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Role Of Partial Transpose And Generalized Choi Maps In Quantum Dynamical Semigroups Involving Separable And Entangled States

Ajit Iqbal Singh
The Indian National Science Academy, New Delhi, ajitis@gmail.com

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ROLE OF PARTIAL TRANSPOSE AND GENERALIZED CHOI MAPS IN QUANTUM DYNAMICAL SEMIGROUPS INVOLVING SEPARABLE AND ENTANGLED STATES*

AJIT IQBAL SINGH†

Abstract. Power symmetric matrices defined and studied by R. Sinkhorn (1981) and their generalization by R.B. Bapat, S.K. Jain and K. Manjunatha Prasad (1999) have been utilized to give positive block matrices with trace one possessing positive partial transpose, the so-called PPT states. Another method to construct such PPT states is given, it uses the form of a matrix unitarily equivalent to its transpose obtained by S.R. Garcia and J.E. Tener (2012). Evolvement or suppression of separability or entanglement of various levels for a quantum dynamical semigroup of completely positive maps has been studied using Choi-Jamiolkowsky matrix of such maps and the famous Horodecki's criteria (1996). A Trichotomy Theorem has been proved, and examples have been given that depend mainly on generalized Choi maps and clearly distinguish the levels of entanglement breaking.

Key words. Power symmetric matrices, unitary equivalence to transpose, partial transpose, entanglement, generalized Choi map, quantum dynamical semigroup.

AMS subject classifications. 05B20, 15B48, 15B57, 46L07, 47A80, 47D06, 81P40

Dedicated to Professor R.B. Bapat on the occasion of his 60th birthday

1. Introduction.

Quantum inseparability or entanglement plays a significant role in quantum communication. The concept goes back to A. Einstein, E. Schrödinger and their contemporaries way back in the 1930s. Important practical applications have been envisaged in recent years by computer scientists, mathematicians and physicists. Various necessary and sufficient conditions were given by M. Horodecki, P. Horodecki and R. Horodecki [45], A. Peres [71], for instance. B.M. Terhal and P. Horodecki [89] came up with different levels in terms of Schmidt numbers. E. Størmer [88] has strengthened and formulated the theory in the context of operator algebras. The dynamics of entanglement in continuous variable open systems with particular emphasis on Gaussian states has also been well studied, but we will not go into that in this paper. We confine our attention mainly to bipartite finite-dimensional setup in this article.

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†INSA Honorary Scientist, The Indian National Science Academy, New Delhi 110002, India (ajitis@gmail.com)

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The next section is devoted to the basics of separable and entangled states and maps as well as of quantum dynamical semigroups in forms suitable for our purpose from standard well known books, monographs, notes, survey articles and research papers. Particular emphasis is on the so-called Choi maps given by M.D. Choi ([12], [14]) and their generalisations introduced and studied mainly by S.J. Cho, S.-H. Kye and S.G. Lee in [11]. These so-called generalized Choi maps have received a lot of attention and have been developed further by many authors. This section also includes a few simple new results. In the third section we give methods to construct PPT states that use power symmetric matrices due to R. Sinkhorn [81] and their generalization by R.B. Bapat, S.K. Jain and K. Manjunatha Prasad [5]. Techniques developed by D. Choudhury [16] and D. Guillot, A. Khare and B. Rajaratnam [34] help to some extent. The fourth section is devoted to the concept of unitary equivalence of a matrix to its transpose and its general form obtained by S.R. Garcia and J.E. Tener [31]. We formulate a variant to be exploited to give more methods to construct PPT states. Finally, in the last section, we come to our main objective of presenting a study of evolvement or suppression of separability or entanglement of various levels for a quantum dynamical semigroup of completely positive maps, in particular, the levels of entanglement breaking for a semigroup of quantum channels.

An appendix containing a poetic felicitation to R.B. Bapat and abstract of the actual expository talk at the conference ICLAA 2014 is given at the end.

2. Basics of separable and entangled states and maps and quantum dynamical semigroups. This section is divided in eight parts A to H. Simple new results and concepts can be found in parts F, G and H. For details an interested reader can refer to standard sources such as those specified from place to place. On the other hand, these can simply be skipped by workers in the area.

2.A. Notation. Let $\mathbb{N}$ be the set of natural numbers $1, 2, \ldots ; \mathbb{Z}$ the set of integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ that of complex numbers. For $n \in \mathbb{N}$, let M_n denote the C^*-algebra of $n \times n$ complex matrices. For $1 \leq j,\, k \leq n$, let E_{jk} be the elementary $n \times n$ matrix with 1 at the (j, k)-th place and zero elsewhere. Let I_n denote the identity matrix present in M_n. For $A \in M_n$, A^t and A^*(or $A^\dagger$) denote the transpose of A and the adjoint of A respectively. Let τ denote the transpose map on M_n to itself taking A to A^t. Let M_n^+ denote the positive cone of positive matrices in M_n, viz., the set of positive semi-definite matrices in M_n. A density matrix is an A in M_n^+ with $\operatorname{tr} A = 1$, where tr denotes the trace. A density ρ gives rise to a positive functional ω_ρ on M_n with $\omega_\rho(I_n) = 1$, also called a state, given by $\omega_\rho(X) = \operatorname{tr}(\rho X)$ for X in M_n. In fact, the correspondence $\rho \to \omega_\rho$ is bijective and we often use the name state for ρ as well. If the state ρ has rank one, then it is called a pure state, otherwise it is called a mixed state. A pure state ρ can be given by

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any unit vector in its range; therefore, a unit vector in $\mathbb{C}^n$ is also called a state. The context makes it clear as to which interpretation of the term state is being meant.

2.B. Separable and entangled states and partial transpose. For $n, m \in \mathbb{N}$, we consider the tensor product $H = \mathbb{C}^m \otimes \mathbb{C}^n$. A state on H can be viewed as an $m \times n$ state and can be represented as $\rho = [A_{jk}]_{1\leq j,k\leq m}$ with $n \times n$ matrices A_{jk} acting on $\mathbb{C}^n$. The state ρ may be called a bipartite state.

(i) The density ρ is said to be *separable* if it is in $M_m^+ \otimes M_n^+$ in the sense that $\rho = \sum_{i=1}^{k} p_i \rho_i \otimes \widetilde{\rho}_i$ where ρ_i and $\widetilde{\rho}_i$ are states on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $p_i > 0$.
   (a) Choi [12] gave examples of non-separable states and also necessary conditions for ρ to be separable; for example, its *partial transpose* $\rho^{PT} = [A_{jk}^t]$ is a state. In the literature the condition is known as the *Peres test* because of significant work by Peres [71] or *Positive Partial transpose* (PPT) and, thus, states that satisfy it can be called Peres states or *PPTS*.
   (b) A pure state ρ is separable if and only if each vector ξ in its range is a product vector, i.e. ξ has the form $\eta \otimes \zeta$ with $\eta \in \mathbb{C}^m$ and $\zeta \in \mathbb{C}^n$; in short, ρ is a pure product state. Further, a separable mixed state is a convex combination of such pure product states.

(ii) Non-separable states are called *entangled states.* The acronym *PPTES* is also used for an entangled PPT state in the literature with interesting applications to Quantum Information theory appearing mainly in Physics Journals, to name a few, [9], [10], [18], [23], [26], [35], [37], [45], [46] [47], [54], [59], [73], [82].

For an expository account of this and further developments one may consult ([27], [40], [41], [42], [44], [48], [64], [65], [70], [69], [83], [90]).

2.C. Separability à la Horodecki et al. M. Horodecki, P. Horodecki and R. Horodecki [45] provided necessary and sufficient conditions for the separability of mixed states and gave examples to illustrate them.

(i) *Theorem* ([45], Theorem 2) : Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces of finite dimension and ρ a state acting on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ i.e., ρ is a linear operator acting on $\mathcal{H}$ with $\operatorname{tr} \rho = 1$ and $\operatorname{tr} \rho P \geq 0$ for any projection P. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the sets of linear operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then ρ is *separable* if and only if for any positive (linear) map $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$ the operator $(I \otimes \Lambda)\rho$ is positive.

(ii) ([45], Remark on p.5) says that one can put $\widetilde{\Lambda} \otimes \Lambda$ or $\widetilde{\Lambda} \otimes I$ instead of $I \otimes \Lambda$ (involving any positive (linear maps) $\widetilde{\Lambda} : \mathcal{A}_1 \to \mathcal{A}_2$, $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$). The same applies to the PPT condition.

