathbb{L}(\lambda)x = 0$, of (1.3). Thus, for any $x \in \mathbb{C}^n$,

$$d(x) := (iGx, x)^2 - 4(x, x)(L_2x, x) \ \in \mathbb{R}, \tag{2.1}$$

is defined to be the *discriminant functional*. Then $\{x \in \mathbb{C}^n : \ d(x) \geq 0\}$ contains all eigenvectors of the real spectrum of $\mathbb{L}$, and $\{x \in \mathbb{C}^n : \ d(x) < 0\}$ contains all eigenvectors of the non-real spectrum of $\mathbb{L}$.

The set of x for which $d(x) = 0$ may be described as a "pointed cone", $K \subset \mathbb{C}^n$. The cone has vertex at the origin. Clearly, if $d(x) = 0$ then $d(\alpha x) = 0$ for all $\alpha \in \mathbb{C}$. In general, this cone provides a "boundary" in $\mathbb{C}^n$ separating a zone containing two distinct real roots for $(\mathbb{L}(\lambda)x, x) = x^*\mathbb{L}(\lambda)x = 0$ (when $d(x) > 0$, the "interior") from a zone containing a non-real conjugate pair of roots (when $d(x) < 0$, the "exterior").

Following Abramov we name[1] the open and closed interiors of K as follows:

$$\mathbb{G} := \{x \in \mathbb{C}^n : \ d(x) > 0\} \ \subset \ \mathbb{G}' := \{x \in \mathbb{C}^n : \ x \neq 0, \ \ d(x) \geq 0\}. \tag{2.2}$$

[1] There seems to be a misprint at this point in [1].

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DEFINITION 2.1. [2] We say that a gyroscopic system (1.2) is *strongly stable* if $d(x) > 0$ for all nonzero $x \in \mathbb{C}^n$ (i.e., if $\mathbb{G} = \mathbb{G}' = \mathbb{C}^n \setminus \{0\}$).

Thus, if the system is strongly stable then $\mathbb{G} = \mathbb{C}^n \backslash \{0\}$. In particular,

$$\sigma(\mathbb{L}) \subset NR(\mathbb{L}) \subset \mathbb{R},$$

all eigenvalues are real and semisimple (as we shall see in Proposition 4.2), and the corresponding eigenvectors are inside K. So the system is stable in the sense of Definition 1.2.

If the system is *not* strongly stable there is at least one $x \neq 0$ such that $d(x) \leq 0$. Consequently, there could be *either* a multiple real eigenvalue (when $x \neq 0$ and $d(x) = 0$ so that x is on the cone K), *or* a conjugate pair of non-real eigenvalues (when $d(x) < 0$ so that x is outside K), or both.

It is important to note that in the special case $L_2^* = L_2 \in \mathbb{R}^{n\times n}$, there is Hamiltonian symmetry of the spectrum, and this implies that nonzero real eigenvalues occur in positive/negative pairs. Similarly for pairs and quadruples of non-real eigenvalues.

Recalling (2.2), the scalar quadratic formula, and Definition 1.3, we define the real-valued functionals $p_\pm(x)$ on $\mathbb{G}'$ by

$$2p_\pm(x) = \frac{-(iGx, x) \pm \sqrt{d(x)}}{(x, x)}. \tag{2.3}$$

(This is consistent with the notation of [1].) If x_0 is an eigenvector associated with eigenvalue λ, then $\mathbb{L}(\lambda)x_0 = 0$, $x_0 \neq 0$, so that $x_0^*\mathbb{L}(\lambda)x_0 = 0$, and at least one of $p_\pm(x_0)$ is an eigenvalue. Furthermore, $p_\pm(x_0) \in \mathbb{R}$ if $d(x_0) \geq 0$, i.e., if $x_0 \in \mathbb{G}'$, the zone in which there are two (possibly coincident) real zeros for $x^*\mathbb{L}(\lambda)x$.

Then the bounds, $k'_- \leq k_- \leq k_+ \leq k'_+$ are defined by

$$k'_- := \inf_{\mathbb{G}'} p_-, \quad k_- := \inf_{\mathbb{G}} p_-, \quad k_+ := \sup_{\mathbb{G}} p_+, \quad k'_+ := \sup_{\mathbb{G}'} p_+. \tag{2.4}$$

Following Abramov, we define systems $\mathbb{L}(\lambda)$ of *waveguide-type* in terms of five hypotheses concerning these four bounds. The first hypothesis is ensured by assuming that $\mathbb{L}$ is monic. The next two are guaranteed simply because we pose the problem on finite-dimensional space. The fourth and fifth are:

$$\mathbb{G} \neq \emptyset \quad \text{and} \quad \mathbb{G} \neq \mathbb{C}^n \setminus 0, \tag{2.5}$$

$$-\infty < k'_- \quad \text{and} \quad k'_+ < \infty. \tag{2.6}$$

[2] As there is no damping in the system, we avoid the term "overdamped", cf. Chapter 13 of [6].

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The first statement of (2.5) is necessary for the existence of *some* real spectrum.

DEFINITION 2.2. A (finite-dimensional) gyroscopic system

$$\mathbb{L}(\lambda) = I\lambda^2 + iG\lambda + L_2, \quad \lambda \in \mathbb{C}, \tag{2.7}$$

(with $G^T = -G \in \mathbb{R}^{n\times n}$ and $L_2^* = L_2 \in \mathbb{C}^{n\times n}$) has *waveguide-type* if conditions (2.5) and (2.6) are satisfied.

For our problems, posed on finite-dimensional space, the conditions (2.6) are automatically satisfied. Conditions (2.5) admit the existence of a real eigenvalue, and ensure that the system is *not* strongly stable (in the sense of Definition 2.1).

Thus, a system (1.2) of waveguide-type is certainly not strongly stable. In terms of the discriminant functional we have:

PROPOSITION 2.3. *A gyroscopic system* (2.7) *has waveguide-type if and only if:*

(a) *There is a nonzero* $x \in \mathbb{C}^n$ *such that* $d(x) > 0$*, and*
(b) *There is a nonzero* $y \in \mathbb{C}^n$ *such that* $d(y) \le 0$*, i.e., the system is not strongly stable.*

In particular, if there is an eigenpair λ_1, x with $\lambda_1 \in \mathbb{R}$, $d(x) > 0$, *and* an eigenpair λ_2, y with $\lambda_2 \notin \mathbb{R}$, $d(y) < 0$, then the eigenvectors x and y satisfy Proposition 2.3. Thus, *in general*, it can be said that "waveguide-type" is primarily concerned with systems having both real and non-real eigenvalues. Nevertheless, we shall see in Examples 3.3 and 5.1, that there are systems with entirely real spectrum and waveguide-type.

