A note on the spectral radius of a product of companion matrices

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A NOTE ON THE SPECTRAL RADIUS OF A PRODUCT
OF COMPANION MATRICES

E.S. KEY† AND H. VOLKMER†

Abstract. Conditions are given on the coefficients of the characteristic polynomials of a set of
k companion matrices to ensure that the spectral radius of their product is bounded by t^k where
0 < t < 1.

Key words. Companion matrices, Matrix products, Spectral radius.

AMS subject classifications. 15A42, 15B99.

1. Introduction. In a recent paper [2] on population dynamics, A. Blumenthal and B. Fernandez used a bound on the spectral radius of a finite product of companion matrices [2, Lemma 5.5]. In light of the authors' work [4] on products of companion matrices, B. Fernandez inquired of the authors if they could supply a proof of their Lemma 5.5, which we have done in Theorem 1 in Section 3 below. In Section 2, we point out that a special case of our result is connected to the well-known Eneström-Kakaya theorem [1, Theorem 1.2] on the location of zeros of a polynomial.

2. The Eneström-Kakaya theorem. Consider the n by n companion matrix

\[ C = \begin{bmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}. \] (2.1)

Its characteristic polynomial is

\[ p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n. \]

By the Eneström-Kakaya theorem [1, Theorem 1.2], the assumption

\[ 1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0. \] (2.2)
implies that all zeros \( \lambda \) of \( p(\lambda) \) satisfy \( |\lambda| \leq 1 \). Therefore, under assumption (2.2), the spectral radius \( \rho(C) \) of \( C \) is at most 1.

We can prove this result using matrix notation as follows. Let \( A \) be the \( n+1 \) by \( n+1 \) matrix

\[
A = \begin{bmatrix} C & 0 \\ u & 1 \end{bmatrix},
\]

where \( u = (0,0,\ldots,0,1) \) is a row vector of dimension \( n \). Let \( B \) be the \( n+1 \) by \( n+1 \) companion matrix

\[
B = \begin{bmatrix}
1 - a_1 & a_1 - a_2 & a_2 - a_3 & \cdots & a_{n-1} - a_n & a_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix},
\]

and let \( L \) be the \( n+1 \) by \( n+1 \) matrix with 1’s on the main diagonal, \(-1\)’s on the super diagonal and 0 entries everywhere else. By calculation, we verify that

\[
AL = LB
\]

so

\[
L^{-1}AL = B.
\]

Therefore, \( A, B \) are similar and so \( \rho(A) = \rho(B) \). We obtain

\[
\rho(C) \leq \rho(A) = \rho(B) = 1,
\]

where \( B \) is a stochastic matrix [3, page 526], that is, a nonnegative matrix whose row sums are all 1.

3. An extension. We use the same idea to prove the following lemma.

**Lemma 3.1.** Let \( C_i, i = 1,\ldots,k, \) be companion matrices of the form (2.1) with first rows \(-(a_{i1}, a_{i2},\ldots,a_{in})\), respectively. Suppose that

\[
1 \geq a_{i1} \geq a_{i2} \geq \cdots \geq a_{in} \geq 0 \quad \text{for } i = 1,2,\ldots,k.
\]

Then

\[
\rho(C_1C_2\cdots C_k) \leq 1.
\]
Proof. We form the matrices $A_i, B_i$ as before and note that
\[ \rho(C_1C_2 \cdots C_k) \leq \rho(A_1A_2 \cdots A_k) = \rho(B_1B_2 \cdots B_k) = 1 \]
since $B_1B_2 \cdots B_k$ is a row stochastic matrix. \( \square \)

We now obtain our main result.

**Theorem 3.2.** Let $C_i, i = 1, \ldots, k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2}, \ldots, a_{in})$, respectively. Suppose that
\[ a_{i0} := 1 > a_{i1} > a_{i2} > \cdots > a_{in} \geq 0 \text{ for } i = 1, 2, \ldots, k. \]  
(3.2)

Define
\[ t = \max_{i=1}^{k} \max_{j=1}^{n} a_{i,j} \frac{1}{a_{i,j-1}} < 1. \]

Then
\[ \rho(C_1C_2 \cdots C_k) \leq t^k < 1. \]

Proof. We define $\tilde{a}_{i,j} = t^{-j}a_{i,j}$ and corresponding companion matrices $\tilde{C}_i$. Let $W = \text{diag}(1, t^{-1}, \ldots, t^{-n+1})$. Then
\[ C_i = tW\tilde{C}_iW^{-1}. \]

Therefore,
\[ \rho(C_1C_2 \cdots C_k) = t^k \rho(\tilde{C}_1\tilde{C}_2 \cdots \tilde{C}_k) \leq t^k, \]

where we applied Lemma 3.1 to the matrices $\tilde{C}_i$. \( \square \)

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