(iii) Next, ([45], Theorem 3) can be reworded as: a state acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$

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is separable if and only if it is PPT. The proof uses results of Størmer [86] and S. Woronowicz [93].

2.D. Entanglement breaking channels, Separable and entangled maps. We refer mainly to M. Horodecki, P. W. Shor and Ruskai[46], Størmer [88] for this subsection. The theory of completely positive maps is now a folklore. One may consult any standard book containing the topic; we mention some sources referred to here ([1], [3], [8], [12], [13], [14], [15], [27], [30], [67], [86], [87], [93]). Ruskai, Szarek and Werner [74] give an interesting analysis in the simplest set-up of 2×2 matrices with applications to Quantum Information theory using Pauli matrices. However, fundamentals are given in (ii) below for the sake of convenience of the reader.

(i) Horodecki, Shor and Ruskai [46] study entanglement breaking channels. A *quantum channel* is a stochastic map, i.e., a map on M_n to itself which is both completely positive and trace preserving.
  (a) A. S. Holevo [41] introduced channels of the form $\varphi(\rho) = \sum_k R_k \operatorname{Tr}(F_k \rho)$, where each R_k is a density matrix and $\{F_k\}$ form a positive operator valued measure POVM. The expression for φ is called the *Holevo form* in [46].
  (b) ([41], [46], Definition 1) A stochastic map φ is called *entanglement breaking* if $(Id \otimes \varphi)\, A$ is separable for any density matrix A, i.e., any entangled density matrix A is mapped to a separable one.
  (c) For $m, n \in \mathbb{N}$ and a linear map φ on M_n to M_m, the *Choi matrix* C_φ for φ is $C_\varphi = \sum_{j,k} E_{jk} \otimes \varphi(E_{jk}) = (Id \otimes \varphi)\rho \in M_n \otimes M_m$, where $\frac{1}{n}\rho$ is the 1-dimensional projection $\frac{1}{n}\sum_{j,k} E_{jk} \otimes E_{jk}$, the so-called basic maximally entangled state. M.D. Choi ([13]) proved that φ is completely positive if and only if C_φ is positive. Physicists usually use $\frac{1}{n}C_\varphi$ for a trace-preserving completely positive map φ and call it a Jamiolkowski state, (See for instance, [75], [92]) following A. Jamiolkowski [49].
  (d) A part of ([46], Theorem 4) says that the following are equivalent for a channel φ.
    (α) φ is entanglement breaking.
    (β) φ has the Holevo form with F_k positive semi-definite.
    (γ) $\frac{1}{n}C_\varphi$ is separable.
    (δ) $\psi \circ \varphi$ is completely positive for all positivity preserving maps ψ.
    ($\in$) $\varphi \circ \Lambda$ is completely positive for all positivity preserving maps Λ.

(ii) ([88],§1) Let $\mathcal{A}$ be an operator system, i.e., a norm-closed self-adjoint linear space of bounded operators on a Hilbert space $\mathcal{K}$ containing the identity. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$, its operator algebra and $\mathcal{T}(\mathcal{H})$, the space of the trace class operators on $\mathcal{H}$. Let τ be the transpose map on $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{B}(\mathcal{K})$) with respect to some orthonormal basis for $\mathcal{H}$ (respectively $\mathcal{K}$). At times

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for $a \in \mathcal{B}(\mathcal{H})$ or $\mathcal{B}(\mathcal{K})$, $\tau(a)$ will be denoted by a^t.

The BW-topology on the space of bounded linear maps on $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is the topology of bounded pointwise weak convergence, i.e., a net (φ_ν) converges to φ if it is uniformly bounded, and $\varphi_\nu(a) \to \varphi(a)$ weakly for all $a \in \mathcal{A}$. Let $S(\mathcal{H})$ be the BW-closed cone generated by maps of the form

$$x \to \sum_{j=1}^{n} \omega_j(x) a_j$$

where ω_j is a normal state on $\mathcal{B}(\mathcal{H})$ and $a_j \in$ the positive cone $\mathcal{B}(\mathcal{H})^+$. Here, $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

For the sake of convenience we recall some well-known conditions for a linear map φ on $\mathcal{A}$ to $\mathcal{B} = \mathcal{B}(\mathcal{H})$. Here $r \in \mathbb{N}$, and φ is a $*$*-map* in the sense that $\varphi(x^*) = \varphi(x)^*$ for $x \in \mathcal{A}$.

(a) The map φ is said to be r*-positive* if the map $\varphi_r = \varphi \otimes \mathrm{Id} : \mathcal{A} \otimes M_r \to \mathcal{B} \otimes M_r$ is positive.

(b) The map φ is said to be *completely positive* if φ is r-positive for all $r \in \mathbb{N}$.

(c) The map φ is said to be r*-copositive (respectively, completely co-positive)*, if $\tau \circ \varphi$ is r-positive (respectively, completely positive).

(d) The map φ is said to be a *Schwarz* map if

$$\varphi(x^* x) \geq \varphi(x^*)\varphi(x) \text{ for } x \in \mathcal{A} \text{ with } x^* x \in \mathcal{A}\,.$$

(e) The map φ is said to be r-*Schwarz* if φ_r is a Schwarz map i.e.,

$$\begin{aligned} \varphi_r\left([x_{jk}]^* [x_{jk}]\right) \geq \varphi_r\left([x_{jk}]^*\right) \varphi_r\left([x_{jk}]\right) \\ \text{for } x_{jk} \in \mathcal{A}, 1 \leq j,k \leq r \text{ with } x^*_{jk} x_{pq} \in \mathcal{A} \\ \text{for } 1 \leq j,k,p,q \leq r. \end{aligned}$$

(f) For a C^*-algebra $\mathcal{A}$, and φ unital in the sense that φ takes the identity of $\mathcal{A}$ to that of $\mathcal{B}(\mathcal{H})$, φ is 2-positive implies that φ is a Schwarz map. As a consequence, a unital φ is completely positive if and only if φ is r-Schwarz for each r.

(g) For a unital φ, the inequality in (d) above is satisfied for normal elements x when φ is positive.

(h) Many more assorted inequalities hold for positive maps (see [14], for instance).

(i) For φ satisfying any condition as in (a) to (e), ψ completely positive on $\mathcal{B}(\mathcal{H})$ to itself, and Λ completely positive on $\mathcal{A}$ to itself, the maps $\varphi \circ \Lambda$ and $\psi \circ \varphi$ satisfy the corresponding condition.

(j) Similar terminology as in (c) above applies to other conditions like (d) and (e).

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(iii) ([88], Lemma 1) sets up an isometric isomorphism $\varphi \to \widetilde{\varphi}$ between the set $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ of bounded linear maps of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and the dual $(\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H}))^*$ of the projective tensor product of $\mathcal{A}$ and $\mathcal{T}(\mathcal{H})$ given by

$$\widetilde{\varphi}(a \otimes b) = \operatorname{Tr}(\varphi(a)b^t)$$

where Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$ taking the value 1 on minimal projections. Furthermore, φ is a positive linear operator if and only if $\widetilde{\varphi}$ is positive on the cone $\mathcal{A}^+\widehat{\otimes}\mathcal{T}(H)^+$ generated by operators of the form $a \otimes b$ with a and b positive.

(iv) As noted in [88], p. 2305, it follows from ([87], Theorem 3.2) that φ is completely positive if and only if $\widetilde{\varphi}$ is positive on the cone $(\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H}))^+$, the closure of the positive operators in the algebraic tensor product $\mathcal{A} \odot \mathcal{T}(\mathcal{H})$.

(v) A positive linear functional ρ on $\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H})$ is said to be *separable* if it belongs to the norm closure of positive sums of states of the form $\sigma \otimes \omega$ where σ is a state of $\mathcal{A}$ and ω a normal state of $\mathcal{B}(\mathcal{H})$. Otherwise ρ is called *entangled*.

(vi) A part of ([88], Theorem 2) says that the following are equivalent for a $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$.
   (a) $\widetilde{\varphi}$ is a separable positive linear functional.
   (b) φ is a BW-limit of maps of the form $x \to \sum_{j=1}^{n} \omega_j(x)b_j$ with ω_j a state of $\mathcal{A}$ and $b_j \in \mathcal{B}(\mathcal{H})^+$.