Observe that, in condition (a), x may or may not be an eigenvector. Example 3.5 (below) is a case in which condition (a) is not satisfied at any eigenvector but there is a nonzero x for which $d(x) = 0$.

Also, if $d(y) \le 0$ at an eigenvector y, then item (b) implies that there is *either* a multiple real eigenvalue[3] (when $d(y) = 0$), or there is a non-real eigenvalue (when $d(y) < 0$). However, this inequality may well be satisfied at a y which is *not* an eigenvector. The beauty of the criterion of Proposition 2.3 lies in the fact that it does not require the calculation of spectral properties. Our concern is with the spectral properties implied by the criterion. Note the following simple corollary of the proposition:

If $L_2 < 0$ then $\mathbb{L}(\lambda)$ of (1.2) is *not* of waveguide-type. This is simply because $L_2 < 0$ implies $d(x) > 0$ for all $x \ne 0$. In other words, $L_2 < 0$ implies strong stability.

[3]It can be shown by example that d(y)=0 may occur whether the associated real eigenvalue is semisimple or not.

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3. Spectra of systems of waveguide-type. We first observe that, for systems of waveguide-type, *the real spectrum is not empty.* To see this observe that, in Proposition 2.3(a), the conditon $d(x) > 0$ for some x ensures that x determines a *real* point in the numerical range of $\mathbb{L}$ (see (1.3)). Thus, we immediately have $NR(\mathbb{L}) \neq \phi$. But then, since $NR(\mathbb{L})$ is bounded it must have real frontier points, and it follows from Theorem 10.15 of [6] that there must be real spectrum.

A major result of this subject is more explicit and asserts that, for systems of waveguide-type, k_- and k_+ of (2.4) are, in fact, real eigenvalues. Thus, it is necessary for systems of waveguide-type that at least one eigenvalue be real. Indeed, *all* eigenvalues may be real[4]. We take advantage of the definitions introduced in Subsections 1.1-1.4 to state a variant of Abramov's theorem and to expand on the role of the eigenfunctions - which are readily visualized. Proof of the theorem is postponed to an Appendix.

Theorem 3.1. *If a gyroscopic system $\mathbb{L}(\lambda)$ of (2.7) has waveguide-type then k_- and k_+ of (2.4) are real eigenvalues. Furthermore:*

(a) *There is an eigenfunction $\mu_r(\lambda)$ with $\mu_r(k_-) = 0$ and $\mu_r^{(1)}(k_-) \leq 0$, and the sign-characteristic for k_- contains no +1's.*
(b) *There is an eigenfunction $\mu_s(\lambda)$ with $\mu_s(k_+) = 0$ and $\mu_s^{(1)}(k_+) \geq 0$ and the sign-characteristic for k_+ contains no -1's.*

Thus, waveguide-type implies the existence of *two* (possibly coincident) real eigenvalues. Proposition 2.3 is not difficult to apply, but it may be useful to provide relatively simple sufficient conditions on the spectrum ensuring waveguide-type. We have:

Proposition 3.2. *If $\mathbb{L}(\lambda)$ of (2.7) has the following properties:*

(a) *there is a real eigenvalue $\lambda_1 \in \mathbb{R}$,*
(b) *there is at least one* linear *elementary divisor associated with λ_1,*
(c) *there exists an eigenvalue $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$,*

then the system has waveguide-type.

Proof. The fact that there exists an eigenvalue $\lambda_1 \in \mathbb{R}$ with a *linear* elementary divisor means that $\mathbb{L}(\lambda)$ has an eigencurve $\mu(\lambda)$ defined on $\mathbb{R}$ with a simple zero at λ_1 (see Section 1.2). Furthermore, because λ_1 is a *simple* zero of $\mu(\lambda)$, there is an eigenvector x of $\mathbb{L}(\lambda_1)$ for which

$$\mu^{(1)}(\lambda_1) = x^*\mathbb{L}^{(1)}(\lambda_1)x \neq 0,$$

[4]Since a direct sum of waveguide systems is again of waveguide-type, we can easily construct waveguide systems with prescribed spectrum. Example 4.3 is of this kind.

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A publication of the International Linear Algebra Society
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(see Theorem 12.5.2 of [8], for example). With this choice of x, and because $\lambda_1 \in \mathbb{R}$, we satisfy condition (a) of Proposition 2.3.

Condition (b) of that proposition follows from our hypothesis (c). Thus, it follows from Proposition 2.3 that the system has waveguide-type. □

The first example seems elementary, but is instructive.

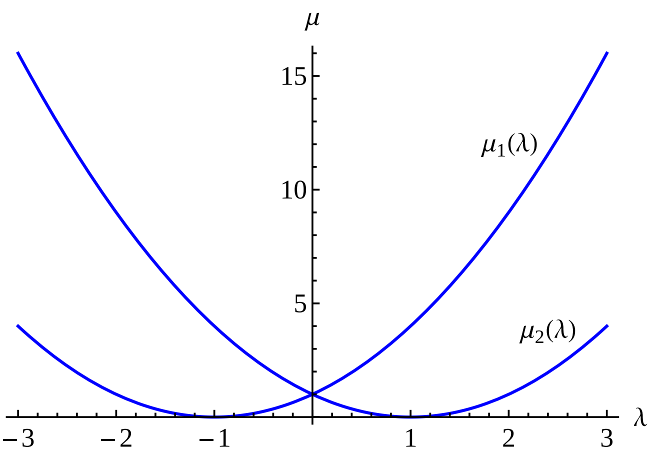

Fig. 3.1. *Eigencurves for Example 3.3 at $g = 2$.*

Example 3.3. Let $G = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$, $g \geq 0$, and $L_2 = I_2$. It is found that the spectrum of $\mathbb{L}(\lambda)$ is real for $g \geq 2$ and, otherwise, there are no real eigenvalues. Also[5], with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$,

$$d(x) = (iGx, x)^2 - 4(x, x)(L_2x, x) = -g^2(\bar{x}_1x_2 - x_1\bar{x}_2)^2 - 4\|x\|^4, \tag{3.1}$$

(and notice that $\bar{x}_1x_2 - x_1\bar{x}_2$ is pure-imaginary). When $g > 2$ all eigenvalues are real and distinct. Indeed, there are four simple real eigenvalues which, when listed in increasing order have types -, +, -, +, respectively. When $g = 2$ there are defective eigenvalues +1 and -1, each with algebraic multiplicity two and neutral type.

The eigencurves are shown in Figure 3.1 for the special case $g = 2$. In this case. we have (in Theorem 3.1) $k_- = -1$, $k_+ = +1$ and $\mu_1(k_-) = \mu_2(k_+) = 0$.