(vii) Definition. A completely positive map φ in $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ will be called *separable* (respectively, *entangled)* if $\widetilde{\varphi}$ is so. A separable map may also be called *entanglement breaking*, in view of (i)(d) above for special φ's, viz., channels, and we do so at times in what follows.

(viii) ([88], Corollary 3) can now be reworded as : Let $\mathcal{H}$ be separable and $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be positive. If $\varphi(\mathcal{A})$ is contained in an abelian C^*-algebra then φ is separable.

(ix) We shall say a positive linear functional ρ on $\mathcal{A}\widehat{\otimes}\mathcal{T}(\mathcal{H})$ is *PPT* (i.e. satisfies the Peres condition) if $\rho \circ (Id \otimes \tau))$ is positive. In line with (ii) above $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ will be said to be *PPT* if $\widetilde{\varphi}$ is so. ([88], Proposition 4) can now be interpreted as: φ is PPT if and only if φ is both completely positive and completely co-positive.

We are now ready to prove a simple result in line with item (i)(d) above.

Theorem 2.1. *Let $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a separable map. Then for any positive bounded map ψ on $\mathcal{B}(\mathcal{H})$ to itself and any positive unital map Λ on $\mathcal{A}$ to itself, the maps $\psi \circ \varphi$ and $\varphi \circ \Lambda$ are both separable.*

Proof. We note that for a state ω of $\mathcal{A}$ and $b \in \mathcal{B}(\mathcal{H})^+$, $\omega \circ \Lambda$ is a state of $\mathcal{A}$ and $\psi(b) \in \mathcal{B}(\mathcal{H})^+$. We can now apply the item 2.D(vi) above for φ and conclude that maps $\varphi \circ \Lambda$ and $\psi \circ \varphi$ satisfy the condition (b) in 2.D(vi). By 2.D(vi) we obtain that

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$\varphi \circ \Lambda$ and $\psi \circ \varphi$ are separable.□

Corollary 2.2. *Let φ be a completely positive map on $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. If φ is separable then for each positive bounded map ψ on $\mathcal{B}(\mathcal{H})$ to itself and each positive unital map Λ on $\mathcal{A}$ to itself, $\psi \circ \varphi$ and $\varphi \circ \Lambda$ are completely positive.*

2.E. Horodecki's-Størmer Theorem. Let $m, n \in \mathbb{N}$, $\mathcal{A} = M_n$, $\mathcal{H} = \mathcal{C}^m$, $\varphi \in \mathcal{B}(A, \mathcal{B}(\mathcal{H}))$, and C_φ the Choi matrix for φ. The map $\varphi^t = \tau \circ \varphi \circ \tau$ (where τ is the transpose map in either M_n or M_m) is completely positive if and only if φ is so. ([88], Lemma 5) says that C_{φ^t} is the density matrix for $\widetilde{\varphi}$.

Størmer continues with his study and gives, amongst other things, his infinite-dimensional extension of Horodecki's Theorem, which we may call Horodecki's-Størmer Theorem, and also methods to construct PPTES. We shall not go into details here (cf. [43], [52], [60], [84]).

2.F. Pure Product states and Schmidt number.

(i) P. Horodecki [47] proved that a separable state on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (with $\mathcal{H}_1$ and $\mathcal{H}_2$ both finite-dimensional) can be written as a convex combination of N pure product states with $N \leq (\dim \mathcal{H})^2$ and gave a new separability criterion in terms of the range of the density matrix. This was carried further in different ways by several authors (cf. [26], [35], [53], [54], [68], [89], [91]).

(ii) K. R. Parthasarathy [68] called a subspace of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_r$ *completely entangled* if it contains no non-zero product vector of the form $u_1 \otimes u_2 \otimes \cdots \otimes u_r$ and gave a concrete example of a space attaining the maximal dimension of such a space. He also introduced a more delicate notion of perfectly entangled subspace for a multipartite quantum system. The notion of completely entangled subspaces is related to notions of unextendible product bases and uncompletable product bases, which are well studied by C.H. Bennett et al [6], D.P. Di Vincenzo et al [28] and further studied by N. Alon and L. Lovász [2], A.O. Pittenger [72], B.V.R. Bhat [7], L. Skowronek [82], N. Johnston [50, 51], R. Sengupta, Arvind and the author [79] and many more authors. They have been used to construct entangled PPT densities on one hand and non-PPT ones on the other.

(iii) B.M. Terhal and P. Horodecki [89] extended the notion of the Schmidt rank or number of a pure state to the domain of bipartite density matrices.

   (a) The Schmidt rank (or number) of a pure bipartite state ξ is the number of non zero co-efficients in the essentially unique Schmidt form $\xi = \sum_j \lambda_j \eta_j \otimes \zeta_j$ with η_j's and ζ_j's forming orthonormal sets in their respective spaces. For a mixed bipartite state ρ, its Schmidt rank is the maximum Schmidt rank (or number) in an optimal pure state decomposition of ρ.

   (b) To motivate our next notions, we quote their characterization viz., Theorem

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1 [89].

Let ρ be a density matrix on $\mathcal{H}_n \otimes \mathcal{H}_n$, i.e., $\mathbb{C}^n \otimes \mathbb{C}^n$. The density matrix has the Schmidt number at least $r+1$ if and only if there exists an r-positive linear map $\Lambda : M_n \to M_n$ such that $(I \otimes \Lambda)(\rho) \ngeq 0$.

Definition 2.3. Let $\mathcal{A}$, $\mathcal{B}(\mathcal{H})$ etc. be as in 2.D and $\varphi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a completely positive map. We say it has *Schmidt number at least* $r+1$ if either there exists an r-positive linear map $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ \varphi$ is not completely positive or there exists an r-positive linear map $\Lambda : \mathcal{A} \to \mathcal{A}$ such that $\varphi \circ \Lambda$ is not completely positive.

Remarkable progress on the topic has been made in recent years as can be gauged from ([17], [19], [20], [21], [22], [23], [24], [44], [52], [55], [58], [60], [62], [63], [76], [85]) The name "partially entanglement breaking " has been associated with maps having Schmidt number $< n$ by some authors. We shall not go into details in this paper.

2.G. Quantum dynamical semigroups. We may refer to any standard source for this folklore material, particularly ([25], [29], [30], [33], [61], [67], [92]) mentioned in the list of references, if we like, rather than original sources.

(i) Algebraically speaking, a dynamical system is a family $(T(t))_{t\geq 0}$ or, for short $(T_t)_{t\geq 0}$, of mappings on a set $\mathcal{X}$ satisfying

$$\begin{aligned} T(t+s) &= T(t)T(s) \text{ for all } t, s \geq 0 \\ T(0) &= Id. \end{aligned}$$

In fact even if we just confine our attention to the first condition, we have $T_0^2 = T_0$, the range $\mathcal{R}_0$ of T_0 contains the range $\mathcal{R}_t$ of T_t for all t, and T_0 restricted to $\mathcal{R}_0$ is the identity. So we may replace $\mathcal{X}$ by $\mathcal{R}_0$, and then the second condition holds for $(S_t)_{t\geq 0}$, where $S_t = T_t|\mathcal{R}_0$. Then $T_t = S_t T_0 = T_t T_0 = T_0 T_t$ and $S_t S_s = S_s S_t = S_{t+s}$. Thus, $(S_t)_{t\geq 0}$ is a dynamical system and we call $(T_t)_{t\geq 0}$ a T_0-*constricted dynamical system.*

(ii) Usually $\mathcal{X}$ is taken to be a Banach space, T_t a bounded linear operator on $\mathcal{X}$ for each t and the system to be strongly continuous on $\mathcal{X}$. Then $(T_t)_{t\geq 0}$ is called a *strongly continuous* (*one parameter*) *semigroup* or a C_0-*semigroup.* Again, if we relax the condition $T_0 = Id$, we call $(T_t)_{t\geq 0}$ a T_0-*constricted* C_0-*semigroup.* We note that the continuity of T_0 and the fact that $T_0 x = x$ for x in $\mathcal{R}_0$ forces $T_0 x = x$ for x in the closure $\bar{\mathcal{R}}_0$ of $\mathcal{R}_0$. This, in turn, gives that $\bar{\mathcal{R}}_0 \subset \mathcal{R}_0$. Hence $\mathcal{R}_0$ is closed and, therefore, a Banach space.