If $g > 2$ the two arcs of Figure 3.1 are displaced "bodily" downwards and there are four distinct real eigenvalues. It is easily seen that:

(a) If $y \in \mathbb{R}^2$ then $d(y) < 0$,

[5]Throughout this paper we use the Euclidean norm on $\mathbb{C}^n$, and the induced operator norm on $\mathbb{C}^{n\times n}$.

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(b) if $x = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ then $d(x) > 0$,

and, by Proposition 2.3, the system *has waveguide-type.*

If $g < 2$ the two arcs of Figure 3.1 are displaced "bodily" upwards and there are no real eigenvalues so (by Theorem 3.1) the system is *not of waveguide-type.*

When $g = 2$ it is easily verified that, since $L_2 = I$,

$$\begin{aligned} d(x) &= (x^*(iG)x)^2 - 4\|x\|^2(L_2x, x), \\ &= \{x^*(iG + 2I)x\}\{x^*(iG - 2I)x\}. \end{aligned} \tag{3.2}$$

But then we have $iG + 2I \geq 0$ and $iG - 2I \leq 0$ and it follows that $d(x) \leq 0$ for all $x \in \mathbb{C}^2$. Thus, when $g = 2$, the system is *not of waveguide-type.*

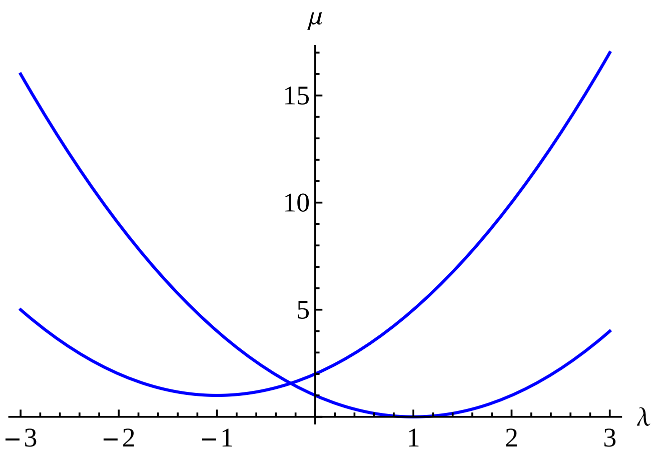

FIG. 3.2. *Eigencurves for Example 3.4.*

EXAMPLE 3.4. Let $\mathbb{L}(\lambda) = I_2\lambda^2 + iG\lambda + L_2$ where

$$G = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 3/2 & i/2 \\ -i/2 & 3/2 \end{bmatrix}.$$

The corresponding eigencurves are shown in Figure 3.2. Note the absence of Hamiltonian symmetry. The double eigenvalue 1 is not semisimple and there is a non-real conjugate pair of eigenvalues. Note that the sufficient conditions of Proposition 3.2 do not apply.

After some calculation it is found that, in (2.1), we have $d(x) \leq 0$ for all nonzero x and, by Proposition 2.3, the system is not of waveguide-type.

We have seen in Theorem 3.1 that systems of waveguide-type have at least two (possibly coincident) real eigenvalues. In contrast with this theorem, the following

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A publication of the International Linear Algebra Society
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example could be described as “contrived”. It shows that a system of the form (1.2) may have two real eigenvalues and *not* have waveguide-type.

EXAMPLE 3.5. (This is Example 4 of [12].) Let

$$G = \begin{bmatrix} 0 & \sqrt{4+\sqrt{12}} \\ -\sqrt{4+\sqrt{12}} & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then it is found that the spectrum consists of two *defective* real eigenvalues, each with algebraic multiplicity two (see Figure 3.3). It follows that, at an associated eigenvector y we have $d(y) = 0$ and condition (b) of Proposition 2.3 is satisfied.

Furthermore, it can be shown that $d(x) \leq 0$ for all $x \neq 0$. Thus, condition (a) of Proposition 2.3 cannot be satisfied, and the system *does not* have waveguide-type. Notice also that, in the context of Theorem 3.1, the sign-characteristic of each real eigenvalue (see Definition 1.3(d)) is the singleton $\{0\}$. The real eigenvalues are at $\lambda = \pm 3^{1/4}$.

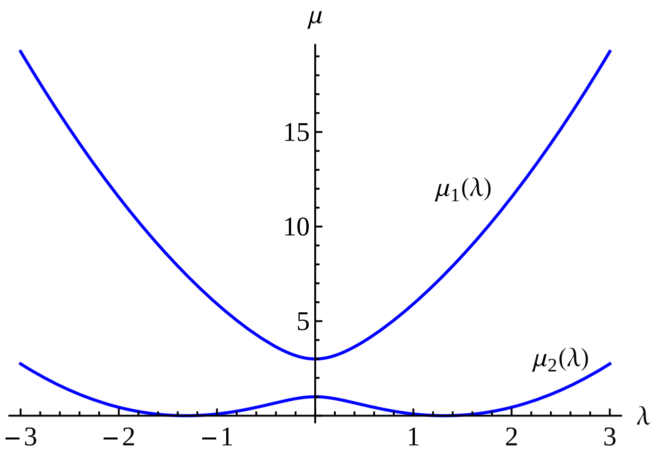

FIG. 3.3. *Eigencurves for Example 3.5.*

Suppose now that we make a *small* perturbation of L_2:

$$L_2 = \begin{bmatrix} 2+\varepsilon & 1 \\ 1 & 2+\varepsilon \end{bmatrix}$$

and consider the perturbed eigenfunctions $\mu_1(\lambda, \varepsilon)$, $\mu_2(\lambda, \varepsilon)$. It is found that (with $g = \sqrt{4+\sqrt{12}}$) we may write

$$\mu_1(\lambda, \varepsilon) = \lambda^2 + 2 + \varepsilon + (g^2\lambda^2 + 1)^{1/2},$$
$$\mu_2(\lambda, \varepsilon) = \lambda^2 + 2 + \varepsilon - (g^2\lambda^2 + 1)^{1/2}.$$

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If $\varepsilon > 0$ both curves of Figure 3.3 are displaced upwards, there are no real eigenvalues and, by Theorem 3.1, the system is *not of waveguide-type.*

If $\varepsilon < 0$ the lower curve of Figure 3.3 is displaced downwards, four simple real eigenvalues are generated, and the system *has waveguide-type.*

4. Semisimple eigenvalues. Semisimple eigenvalues may, of course, arise whether a system has waveguide-type or not and, in this section, we first discuss a role played by the discriminant in the geometry of the eigenfunctions. As in (1.2), let

$$\mathbb{L}(\lambda) = I\lambda^2 + iG\lambda + L_2, \quad G^T = -G \in \mathbb{R}^{n\times n}, \quad L_2^* = L_2 \in \mathbb{C}^{n\times n}$$

and define $d(x)$ as in (2.1). For an eigenvalue λ_0 of $\mathbb{L}(\lambda)$ with eigenvector x_0, $||x_0||^2 = x_0^* x_0 = 1$, we have

$$\mathbb{L}(\lambda_0)x_0 = (I\lambda_0^2 + iG\lambda_0 + L_2)x_0 = 0. \tag{4.1}$$

Then

$$\lambda_0^2 + (iGx_0, x_0)\lambda_0 + (L_2x_0, x_0) = 0$$

and so, using (2.1),

$$2\lambda_0 = -(iGx_0, x_0) \pm \sqrt{d(x_0)}. \tag{4.2}$$

If eigenvector $x_0 \in \mathbb{G}'$ of (2.2) then $d(x_0) \geq 0$ and λ_0 takes a *real* value consistent with (4.2).

Now let $\mu(\lambda)$ be an analytic eigenfunction for $\mathbb{L}(\lambda)$ for which $\mu(\lambda_0) = 0$ and (as in Subsection 1.2)

$$\mathbb{L}(\lambda)x(\lambda) = \mu(\lambda)x(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.3}$$

with $\|x(\lambda)\|^2 = x(\lambda)^* x(\lambda) = 1$. Thus, (4.1) holds and, differentiating, we also have

$$\mathbb{L}^{(1)}(\lambda)x(\lambda) + \mathbb{L}(\lambda)x^{(1)}(\lambda) = \mu^{(1)}(\lambda)x(\lambda) + \mu(\lambda)x^{(1)}(\lambda) \tag{4.4}$$

on a real neighbourhood of λ_0.

It is known (Theorem 12.2.1 of [8]) that, because all elementary divisors associated with λ_0 are linear, $x_0 = x(\lambda_0)$ can be chosen in (4.1) so that

$$x_0^* L^{(1)}(\lambda_0)x_0 = x_0^*(2I\lambda_0 + iG)x_0 \neq 0.$$

Indeed, if $L^{(1)}(\lambda_0)$ has p_+ positive eigenvalues and p_- negative eigenvalues, then there are p_+ (resp., p_-) linearly independent eigenvectors associated with λ_0 for which

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$x_0^* L^{(1)}(\lambda_0) x_0 > 0$ (resp., $x_0^* L^{(1)}(\lambda_0) x_0 < 0$). Consequently, there are p_+ (resp., p_-) choices of x_0 for which $\mu^{(1)}(\lambda_0) > 0$ (resp., $\mu^{(1)}(\lambda_0) < 0$).

Premultiply (4.4) by $x(\lambda)^*$ and set $\lambda = \lambda_0$, $x_0 = x(\lambda_0)$ (as in (4.1)). Then, using (4.2) and $\mu(\lambda_0) = 0$, we obtain

$$\begin{aligned} \mu^{(1)}(\lambda_0) &= x_0^*(2I\lambda_0 + iG)x_0, \\ &= 2\lambda_0 + x_0^*(iG)x_0, \\ &= \pm\sqrt{d(x_0)}, \end{aligned} \tag{4.5}$$

and note that, if Ker $\mathbb{L}(\lambda_0)$ has dimension $\delta \geq 1$ then $\mu^{(1)}(\lambda_0)$ may take as many as δ distinct values determined by the discriminant.

Furthermore, (in contrast with Example 3.5) when λ_0 is semisimple these values are nonzero. Indeed, the sign of $\mu^{(1)}(\lambda_0)$ determines a member of the *sign characteristic* of $\mathbb{L}(\lambda)$, either +1 or -1. There are precisely δ such signs and they are stable under small perturbations of $\mathbb{L}(\lambda)$ (see Section 5.9 and Proposition 12.2.1 of [8]). Thus, the discriminant plays a role in the geometry of the eigenfunctions. In particular, (4.5) provides a geometric interpretation for the magnitude of the discriminant functional evaluated at an eigenvector: The slope of an eigencurve at an eigenvalue (see Subsection 1.2) is determined by the square root of the discriminant at a corresponding eigenvector x_0 for which $||x_0|| = 1$.

Proposition 4.1. *Let λ_0 be a semisimple real eigenvalue of $\mathbb{L}(\lambda)$ of* (1.2) *with algebraic multiplicty $\delta \geq 1$. Then exactly δ eigencurves have zeros at λ_0. These zeros are all simple and the slopes of these eigencurves at λ_0 are determined by the discriminant as in* (4.5).

For comparison with waveguide-type, the significance of strong stability is clarified in the next proposition (and recall that systems with strong stability do not have waveguide-type). Illustrations for systems of category (2) appear in Examples 3.3 and 5.1.

Proposition 4.2. *Consider the following statements:*

(1) *The gyroscopic system $\mathbb{L}(\lambda)$ of* (4.1) *is strongly stable.*
(2) *All eigenvalues of $\mathbb{L}(\lambda)$ of* (4.1) *are real and semisimple.*
(3) *All solutions of the differential system*

$$Ix^{(2)}(t) - Gx^{(1)}(t) + L_2 x(t) = 0$$

are bounded on the real line.

Then (1) $\Rightarrow$ (2) *and* (2) $\Leftrightarrow$ (3).

Proof. Let λ_0 be a real eigenvalue of $\mathbb{L}(\lambda)$ of (1.2) and let $\mathbb{L}(\lambda)$ be strongly stable.

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Then $d(x) > 0$ for all $x \neq 0$. It follows immediately from (4.5) (and the discussion of eigencurves) that for each eigencurve $\mu_j(\lambda)$ with $\mu_j(\lambda_0) = 0$, we have $\mu_j^{(1)}(\lambda_0) \neq 0$. This implies that λ_0 is a semisimple real eigenvalue of $\mathbb{L}(\lambda)$ and, hence, (1) $\Rightarrow$ (2).

The equivalence of (2) and (3) follows from Theorem 2.3 (p. 155) of [7]. ◻

Concerning the converse statement, (2)$\Rightarrow$(1) in Proposition 4.2, the semisimple property ensures that $\mu^{(1)}(\lambda_0) \neq 0$ for each eigencurve with $\mu_j(\lambda_0) = 0$ and then, from (4.5), $d(x_0) > 0$ for corresponding eigenvectors x_0. But this does not guarantee $d(x) > 0$ for all $x \neq 0$, so the system is not necessarily strongly stable. This is confirmed by an example of Müller [15], which appears again as Example 7 of [12]. More detail will be given in the next section.

In the two following examples, L_2 is real-symmetric, so that the spectra have Hamiltonian symmetry. Consequently, the real-analytic eigenfunctions $\mu(\lambda)$ (see (4.3)) are even functions of λ. Example 4.3 has waveguide-type and Example 4.4 is not of waveguide-type.