(iii) When $\mathcal{X}$ is an operator system $\mathcal{A}$, and maps T_t satisfy conditions like those in the item 2.D(ii) (a) to (e) above, we term a C_0-semigroup $(T_t)_{t\geq 0}$ as a *quantum dynamical semigroup or system.* In practice, T_t's are all taken to be completely positive and $\mathcal{A}$ to be M_n or $\mathcal{B}(\mathcal{H})$. Once again, the term T_0-constricted quantum

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dynamical semigroup will be used when we relax the condition $T_0 = Id$.

(iv) For a C_0-semigroup $(T_t)_{t\geq 0}$, the *infinitesimal generator* L is the operator which has the domain

$$D(L) = \left\{ x \in \mathcal{X} : \lim_{t\to 0+} \frac{1}{t}(T_t x - x) \text{ exists} \right\}$$

and is given by $Lx = \lim\limits_{t\to 0+} \frac{1}{t}(T_t x - x)$ for x in $D(L)$. Then L is a closed and densely-defined linear operator that determines the semigroup uniquely.

(v) For an $A \in \mathcal{B}(\mathcal{X})$, $T_t = \exp(tA)$, L coincides with A. For this reason T_t as in (iv) above is written as $\exp(tL) = e^{tL}$ as well.

(vi) When T_t's satisfy any of the conditions in 2.D(ii) (a) to (e) or corresponding "co" parts as indicated in 2.D(ii) (j) above, L satisfies a corresponding variant of the condition. Fundamental theoretical work in this direction is by V. Gorini, A. Kossakowski and E. C. G. Sudarashan [33] and G. Lindblad [61], though history can be traced back to specific irreversible processes or quantum stochastic processes of open systems by many like R. V. Kadison or E. B. Davies. For further basic developments one can see [25], [30] and [67].

(vii) It follows from the proof of and the Proposition itself on p.73 [29] that if there exists some $t_0 > 0$ such that $T(t_0)$ is invertible, then

   (a) for $0 \leq t < t_0$, $T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t)$ and for $t = nt_0 + s$ for $n \in \mathbb{N}$, $s \in [0, t_0)$, $T(t) = T(t_0)^n T(s)$, and therefore, $T(t)$ is invertible for all $t \geq 0$; $T(t)^{-1} = T(t_0 - t)T(t_0)^{-1}$ for $0 \leq t < t_0$ and $T(t)^{-1} = T(s)^{-1}T(t_0)^{-n}$ for $t = nt_0 + s$ for $n \in \mathbb{N}$, $s \in [0, t_0)$;

   (b) $(T(t))_{t\geq 0}$ can be embedded in a group $(T(t))_{t\in\mathbb{R}}$ on $\mathcal{X}$.

THEOREM 2.4. *Let $(T_t)_{t\geq 0}$ be a quantum dynamical system of completely positive maps. If there exists some $t_0 > 0$ such that $T(t_0)$ is invertible and $T(t_0)^{-1}$ satisfies any of the conditions 2.D(ii) (a) to (c) then each $T(t)^{-1}$ satisfies the corresponding condition.*

Proof. This is obvious from (vii)(a) just above. □

2.H. Separability and entanglement of generalized Choi maps and use in Quantum Information theory. We begin with a general set-up.

(i) **A foliation:** Let D_n be the linear span of $\{E_{jj} : 1 \leq j \leq n\}$ and F_n the linear span of $\{E_{jk} : 1 \leq j \neq k \leq n\}$. We note that as a linear space $M_n = D_n \oplus F_n$. Also any linear map Λ on M_n to itself can be expressed in the form $\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$ where $\Lambda_{11} : D_n \to D_n$, $\Lambda_{12} : F_n \to D_n$, $\Lambda_{21} : D_n \to F_n$ and $\Lambda_{22} : F_n \to F_n$ are linear maps.

   (a) Let C_Λ be the so-called Choi matrix of map Λ. It is given by the block matrix $[\Lambda(E_{jk})]$ written as an $n^2 \times n^2$ matrix with entries in $\mathbb{C}$, in fact. The

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diagonal of C_Λ is same as the diagonal of the block matrix with $\Lambda_{11}(E_{jj})$ at the jj^{th} block. As a consequence, $\operatorname{tr}(C_\Lambda) = \sum_{j=1}^{n} \operatorname{tr}\Lambda_{11}(E_{jj}) = \operatorname{tr}\Lambda_{11}(I_n) = \operatorname{tr}\Lambda(I_n)$. So C_Λ is a density matrix if and only if Λ is completely positive with $\operatorname{tr}\Lambda_{11}(I_n) = 1$. See ([12], [13], [14], [15]) for more details.

(b) We consider the class $\mathcal{L}$ of maps Λ with $\Lambda_{12} = 0$ and $\Lambda_{21} = 0$ and write $\Lambda = \Lambda_1 \oplus \Lambda_2$ with $\Lambda_1 = \Lambda_{11}$ and $\Lambda_2 = \Lambda_{22}$. Addition and product of maps in $\mathcal{L}$ is component-wise and as a consequence, for $\Lambda \in \mathcal{L}$, $e^\Lambda = e^{\Lambda_1} \oplus e^{\Lambda_2}$.

(c) A large number of examples in the study of positive, k-positive and completely positive maps, of dynamical semigroups and of separability, entanglement and Schmidt number for density matrices are in $\mathcal{L}$. One may observe this tendency in [11], [12], [14], [15], [17], [18], [19], [20], [21], [22], [23], [24], [26], [36], [38], [45], [47], [56], [57], [62], [66], [73], [74], [76], [77], [78], [89], for instance.

Such maps have been tested for more properties like separability, entanglement and Schmidt numbers and used to construct Entanglement witnesses, Entanglement breaking or partially entanglement breaking channels by some of them. We shall not go into details here.

(d) Quite often Λ_2 is just in the one-dimensional linear space spanned by $\mathcal{I}_{F_n}$, the identity operator on F_n, or else in the two dimensional algebra generated by $\mathcal{I}_{F_n}$ and the restriction τ_{F_n} of the transpose map τ on M_n. Also Λ_1's are usually taken to be upper (or lower) triangular matrices (cf.[56]) or matrices with rows being just permutations of each other ([11], [14]).

We confine our attention to only one class and give below relevant known results or minor riders for further use in Section 5.

(ii) **Generalized Choi maps:** We reformulate generalized Choi maps as presented in Cho, Kye and Lee [11] and also give little variants of them below.

(a) For $a, b, c \in \mathbb{C}$, let

$$D(a,b,c) = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

The set $\mathcal{D} = \{D(a,b,c) : a,b,c \in \mathbb{C}\}$ is a commutative semigroup with identity $D(1,0,0) = I_3$ simply because $D(a',b',c')\, D(a,b,c) = D(a'a+b'c+c'b, c'c+a'b+b'a, b'b+c'a+a'c)$. Further $\mathcal{D} \cap GL(3,\mathbb{C})$ is a subgroup of the general linear group $GL(3,\mathbb{C})$. To see this it is enough to note that

$$\begin{aligned} D(a,b,c)D(a^2-bc, c^2-ab, b^2-ac) &= (a^3+b^3+c^3-3abc)I_3 \\ &= \det D(a,b,c) I_3. \end{aligned}$$

(b) For $a, b, c \in \mathbb{R}_+ = [0,\infty)$, the map $\Phi[a,b,c]$ defined on p.214 [11] is the same

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as $D(a-1,b,c)\oplus(-\mathcal{I}_{F_3})$. We prefer to consider for $a,b,c\in\mathbb{R}_+$, the variants

$$\rho[a,b,c]=\Phi[a+1,b,c]=D(a,b,c)\oplus(-\mathcal{I}_{F_3})$$

and generalizations,

$$\rho[a,b,c,d]=D(a,b,c)\oplus d\mathcal{I}_{F_3},\quad \text{and}$$

$$\tau[a,b,c,d]=D(a,b,c)\oplus d\,\tau_{F_3}=\rho[a,b,c,d]\,\tau=\,\tau\rho[a,b,c,d].$$

(c) We note that $\rho[a,b,c,d]$ is unital if and only if $a+b+c=1$ if and only if $\tau[a,b,c,d]$ is unital. Also $\rho[a,b,c,d]$ is trace-preserving if and only if $a+b+c=1$ if and only if $\tau[a,b,c,d]$ is trace-preserving.
The map $\rho[a,b,c,d]$ is a $*$-map if and only if $a,b,c,d\in\mathbb{R}$ if and only if $\tau[a,b,c,d]$ is a $*$-map.
Finally, if $\rho[a,b,c,d]$ or $\tau[a,b,c,d]$ is a positive map then a,b,c are all non-negative simply because the image of E_{11} is $\begin{bmatrix}a&0&0\\0&c&0\\0&0&b\end{bmatrix}$.

(d) We note that $\rho[1,0,\mu]$ with $\mu\geq 1$ is the same as the map Φ in ([14], Appendix B, Example).

For the sake of convenience we recast the results in [11] in our notation.

(iii) **Properties of $\rho[a,b,c]$, $\rho[a,b,c,d]$ and $\tau[a,b,c,d]$.**
Let $a,b,c\in\mathbb{R}_+,\ d\in\mathbb{R}$.

(a) By Theorem 2.1 [11], the map $\rho[a,b,c]$ is positive if and only if $a+b+c\geq 2$ together with $bc\geq(1-a)^2$ in case $0\leq a\leq 1$.

(b) By [11], Lemma 3.1, the map $\rho[a,b,c,d]$ is completely positive if and only if $a\geq d$ and $a\geq -2d$.
In particular, $\rho[a,b,c]$ is completely positive if and only if $a\geq 2$. This is Proposition 3.2 [11].

(c) By [11] Lemma 3.1, second part, $\tau[a,b,c,d]$ is completely positive if and only if $bc\geq d^2$. As a consequence, $\rho[a,b,c,d]$ is positive if $bc\geq d^2$.
In particular, $\tau\circ\rho[a,b,c]$ is completely positive (i.e. $\rho[a,b,c]$ is completely copositive) if and only if $bc\geq 1$. This is a part of Proposition 3.3 of [11].

(d) Theorem 3.4 [11] gives that for $0\leq a<2$, $\rho[a,b,c]$ is decomposable if and only if $bc\geq(1-\frac{a}{2})^2$.
This combined with (b) and (c) gives that $\rho[0,b,c]$ is not completely positive and it is decomposable if and only if $bc\geq 1$ if and only if it is completely copositive.
Further, for $0<a<2$ and $bc<1$, $\rho[a,b,c]$ is neither completely positive nor completely copositive, but is nevertheless decomposable if and only if $bc\geq(1-\frac{a}{2})^2$.

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(e) Theorem 4.2 [11] says that $\rho[a,b,c]$ is 2-positive if and only if either $a \geq 2$ or $1 \leq a < 2$ and $bc = (2-a)(b+c) > 0$. Further, taking $a = 1$, $b = 2 = c$, we get the well-known example of a 2-positive map that is not completely positive given by Choi in [12] as remarked on p. 214 [11] as well.

Remark 2.5. Corollary 2.2 combined with the above remarks can give us a multitude of completely positive maps and states that are PPT, non-PPT or PPTE. We illustrate this by recording a few special cases for further use.

Theorem 2.6. *Let $a, b, c, d \in \mathbb{R}$ with $a, b, c \geq 0$. Let $\varphi = \rho[a,b,c,d]$ and $\psi = \tau[a,b,c,d]$.*

(i) *If $d = 0$, then $\varphi = \psi$ is a separable map.*

(ii) *Let $d > 0$.*
  (a) *If $a \geq d$ but $bc < d^2$, then φ is a non-PPT completely positive map.*
  (b) *If $bc \geq d^2$ but $a < d$, then ψ is a non-PPT completely positive map.*
  (c) *If $a \geq d$, $bc \geq d^2$, then φ and ψ are PPT maps.*
  (d) *If $a \geq d$, $bc \geq d^2$ but $a + b < 2d$ or $a + c < 2d$, then φ and ψ are PPTE maps.*
  In particular, this is so if $a = 1 = d$, $0 < b < 1$, $c = \frac{1}{b}$.
  (e) *If $a + 2(b + c) < 2d \leq 2a$ then φ has Schmidt number 3 and it is non-PPT. In particular, it is so if $b + c = 1 - a$, and, $\frac{2}{3} < a \leq 1$ together with $1 - \frac{a}{2} < d \leq a$, or, equivalently, $\frac{1}{2} < d \leq 1$ together with $d \leq a \leq 1$ for $d > \frac{2}{3}$ whereas $2(1-d) < a \leq 1$ for $\frac{1}{2} < d \leq \frac{2}{3}$.*

(iii) *Let $d < 0$.*
  (a) *If $a \geq -2d = 2|d|$ but $bc < d^2$, then φ is a non-PPT completely positive map.*
  (b) *If $bc \geq d^2$ but $a < 2|d|$, then ψ is a non-PPT completely positive map.*
  (c) *If $a \geq 2|d|$, $bc \geq d^2$, then φ and ψ are PPT maps.*

Proof. Parts (i), (ii) (a), (ii)(b), (ii)(c), (iii)(a), (iii)(b) follow immediately from (iii) above. For (ii)(d) we consider the map $\xi = \rho[a',b',c'] \equiv \rho[a',b',c',-1]$ with a', b', c' to be suitably chosen yet to be specified. We have

$$\xi\varphi = \rho[a'a + b'c + c'b, c'c + a'b + b'a, b'b + c'a + a'c, -d].$$

By item (iii)(b), $\xi\varphi$ is completely positive if and only if $a'a + b'c + c'b \geq 2d$. If $a+b < 2d$ then we take $a' = 1 = c'$ and $b' = 0$. Then $a'+b'+c' = 2$, $b'c' = 0 = (1-a')^2$ and $a'a + b'c + c'b = a + b < 2d$. So by Item (iii)(a) and (b), ξ is positive and $\xi\varphi$ is not completely positive. Similarly, in case $a + c < 2d$, we take $a' = 1 = b'$ and $c' = 0$ and conclude that ξ is positive but $\xi\varphi$ is not completely positive. By Corollary 2.2, φ and ψ are not separable. So φ and ψ are PPTE maps.

For (ii)(e), we may consider $\eta = \rho[1, 2, 2]$, which as noted in item (iii)(e) above, is

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2-positive but not completely positive. We have $\eta\phi = \rho[a+2(b+c), 2c+b+2a, 2b+c+2a, -d]$, which , by item (iii)(b) can not be completely positive, simply because $a+2(b+c) < -2(-d)$. In this case $bc \leq (\frac{b+c}{2})^2 < (\frac{2d-a}{4})^2 < d^2$. So ϕ is non-PPT. Rest is simple computation. ∎

Theorem 2.7. *Let $a, b, c, d \in \mathbb{R}$ with $a, b, c \geq 0$ and $a+b+c = 1$. Let $\varphi = \rho[a, b, c, d]$, $\psi = \tau[a, b, c, d]$. Let $A = \frac{1}{3}C_\varphi$ and $B = \frac{1}{3}C_\psi$ be the Choi-Jamiolkowski matrices of φ and ψ respectively.*

(i) *If $d = 0$ then $A = B$ is a separable state.*
(ii) *Let $d > 0$.*
  (a) *If $a \geq d$ but $bc < d^2$, then A is a non-PPT state.*
  (b) *If $bc \geq d^2$ but $a < d$, then B is a non-PPT state.*
  (c) *If $a \geq d$, $bc \geq d^2$ then A and B are PPT states.*
  (d) *If $a \geq d$, $bc \geq d^2$ but $a+b < 2d$ or $a+c < 2d$ then A and B are PPTES. In particular, this is true if for an arbitrary $0 < \beta < 1$, we take $\lambda = \beta/(\beta^2+\beta+1)$, $a = \lambda = d$, $b = \lambda\beta$, $c = \frac{\lambda}{\beta}$.*
  (e) *If $d \leq a < 2d$ and $2(b+c) < 2d - a$ then A has Schmidt number 3 and it is non-PPT. This is equivalent to requiring: $\frac{2}{3} < a \leq 1$ together with $1 - \frac{a}{2} < d \leq a$, or, equivalently, $\frac{1}{2} < d \leq 1$ together with $d \leq a \leq 1$ for $d > \frac{2}{3}$, whereas, $2(1-d) < a \leq 1$ for $\frac{1}{2} < d \leq \frac{2}{3}$, and, then taking b and $c \geq 0$ with $b + c = 1 - a$.*
(iii) *Let $d < 0$.*
  (a) *If $a \geq -2d = 2|d|$ but $bc < d^2$ then A is a non-PPT state.*
  (b) *If $bc \geq d^2$ but $a < 2|d|$, then B is a non-PPT state.*
  (c) *If $a \geq 2|d|$, $bc \geq d^2$, then A and B are PPT states.*

Proof. This follows immediately from Theorem 2.6 above.∎

Remark 2.8. This concerns the range and the rank of $C_{\rho[a,b,c,d]}$ and $C_{\tau[a,b,c,d]}$.