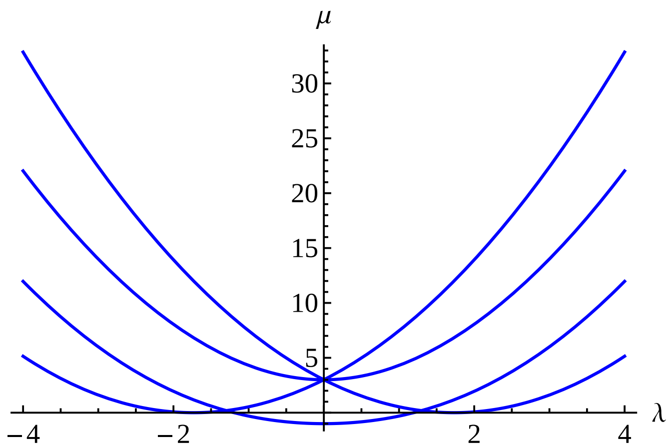

Fig. 4.1. *Eigencurves for Example 4.3.*

Example 4.3. Let

$$G = \frac{2\sqrt{3}}{3}\begin{bmatrix} 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then $\mathbb{L}$ has (truncated) real eigenvalues

$$-1.732, \quad -1.732, \quad -1.196, \quad 1.196, \quad 1.732, \quad 1.732,$$

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A publication of the International Linear Algebra Society
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and non-real eigenvalues $\pm 1.4481i$ (see Figure 4.1). Furthermore, the double real eigenvalues are not semisimple. It follows immediately from Proposition 3.2 that the system has waveguide-type.

The presence of non-real eigenvalues and of real eigenvalues which are *not* semisimple show that the system is not strongly stable.

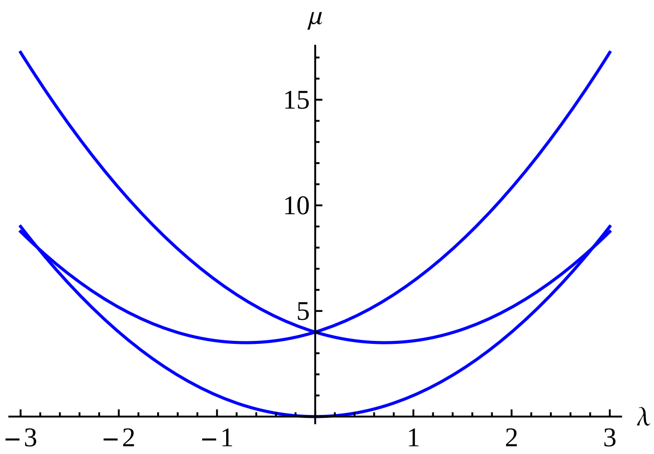

FIG. 4.2. *Eigencurves of Example 4.4.*

EXAMPLE 4.4. Let

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

Then $\mathbb{L}(\lambda)$ of (2.7) has the six (truncated) eigenvalues

$$\pm 0.7071 \pm i(1.8708), \quad 0, \quad 0,$$

and the zero eigenvalue is defective, with algebraic and geometric multiplicites two and one, respectively. Notice that the hypothesis (b) of Proposition 3.2 is not satisfied.

Then observe that $L_2 = -2G^2$ so that we may write

$$\mathbb{L}(\lambda) = I\lambda^2 + iG\lambda + 2(iG)^2 \tag{4.6}$$

From the definition (2.1),

$$\begin{aligned} d(x) &= (iGx, x)^2 - 4||x||^2(-2G^2x, x) \\ &= 8||x||^2(G^2x, x) - (Gx, x)^2. \end{aligned} \tag{4.7}$$

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A publication of the International Linear Algebra Society
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Then we have $(G^2x, x) = (-G^TGx, x) = -(Gx, Gx) = -||Gx||^2$, so that

$$d(x) = -8||x||^2||Gx||^2 - (Gx, x)^2,$$

and it follows that $d(x) \leq 0$ for all $x \neq 0$. Thus, by Proposition 2.3, the system does not have waveguide-type.

Examples 3.3 and 4.3 illustrate the fact that, for Hamiltonian systems of the form (1.2), *waveguide-type* admits either semisimple or defective real eigenvalues. In contrast, the multiple real eigenvalues of Examples 3.5 and 4.4 are defective and the systems are not of waveguide-type.

Proposition 4.5. (a) *Systems of the form* (1.2) *with waveguide-type have real spectrum and eigenvalues may be either semisimple or defective.*
(b) *There are systems of the form* (1.2) *with* $\sigma(\mathbb{L})$ *contained in* $\mathbb{R}$*, all eigenvalues are defective, and the system is not of waveguide-type.*

5. Waveguide-type and real spectrum. As we have seen, an important property of systems of waveguide-type is the guarantee of two real eigenvalues (Theorem 3.1 and Proposition 4.5) and it is natural to ask: Does the existence of real spectrum imply waveguide-type? Or strong stability? We have seen in Examples 3.5 and 4.4 that the presence of real spectrum need not imply waveguide-type, but they both include defective eigenvalues. In contrast, the following example of P.C. Müller [15] (see also [12]) has entirely real semisimple spectrum and is of waveguide-type.

By definition, systems of waveguide-type cannot be strongly stable - because there is a y such that $d(y) \leq 0$. However, it is possible that $d(x) > 0$ for all *eigenvectors* and an x for which $d(x) \leq 0$ must be sought elsewhere. Müller's example is of this kind. (Notice also that, in this case, all eigenvalues are real and simple (unrepeated), so Proposition 3.2 does not apply.) We see "at a glance" (Figure 5.1) that all eigenvalues are real and semisimple.

The *numerical range*, $NR(\mathbb{L})$, contains non-real points and, even though $d(x) > 0$ at all eigenvectors x, there are other vectors, y, for which $d(y) \leq 0$, to show that the system is not strongly stable (see Proposition 4.2). Indeed, Figure 7 of [12] shows that there are non-real points in $NR(\mathbb{L})$ (i.e. there are vectors y for which $d(y) < 0$). Then Proposition 2.3 can be applied to establish waveguide-type .