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(i) We first note that

$$C_{\rho[a,b,c,d]} = \begin{bmatrix} a & 0 & 0 & 0 & d & 0 & 0 & 0 & d \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & a & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ d & 0 & 0 & 0 & d & 0 & 0 & 0 & a \end{bmatrix},$$

$$C_{\tau[a,b,c,d]} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & d & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

(ii) By (i), the range of $C_{\rho[a,b,c,d]}$ is the linear span of

$$\begin{aligned} &ae_1 + de_5 + de_9, \quad ce_2, \quad be_3, \quad be_4, \\ &de_1 + ae_5 + de_9, \quad ce_6, \quad ce_7, \quad be_8 \quad \text{and} \\ &de_1 + de_5 + ae_9. \end{aligned}$$

Here e_j, $1 \le j \le 9$ is the arrangement of product vectors $e_p^3 \otimes e_q^3$ in the lexicographic order, $(e_p^3 : p = 1, 2, 3)$ being the standard ordered basis of $\mathbb{C}^3$.

Now the matrix $\begin{bmatrix} a & d & d \\ d & a & d \\ d & d & a \end{bmatrix}$ has determinant $= (a-d)^2(a+2d)$.

Therefore $C_{\rho[a,b,c,d]}$ attains all ranks from 0 to 9 depending on the values of a, b, c, d in $\mathbb{C}$. For instance, for a, b, c all non-zero, the matrix $C_{\rho[a,b,c,d]}$ has rank 7 if $a = d$, rank 8 if $a = -2d$ and rank 9 if $a \neq d$ and $a \neq -2d$.

(iii) Next, the range of $C_{\tau[a,b,c,d]}$ is the linear span of ae_1, $ce_2 + de_4$, $be_3 + de_7$, $de_2 + be_4$, ae_5, $ce_6 + de_8$, $de_3 + ce_7$, $de_6 + be_8$ and ae_9. The matrices $\begin{bmatrix} c & d \\ d & b \end{bmatrix}$ and $\begin{bmatrix} b & d \\ d & c \end{bmatrix}$ both have determinant $= bc - d^2$. So the matrix $C_{\tau[a,b,c,d]}$ has rank 9 if $a \neq 0 \neq bc - d^2$, rank 6 if $a \neq 0 = bc - d^2$ and $d \neq 0$, rank 6 if $a \neq 0 = bc - d^2 = d$ and $b^2 + c^2 \neq 0$, and rank 3 if $a \neq 0 = b = c = d$.

Remark 2.9. Let $a, b, c \geq 0$ and $d \in \mathbb{R}$ with $d \neq 0$. Let $A = \frac{1}{3} C_{\rho[a,b,c,d]}$ and

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$B = \frac{1}{3}C_{\tau[a,b,c,d]}$. We refer to their expanded form coming from Remark 2.8 above.

(i) In view of Item (iii)(b) or Theorem 2.7, A is a density matrix if and only if $a+b+c=1$ and $a \geq \max\{d, -2d\}$. We now consider only this case. Then $a > 0$. Further, from the expanded form of A, it is clear that it is enough to look at its only non-trivial sub-block $A_1 = \frac{1}{3}\begin{bmatrix} a & d & d \\ d & a & d \\ d & d & a \end{bmatrix}$ acting on the span of e_1, e_5 and e_9. Now $3\mathrm{A}_1$ has eigenvalues $a+2d$, $a-d$, $a-d$. Since $d \neq 0$, we have $a+2d \neq a-d$. Let $\xi = \frac{1}{\sqrt{3}}(e_1+e_5+e_9)$. Then $A_1 = \frac{1}{3}(a+2d)P_\xi + \frac{1}{3}(a-d)P$, P_ξ is the projection determined by ξ and P, the (orthogonal) projection on $\xi^\perp$. Also $\xi^\perp$ is the linear span of $e_1 - e_5$ and $e_5 - e_9$ and it contains no non-zero product vectors.

(ii) By Item (iii)(b) or Theorem 2.7, we have that B is a density matrix if and only if $a+b+c=1$ and $bc \geq d^2$. We now consider only this case. Then $b > 0$, $c > 0$. Further, it is clear from the expanded form of B that it is enough to consider its non-trivial sub-blocks $B_1 = \frac{1}{3}\begin{bmatrix} c & d \\ d & b \end{bmatrix}$, $B_2 = \frac{1}{3}\begin{bmatrix} b & d \\ d & c \end{bmatrix}$, $B_3 = \frac{1}{3}\begin{bmatrix} c & d \\ d & b \end{bmatrix}$, acting on linear spans L_1, L_2, L_3 respectively of pairs of product vectors (e_2, e_4), (e_3, e_7) and (e_6, e_8) respectively. Because $d \neq 0$, B_1 has distinct eigenvalues. Further, any corresponding eigenvector has the Schmidt rank 2. The same applies to B_2 and B_3 as well.

3. Power Symmetric Matrices and Construction of PPT States. This section has been motivated by the work on "Power symmetric matrics" by Sinkhorn [81] and its generalization by Bapat, Jain and Prasad [5] on one hand and preservation of positivity of a block matrix under taking powers of blocks, the so-called Schur or Hadamard product by Choudhury [16] and Guillot, Khare and Rajaratnam [34] on the other. The purpose is to illustrate the interesting interplay rather than the utmost generality.

3.A. Power symmetric matrices. We first recall the known concepts and results to be used.

(i) Sinkhorn [81] calls a stochastic matrix A *power symmetric* if its transpose A^t equals A^q for some $q \in \mathbb{N}$. The smallest such positive integer q for which this is true is called the symmetric order of A.

Let A be a power symmetric matrix of symmetric order $q > 1$.

(a) Sinkhorn notes that such an A commutes with A^t and, is therefore, normal.

(b) He proves that A is bistochastic by showing that a normal stochastic matrix is bistochastic.

(c) He also shows that A satisfies $A^{q^2} = A$, and, therefore, an eigenvalue λ of

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A is either zero or has modulus one.

(d) For $r \in \mathbb{N}$, let J_r be the bistochastic $r \times r$ matrix having all entries equal to $\frac{1}{r}$. Sinkhorn obtains a general form for A in terms of certain J_r's.

(ii) It may be of interest to note that Sinkhorn is famous for his Diagonal Theorems: If A is an $n \times n$ matrix with strictly positive elements then there exist diagonal matrices D_1 and D_2 with strictly positive elements such that D_1AD_2 is doubly stochastic, D_1 and D_2 are essentially unique i.e., up to a positive multiple by a and $\frac{1}{a}$ respectively (Ann. Math. Statistics, 1964). Further, there exists a unique stochastic matrix of the form DAD, where D is a diagonal matrix with positive diagonal elements (Can. J. Math., 1966).

This stimulated further research by reputed mathematicians like R. Brualdi.

(iii) The property (i)(a) persists even if we take A to be a real matrix instead of stochastic. Further, (i)(c) holds even if we take A to be a complex matrix satisfying $A^t = A^q$ or $A^* = A^q$ for some $q \in \mathbb{N}$ with $q > 1$.

This motivates our next result.

For $q \in \mathbb{N}$ with $q > 1$, let $\mathbb{T}_q^{(1)} = \{\lambda : \lambda^{q-1} = 1\}$ and $\mathbb{T}_q^{(2)} = \{\lambda : \lambda^{q^2-1} = 1 \neq \lambda^{q-1}\}$. We note that $\mathbb{T}_q^{(2)}$ can be expressed as a disjoint union of pairs $T_r = \{\lambda_r, \lambda_r^q\}$, $1 \leq r \leq d_q$, say. A subset S_2 of $\mathbb{T}_q^{(2)}$ will be called *unpaired* if $\#S_2 \cap T_r \leq 1$ for $1 \leq r \leq d_q$.

For the sake of convenience an empty sum will be taken to be zero. Also for an $A = [a_{jk}]$ in M_n, $\bar{A}$ denotes the matrix $[\bar{a}_{jk}]$.

Theorem 3.1. *Let A ba a non-zero real or normal matrix in M_n and $q \in \mathbb{N}$ with $n, q > 1$. Then A satisfies $A^t = A^q$ if and only if there exist subsets S_1 and S_2 of $\mathbb{T}_q^{(1)}$ and $\mathbb{T}_q^{(2)}$ respectively and mutually orthogonal non-zero projections P_s, $s \in S_1 \cup S_2$ such that*

(i) *S_2 is unpaired;*
(ii) *for $s \in S_1$, P_s is a real matrix;*
(iii) *for $s \in S_2$, $\bar{P}_s$ and $P_{s'}$ are orthogonal for $s' \in S_1 \cup S_2$;*
(iv) $A = \sum_{s \in S_1} sP_s + \sum_{s \in S_2} sP_s + \sum_{s \in S_2} s^q \bar{P}_s$.

Proof. The 'if' part follows immediately by taking the transpose of A as in (iv).

For the other part, suppose $A^t = A^q$. Then by 3.A(iii) A is normal and $A^{q^2} = A$. Let $\sum =$ the set of non-zero eigenvalues of A. Because $A \neq 0$, $\sum$ is nonempty. Also $A^{q^2} = A$ forces $\sum$ to be contained in $\{\lambda : \lambda^{q^2-1} = 1\} = \mathbb{T}_q^{(1)} \cup \mathbb{T}_q^{(2)}$. Let $S_1 = \sum \cap \mathbb{T}_q^{(1)}$ and $\sum_2 = \sum \cap \mathbb{T}_q^{(2)}$. Because A is normal, there exist non-zero mutually orthogonal projections $\{P_s : s \in \sum\}$ such that $A = \sum_{s \in \sum} sP_s$. Then $A^t = \sum_{s \in \sum} sP_s^t = \sum_{s \in \sum} s\bar{P}_s$. Also $A^q = \sum_{s \in \sum} s^q P_s$.

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But $A^t = A^q$. So $\sum\limits_{s\in\sum} s\bar{P}_s = \sum\limits_{s\in\sum} s^q P_s$.

So for each $s \in \sum$, there exists $\lambda_s \in \sum$ satisfying $s = \lambda_s^q$ and $\bar{P}_s = P_{\lambda_s}$,

For $s \in S_1$, $\lambda_s = s$ and, therefore, $\bar{P}_s = P_s$, i.e., P_s is a real matrix.

For $s \in \sum_2$, $\bar{P}_s = P_{\lambda_s}$; and also, $\lambda_s \in \sum_2$ with $\bar{P}_{\lambda_s} = P_s$. We can form an unpaired set $S_2 \subset \sum_2$ with $\{s, s^q : s \in S_2\} = \sum_2$.

Hence $A = \sum\limits_{s\in S_1} sP_s + \sum\limits_{s\in S_2} sP_s + \sum\limits_{s\in S_2} s^q\bar{P}_s$ and conditions (i), (ii), (iii) are also satisfied. $\square$

Definition 3.2. Let A be a normal matrix. For a $q \in \mathbb{N}$, $q > 1$, A will be called *q-reflected* if $A^t = A^q$ and it will be called *reflected* if it is q-reflected for some $q \geq 2$.

Remark 3.3. Let A be a non-zero matrix..

(i) If A is power symmetric of symmetric order > 1 then it is reflected.
(ii) If A is reflected then A is a partial isometry.
(iii) Suppose A is reflected. Then the following hold.
   (a) A is positive if and only if A is projection and real.
   (b) A is Hermitian if and only if $A = P - Q$ for some real mutually orthogonal projections P and Q.
   (c) If A is q-reflected, then, in the notation of Theorem 3.1, A satisfies (i) to (iv) of Theorem 3.1 and also,

$$\begin{aligned}\bar{A}A &= \sum_{s\in S_1} P_s + \sum_{s\in S_2} s\bar{s}^q P_s + \sum_{s\in S_2} \bar{s}s^q \bar{P}_s = A\bar{A} \\ &= \sum_{s\in S_1} P_s + \sum_{s\in S_2} \bar{s}^{q-1} P_s + \sum_{s\in S_2} s^{q-1}\bar{P}_s \ ;\end{aligned}$$

   in particular, $\bar{A}A$ is a real q-reflected matrix.

3.B. Generalized power symmetric matrices. We begin with the relevant material from Bapat, Jain and Prasad [5].

(i) As defined by Bapat, Jain and Prasad [5], a *generalized power symmetric* matrix is a stochastic matrix A which satisfies $(A^p)^t = A^q$ for some $p, q \in \mathbb{N}$ with $p < q$.
(ii) Bapat et al [5] obtain several properties of a generalized power symmetric matrix A and also give a general form for A in terms of J_r's. An easy one to be used is, $A^{p^2} = A^{q^2}$ and therefore, an eigenvalue λ of A has to be either zero or satisfy $\lambda^{q^2-p^2} = 1$. We may formulate an analogue of Theorem 3.1 but it is not so neat or revealing.

We make an attempt to utilize contents of this section to construct PPT states.

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For this purpose we give methods to produce classes of positive block matrices that are positive under partial transpose, in short, PPT. The idea is to replace transpose by a power in a broad sense and see if that will be helpful.

3.C. Positivity of block matrices and Schur or Hadamard products. We collect a few results in this direction.

(i) This is a well-known result and one may find it in books like [8] and [40]: Let A, B, C be $n \times n$ matrices. Then the block matrix $R = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ if and only if $A \geq 0$, $C \geq 0$ and there exists a contraction K such that $B = A^{\frac{1}{2}} K C^{\frac{1}{2}}$.

(ii) Choudhury's Theorem. Choudhury [16] considered a block matrix $H = [H_{jk}]$, where H_{jk}'s are normal $n \times n$ matrices for $1 \leq j,\ k \leq m$. Theorem 5 [16] says that if the m^2 matrices $\{H_{jk} : 1 \leq j, k \leq m\}$ are a commuting family and H is positive semi-definite then $H_\alpha = [H_{jk}^\alpha]$ is positive semi-definite for all $\alpha = 1, 2, \ldots$.

(iii) Arbitrary Hadamard powers can be defined for non-negative matrices (cf. [4]). In that spirit, Guillot, Khare and Rajaratanam [34] give interesting further developments, but reveal that actions of replacing α by other numbers or relaxing the conditions limit the possibility of preserving the positive semi-definiteness and that of its application to the construction of PPT states.

Definition 3.4. Let $\mathcal{M}$ be a set of normal $n \times n$ matrices.

(i) For $q \in \mathbb{N}$, $q > 1$, the set $\mathcal{M}$ will be called *q-reflected* if $A^t = A^q$ for each A in $\mathcal{M}$. It will be called *reflected* if it is q-reflected for some $q \geq 2$.

(ii) For $p, q \in \mathbb{N}$ with $p < q$, the set $\mathcal{M}$ will be called *generalized (p, q)-reflected* if $(A^p)^t = A^q$ for each A in $\mathcal{M}$. Further it will be called *generalized reflected* if it is (p, q)-reflected for some $p, q \in \mathbb{N}$ with $1 \leq p < q$.

Theorem 3.5. *Let $\mathcal{H} = \{H_{jk} : 1 \leq j, k \leq m\}$ be a reflected commuting family of matrices such that $H = [H_{jk}]$ is positive. Then H is positive under partial transpose.*

Proof. There exists a $q \in \mathbb{N}$ with $q \geq 2$ such that $\mathcal{H}$ is q-reflected. We first note that it follows from Remark 3.3(iii)(a) that each H_{jj} is real and also a projection. So $H_{jj}^t = H_{jj} = H_{jj}^q$ for $1 \leq j \leq m$. By Choudhury's Theorem 3.C(ii) H_q is positive. But $H_q = H^{PT}$. So H is positive under partial transpose. ▯

Theorem 3.6. *Let B be reflected and $A, C \geq 0$ satisfy $R = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ and B commutes with A or C. If either B or both A and C are real, then R is positive under partial transpose.*

Proof. By 3.C(i) there exists a contraction K such that $B = A^{\frac{1}{2}} K C^{\frac{1}{2}}$. Suppose

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B commutes with A. Then for $q \in \mathbb{N}$, $B^q = A^{\frac{1}{2}} B^{q-1} K C^{\frac{1}{2}}$. By Theorem 3.1, B is a contraction. So $B^{q-1}K$ is a contraction. By 3.C(i), $V = \begin{bmatrix} A & B^q \\ (B^q)^* & C \end{bmatrix} \geq 0$.