Example 5.1. We have:

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & G_1 \\ -G_1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_1 \end{bmatrix},$$

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A publication of the International Linear Algebra Society
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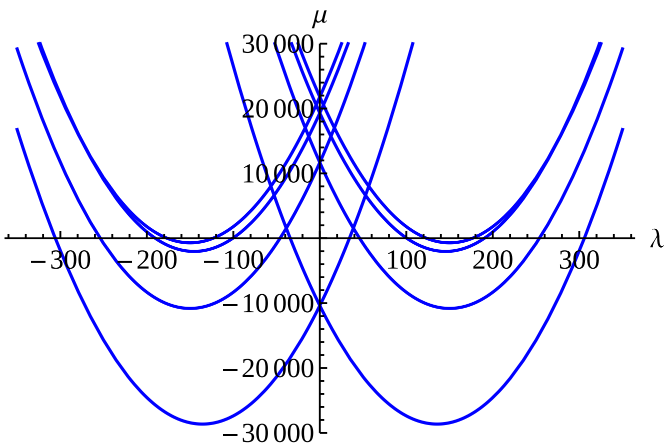

FIG. 5.1. *Eigencurves for Example 5.1.*

where $M_1 = \mathrm{diag}[0.2,\ 0.8,\ 0.2,\ 1/9]$, $G_1 = 150\ \mathrm{diag}[0.4,\ 1.6,\ 0.4,\ 7/36]$, and

$$K_1 = \begin{bmatrix} -2800 & -1200 & 0 & -1200 \\ -1200 & -15600 & -1200 & 0 \\ 0 & -1200 & -2800 & 1200 \\ -1200 & 0 & 1200 & 561.48 \end{bmatrix}.$$

Note that K_1 and K are indefinite.

6. Canonical forms. Canonical forms (in the sense of *Jordan canonical forms*) play an important part in computation with matrix polynomials, and waveguide systems of the form (1.2) are included in the recent analysis of [13]. We indicate the nature of these results here, and observe that they are consistent with the definitions of Sections 1 and 3.

Canonical forms are arrived at using a "linearization" of $\mathbb{L}(\lambda)$ of (1.2), namely linear pencils $\lambda A - C_R$, where

$$A = \begin{bmatrix} iG & I \\ I & 0 \end{bmatrix}, \qquad C_R = \begin{bmatrix} 0 & I \\ -L_2 & -iG \end{bmatrix},$$

Observe that $A^* = A, (AC_R)^* = AC_R$, and $\lambda A - C_R$ is a "linearization" of $\mathbb{L}(\lambda)$ with Hermitian coefficients.

Then (see Theorem 4.3 of [13]), there is a (selfadjoint Jordan) triple (X, J, PX^*), where P and J (a Jordan form for $A^{-1}C_R$) are block diagonal. We sketch the block structure of J and P:

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(a) If α is a *real* eigenvalue with a partial multiplicity *three* (for convenience) and associated *sign characteristic* ϵ, ($\epsilon = +1$ or - 1), then P and PJ have associated blocks on their main diagonal:

$$P_j = \epsilon \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_j J_j = \epsilon \begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 0 \end{bmatrix}.$$

(For a linear elementary divisor the corresponding entries are ϵ, $\epsilon\alpha$.)

(b) If $\beta \neq \bar{\beta}$ is a *non-real* conjugate eigenvalue pair with partial multiplicity *two*, then P and PJ have a diagonal block structure with associated (Hermitian) diagonal blocks:

$$P_k = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_k J_k = \begin{bmatrix} 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 0 \\ 1 & \bar{\beta} & 0 & 0 \\ \bar{\beta} & 0 & 0 & 0 \end{bmatrix}.$$

(For a linear elementary divisor the corresponding entries are $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{smallmatrix}\right]$.) The extension to elementary divisors of general degree is natural, and the reader is referred to [13] for more details.

7. Conclusions. There is a considerable literature on systems of waveguide-type, and the natural mathematical models for such systems are set in spaces of infinite dimension. It is also the case that computational methods are frequently used in the detailed examination of such systems. This generally requires the formulation of systems acting on a finite dimensional space, and such a system should retain essential spectral properties of waveguides. Beginning with a review of the basic spectral properties of matrix-valued functions, we have examined the spectra of finite-dimensional waveguide systems and provided interesting examples.

Acknowledgment. The authors are grateful to the reviewers for thoughtful comments.

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A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

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Appendix A. Abramov's existence theorem.

Theorem A.3 below is a finite-dimensional version of Abramov's theorem [6]. It includes the existence of real eigenvalues for systems of waveguide-type as forecast in Theorem 3.1. The proof is essentially that of Abramov, but it is adapted to the present context and includes some small refinements. The concepts introduced in Section 1.3 play an important role in this discussion. It will be seen that the five items of Theorem A.3 (below) are consistent with our discussion of Examples 3.3–5.1.

Recalling equation (2.3) and Definition 2.2 we make a further subdivision of the real line. Define

$$\delta_- := \inf_{\mathbb{G}} p_+ \quad \delta_+ := \sup_{\mathbb{G}} p_-. \tag{A.1}$$

Then we have:

LEMMA A.1. *If* $\mathbb{L}(\lambda)$ *of* (1.2) *has waveguide-type, then* $\delta_- \le \delta_+$.

Proof. We assume that there is a vector x_2 such that $d(x_2) < 0$ (i.e., $\mathbb{G}' \neq \mathbb{C}^n \setminus \{0\}$). Since $\mathbb{G} \neq \emptyset$, there exist x_1 such that $d(x_1) > 0$. We set $z_t = tx_1 + (1-t)x_2$, $t \in [0,1]$, and consider the polynomial $f(t) = d(z_t)$. Since $f(1) > 0$ and $f(0) < 0$, it has a zero in $(0,1)$. We denote by t_* the zero nearest to 1. Since $f(t) > 0$, $t \in (t_*, 1]$, we have $z_t \in \mathbb{G}$ for t from $(t_*, 1]$. But $d(z_{t_*}) = 0$ and, therefore,

$$\delta_- = \inf_{\mathbb{G}} p_+ \le \lim_{t \to t_*^+} p_+(z_t) = p_+(z_{t_*}) = p_-(z_{t_*}) = \lim_{t \to t_*^+} p_-(z_t) \le \sup_{\mathbb{G}} p_- = \delta_+.$$

The required inequality follows.

Assume now that $\mathbb{G}' = \mathbb{C}^n \setminus \{0\}$. Since $\mathbb{G} \neq \mathbb{C}^n \setminus \{0\}$, there exist a vector $x_3 \neq 0$ such that $d(x_3) = 0$ (i.e., $x_3 \in \mathbb{G}' \setminus \mathbb{G}$). Assume that $x_1 \in \mathbb{G}$. We set $z_\alpha = x_3 + \alpha x_1$, $\alpha \in \mathbb{R}$ and we consider the nonnegative polynomial $d(z_\alpha)$ with leading coefficient $d(x_1) > 0$. Thus, for $\alpha > 0$ sufficiently small, we have $d(z_\alpha) > 0$ and $z_\alpha \in \mathbb{G}$. Since z_α tends to x_3 when α goes to zero and $d(x_3) = 0$, the proof concludes as in the previous case. ∎

Now if $\mathbb{L}(\lambda)$ has waveguide-type then, using (2.4),

$$-\infty < k'_- \le k_- \le \delta_- \le \delta_+ \le k_+ \le k'_+ < +\infty. \tag{A.2}$$

See Figure A.1 below[7].

Abramov's classification of real eigenvalues is based on (A.2) in a natural way:

$$\sigma'_-, \ \ \sigma_-, \ \ \sigma_0, \ \ \sigma_+, \ \ \sigma'_+$$

[6] Proposition 8 of [1].
[7] See also Figure 4 of [5] and Figure 1 of [9].

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A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

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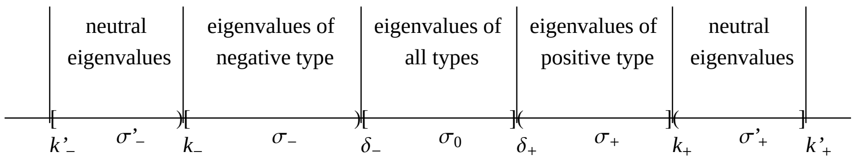

FIG. A.1. *Zones for real eigenvalues.*

are the (possibly empty) sets of real eigenvalues in the five finite intervals of the real line defined by (A.2). Thus, if $\sigma_{\mathbb{R}}$ denotes the set of all real eigenvalues of $\mathbb{L}$, then

$$\sigma'_- := \sigma_{\mathbb{R}} \cap [k'_-, k_-), \quad \sigma_- := \sigma_{\mathbb{R}} \cap [k_-, \delta_-), \quad \sigma_0 := \sigma_{\mathbb{R}} \cap [\delta_-, \delta_+],$$

$$\sigma_+ := \sigma_{\mathbb{R}} \cap (\delta_+, k_+], \quad \sigma'_+ := \sigma_{\mathbb{R}} \cap (k_+, k'_+].$$

If, in the definition of σ'_-, we have $k'_- = k_-$, then we consider the set σ'_- to be empty, and similarly for σ'_+ and k_+, k'_+.

It is our next objective to associate real eigenvalues of the three types in Definition 1.3 with these subintervals of the real line but bear in mind that, as in our Examples 3.3–5.1, some or all of these subsets of real eigenvalues may be empty. In particular, thinking in terms of eigenfunctions for the system (1.2), it is obvious that the right-most real eigenvalue has either positive or neutral type, with a similar property for the left-most real eigenvalue. (See Figures 3.3 and 4.1 above, and also Theorem 3.1 of [14].)

We refer to (2.2) and (2.3) for the definitions of $\mathbb{G}$ and $p_\pm(x)$, and give the following lemma without proof.

LEMMA A.2. *For $\mathbb{L}(\lambda)$ of (1.2) and $\lambda \in \mathbb{R}$ we have:*

(a) *If $(\mathbb{L}(\lambda)x, x) < 0$ then $x \in \mathbb{G}$ and $\lambda \in (p_-(x), p_+(x))$.*
(b) *If $x \in \mathbb{G}'$ then $(\mathbb{L}^{(1)}(p_\pm(x))x, x) = \pm\sqrt{d(x)}$.*
(c) *If $\lambda \in (-\infty, k_-] \cup [k_+, +\infty)$, then $\mathbb{L}(\lambda) \geq 0$.*
(d) *If $\lambda \in (-\infty, k'_-) \cup (k'_+, +\infty)$, then $\mathbb{L}(\lambda) > 0$.*
(e) *The functionals p_+, p_- and d are continuous on $\mathbb{G}'$.*

Recalling Definitions 1.3(a) and 1.3(b), we have:

THEOREM A.3. *If $\mathbb{L}(\lambda)$ of (1.2) has waveguide-type, then:*

(a) *k_+ and k_- are real eigenvalues.*
(b) *If $\sigma_+ \neq \emptyset$, then $k_+ \in \sigma_+$ (and similarly for k_- and σ_-).*
(c) *σ_+ (σ_-) consists of eigenvalues of positive (resp., negative) type.*
(d) *$\sigma'_+ \cup \sigma'_-$ consists of eigenvalues of neutral type.*

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A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

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(e) *A multiple real eigenvalue of mixed type has a neutral eigenvector.*

Proof. (a) We first show that k_+ is an eigenvalue. Since $k_+ = \sup_{\mathbb{G}} p_+$ and the unit sphere in $\mathbb{C}^n$ is compact, it follows from (2.3) that there is a sequence $x_n \subset \mathbb{G}$ such that

$$\|x_n\| = 1, \quad p_+(x_n) \to k_+, \quad \text{and} \quad x_n \to x. \tag{A.3}$$

Since $(\mathbb{L}(p_+(x_n))x_n, x_n) = 0$, it follows that

$$\begin{aligned} |(\mathbb{L}(k_+)x_n, x_n)| &= |(\mathbb{L}(k_+)x_n, x_n) - (\mathbb{L}(p_+(x_n))x_n, x_n)| \\ &= |([\mathbb{L}(k_+) - \mathbb{L}(p_+(x_n))]\, x_n, x_n)| \\ &\leq \|\mathbb{L}(k_+) - \mathbb{L}(p_+(x_n))\|\, \|x_n\|\, \|x_n\| \\ &= \|\mathbb{L}(k_+) - \mathbb{L}(p_+(x_n))\| \end{aligned}$$

and we have

$$\lim_{n\to\infty} (L(k_+)x_n, x_n) = 0.$$

By part (c) of Lemma A.2, $\mathbb{L}(k_+) \geq 0$, hence

$$\|\mathbb{L}(k_+)x_n\| \leq \|\mathbb{L}(k_+)\|^{1/2} (L(k_+)x_n, x_n)$$

and it follows that

$$\mathbb{L}(k_+)x_n \to 0, \quad \mathbb{L}(k_+)x = 0. \tag{A.4}$$

From (A.3), it follows that $x \neq 0$ and consequently k_+ and x form an eigenpair for $\mathbb{L}$. In a similar way one shows that k_- is an eigenvalue.

(b) Follows from (a) and the definitions of σ_+ and σ_-.

(c) We show that σ_+ consists of eigenvalues of positive type. First, we establish that k_+ has an eigenvector of positive type. We select a sequence $\{x_n\}$ with the properties (A.3) and (A.4). From part (e) of Lemma A.2 it follows that

$$d(x) = \lim_{n\to\infty} d(x_n) \geq 0, \quad k_+ = p_+(x).$$

Assume that $d(x) = 0$, then $p_+(x) = p_-(x)$ and

$$k_+ = p_+(x) = p_-(x) = \lim_{n\to\infty} p_-(x_n) \leq \delta_+$$

which contradicts the fact that $\delta_+ < k_+$ (in (c) we accept that $\sigma_+ \neq \emptyset$). Consequently $d(x) > 0$ and $x \in \mathbb{G}$. By part (b) of Lemma A.2, k_+ and x form an eigenpair of positive type.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

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Second, we show that k_+ is an eigenvalue of positive type. Let z be any eigenvector corresponding to k_+. If k_+ and z form an eigenpair of negative type than we have $(\mathbb{L}(k_+)z, z) = 0$ and $(\mathbb{L}^{(1)}(k_+)z, z) < 0$. It follows that $k_+ < p_+(z)$ which contradicts the fact that k_+ is the least upper bound of p_+ on $\mathbb{G}$.

Now we want to show that the eigenpair k_+, z cannot be neutral (see Definition 1.3(b)). If so, we let $z_\alpha = z + \alpha x$ with $\alpha \in \mathbb{R}$, where x is the eigenvector of postive type corresponding to k_+. Then $\mathbb{L}(k_+)z_\alpha = 0$ and we have $z_\alpha \in \mathbb{G}'$ for $\alpha \in \mathbb{R}$. Note that (in (2.1)) $d(z_\alpha)$ is a nonnegative polynomial in α of fourth degree, with leading coefficient $d(x) > 0$. Therefore, for sufficiently small $\alpha \neq 0$, the vector $z_\alpha \in \mathbb{G}$ and $k_+ = p_-(z) = \lim_{\alpha \to 0^+} p_-(z_\alpha) \leq \delta_+$, which leads to a contradiction. Consequently, k_+ is an eigenvalue of positive type.

Let $\lambda \in \sigma_+$, $\lambda \neq k_+$, that is $\delta_+ < \lambda < k_+$, and let z be a corresponding eigenvector. Since $(\mathbb{L}(\lambda)z, z) = 0$, we have $z \in \mathbb{G}'$. The eigenpair λ, z cannot be of negative type, since if $(\mathbb{L}^{(1)}(\lambda)z, z) < 0$ then $z \in \mathbb{G}$ and $\lambda = p_-(z)$, contradicting the fact that $\delta_+ < \lambda$.

Now we show that the eigenpair λ, z cannot be neutral. If so, then $(\mathbb{L}(\lambda)z, z) = 0$ and $(\mathbb{L}^{(1)}(\lambda)z, z) = 0$, so that $\lambda = p_\pm(z)$. Consider an eigenpair k_+, x of positive type, and set $z_\alpha = z + \alpha x$. We can assume that $\mathrm{Re}(\mathbb{L}(\lambda)z, x) \leq 0$. (If not we can replace x by $-x$.) Note that, since $\delta_+ < \lambda < k_+$, we have $(\mathbb{L}(\lambda)x, x) < 0$, and therefore,

$$(\mathbb{L}(\lambda)z_\alpha, z_\alpha) = (\mathbb{L}(\lambda)z, z) + 2\alpha \mathrm{Re}(\mathbb{L}(\lambda)z, x) + \alpha^2(\mathbb{L}(\lambda)x, x) < 0$$

for $\alpha > 0$. We have $z_\alpha \in \mathbb{G}$, for $\alpha > 0$ and we obtain $\lambda = p_-(z) = \lim_{\alpha \to 0^+} p_-(z_\alpha) \leq \delta_+$, which contradicts the fact that $\lambda \in \sigma_+$. Thus, λ is an eigenvalue of positive type.

The case in which $\lambda \in \sigma_-$ can be proved in a similar way.

(d) Let λ, x be an eigenpair with $\lambda \in \sigma'_+$. Then $x \in \mathbb{G}'$. But x cannot be in $\mathbb{G}$ since, by definition, $\lambda > k_+$. So we have $d(x) = 0$ and by part (b) of Lemma A.2 the eigenpair λ, x is neutral. It can be shown in a similar way that eigenvalues in σ'_- are also neutral.

(e) Assume that $\lambda \in \sigma_0$ is an eigenvalue of mixed type. Then λ has eigenvectors of both positive and negative type. Let λ, x and λ, y be eigenpairs of positive and negative type, respectively. Then $z_t = tx + (1-t)y$, $t \in [0, 1]$ is also an eigenvector corresponding to λ. The polynomial $q(t) = (\mathbb{L}^{(1)}\lambda)z_t, z_t)$ has opposite signs at the endpoints of interval $[0, 1]$. Therefore, there exists a $t_* \in (0, 1)$ such that $q(t_*) = (\mathbb{L}^{(1)}(\lambda)z_{t_*}, z_{t_*}) = 0$. Consequently, λ, z_{t_*} is a neutral eigenpair. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 670-692, October 2015

ELA
http://math.technion.ac.il/iic/ela

EXAMPLE A.4. We return to the system discussed in Example 4.3 and add some details to Figure 4.1 (see Figure A.2). Note that, in (2.4) and (A.2),

$$k'_- = -1.732, \quad k_- = -1.196, \quad k_+ = 1.196, \quad k'_+ = 1.732,$$

with $k'_- < k_- < k_+ < k'_+$. Note also that the eigenvalues $k'_- = -1.732$ and $k'_+ = 1.732$ are not semisimple, and items (a)–(e) of Theorem A.3 are illustrated.

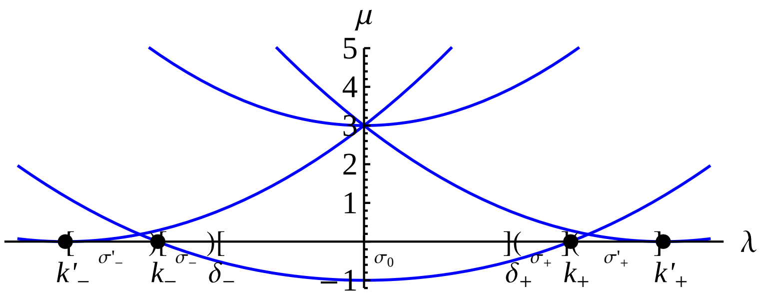

FIG. A.2. *The boundary and the real part of $NR(\mathbb{L})$.*

Appendix B. A Venn diagram.

The Venn diagram of Figure B.1 illustrates the variety of gyroscopic systems which have arisen in our study of equation (1.2). In particular, we have:

$$\{\text{strongly stable systems}\} \subset \{\text{systems with real and semisimple spectrum}\}$$
$$\subset \{\text{systems with all real spectrum}\} \subset \{\text{gyroscopic systems}\}$$

There are systems of waveguide-type in each of the last three subsets.

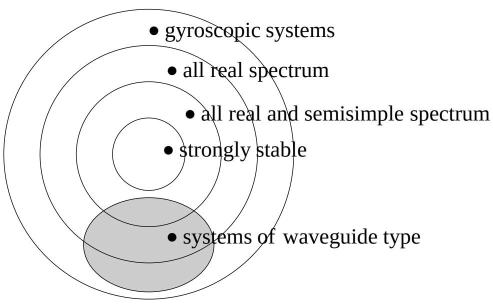

FIG. B.1. *Gyroscopic and waveguide systems.*