Since B is reflected, $B^t = B^q$ for some $q \in \mathbb{N}$. So $\begin{bmatrix} A & B^t \\ (B^t)^* & C \end{bmatrix} = V \geq 0$. Now R^{PT} is V in case both A and C are real, and $\bar{V}$ if B is real. Hence R is positive under partial transpose.

Similar arguments give the result when B commutes with C. ∎

Remark 3.7. Given any $B \in M_n$, there exist real $A, C \geq 0$ with B commuting with A and C such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. All we have to do is to note that $R_B = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ is Hermitian and therefore there exists $\lambda \geq 0$ such that $-\lambda I_{2n} \leq R_B \leq \lambda I_{2n}$. We can take $A = C = \lambda I_n$.

Theorem 3.8. *Let $A, B, C \in M_n$ be such that B is generalized reflected and invertible, $A, C \geq 0$ satisfy $R = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. If B and B^t commute with A or C and also either B or both A and C are real, then R is positive under partial transpose.*

Proof. Because B is generalized reflected there exist $p, q \in \mathbb{N}$ with $p < q$ such that $(B^p)^t = B^q$. Properties for generalized power symmetric matrices noted in 3.B(ii) hold for B. To elaborate, eigenvalues λ of B satisfy $\lambda^{p^2} = \lambda^{q^2}$. Because B is normal this gives $\|B\| = 1$. Now B^{-1} is generalized reflected as well. So $\|B^{-1}\| = 1$ too.

By 3.C(i), the condition $R \geq 0$ gives a contraction K such that $B = A^{\frac{1}{2}} K C^{\frac{1}{2}}$. Now $(B^p)^t = B^q$, So $B^t = ((B^{p-1})^t)^{-1} B^q = ((B^t)^{-1})^{p-1} B^q$. Suppose B, B^t commute with A. Then

$$\begin{aligned} B^t &= ((B^t)^{-1})^{p-1} B^{q-1} B = ((B^t)^{-1})^{p-1} B^{q-1} A^{\frac{1}{2}} K C^{\frac{1}{2}} \\ &= ((B^t)^{-1})^{p-1} A^{\frac{1}{2}} B^{q-1} K C^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} ((B^t)^{-1})^{p-1} B^{q-1} K C^{\frac{1}{2}} . \end{aligned}$$

As already noted B and B^{-1} are contractions. So, $K_1 = ((B^t)^{-1})^{p-1} B^{q-1} K$ is a contraction. By 3.C(i), $V = \begin{bmatrix} A & B^t \\ (B^t)^* & C \end{bmatrix} \geq 0$.

Hence as argued in the proof Theorem 3.6 above R is positive under partial transpose. ∎

Remark 3.9. We can relax the condition of invertibility on B and work with

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the group inverse G of B.

Remark 3.10. Let A, B, C, R be as in 3.C(i) with tr$R$ not equal to zero. H.J. Woerdeman [91] gives conditions under which $\rho = \frac{1}{\mathrm{tr}R}\, R$ is a separable state. We are now in a position to use them, particularly [91], Theorem 3.2 onwards. We record the simplest case:

Let $A = I, B, C, R$ be as in Theorem 3.6 (Theorem 3.8). Then ρ is separable. To see this, let $W = \begin{bmatrix} I & B^* \\ B & C \end{bmatrix}$ and $V = \begin{bmatrix} I & B^t \\ (B^t)^* & C \end{bmatrix}$. Then $W = V$ if B is real and $\bar{V}$ if C is real. So as in the proof of Theorem 3.6 or 3.8 we have $W \geq 0$. We can now apply the discussion in [91].

4. Peres Condition and Unitary Equivalence of a Matrix to its Transpose. We follow the notation and terminology of Garcia and Tener [31] who obtained a canonical decomposition for complex matrices T which are *UET*, i.e., *unitarily equivalent to their transpose* $T^t(UET)$.

4.A. . We collect a few facts from [31] for ready reference.

(i) [31, §1]. In his problem book ([39], Pr. 159) Halmos asks whether every square matrix is UET and in his discussion gives the counterexample $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, which is not UET. Every Toeplitz matrix is UET via the permutation matrix which reverses the order of the standard basic vectors.

(ii) [31, Theorem 1.1]. A matrix T in M_n is UET if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent):
   (a) irreducible complex symmetric matrices (CSMs),
   (b) irreducible skew-Hamiltonian matrices (SHMs) (such matrices are necessarily 8×8 or larger, a SHM is a $2d \times 2d$ block matrix of the form $\begin{pmatrix} A & B \\ D & A^t \end{pmatrix}$ with $B^t = -B$ and $D^t = -D$),
   (c) $2d \times 2d$ blocks of the form $\begin{pmatrix} A & 0 \\ 0 & A^t \end{pmatrix}$ where A is irreducible and neither unitarily equivalent to a complex symmetric matrix (UECSM) nor unitarily equivalent to a skew-Hamiltonian matrix (UESHM) (such matrices are necessarily 6×6 or larger).

   Moreover, the unitary orbits of the three classes described above are pairwise disjoint.

(iii) ([31], Corollary 2.3). If T is UET and has order $n \times n$ with $n \leq 7$, then T is UECSM.

(iv) ([31], 8.3 and 8.4). S is UET if and only if S is unitarily equivalent to a matrix T

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that satisfies $TQ = QT^t$, where Q is a unitary matrix of the special form (some of the blocks may be absent and empty blocks are all zero):

$$Q = \left(\begin{array}{c|c|cc|c|cc} Q_+ & & & & & & \\ \hline & Q_- & & & & & \\ \hline & & 0 & \lambda_1 X_1^t & & & \\ & & X_1 & 0 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & 0 & \lambda_r X_r^t \\ & & & & & X_r & 0 \end{array}\right),$$

where
(a) $Q_+ = Q_+^t$ is complex symmetric and unitary,
(b) $Q_- = -Q_-^t$ is skewsymmetric and unitary,
(c) $\lambda_i \neq \pm 1$ and X_i is unitary for $i = 1, 2, \ldots, r$.

(v) ([31], 8.5). Given Q as in (iv) above, T is as in (iv) above if and only if

$$T = \left(\begin{array}{c|c|cc|c|cc} T_+ & & & & & & \\ & T_- & & & & & \\ \hline & & A_1 & 0 & & & \\ & & 0 & X_1 A_1^t X_1^* & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & A_r & 0 \\ & & & & & 0 & X_r A_r^t X_r^* \end{array}\right),$$

where
(a) $T_+ = Q_+ T_+^t Q_+^*$ (such a T_+ is UECSM),
(b) $T_- = Q_- T_-^t Q_-^*$ (such a T_- is UESHM),
(c) $A_1, \ldots, A_r$ are arbitrary.
In fact this is the final step of the proof of (ii) in [31].

It is my pleasure to thank my students Priyanka Grover and Tanvi Jain. Priyanka came to discuss UET in some other context and Tanvi found a former version of [31] on the internet for that context.

Definition 4.1. A tuple $(Y_1, \ldots, Y_s)$ of $n \times n$ matrices is said to be *collectively unitarily equivalent to the respective transposes (CUET)* if there is a unitary U with $Y_j = UY_j^t U^*$ for $1 \leq j \leq s$.

Remark 4.2. W.B. Arveson [3, Lemma A.3.4] gives that $\left(\begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & -\lambda & 0 \end{pmatrix}\right)$ is not CUET where λ is a non-real complex number and μ is a complex number with

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$|\mu| = (1 + |\lambda|^2)^{\frac{1}{2}}$.

Remark 4.3. Discussion in 4.A above tells us how to construct CUET tuples viz., choose a Q as in item 4.A(iv) and then Y_j's as in 4.A(v) by varying T_+, T_-, A_k for $1 \le k \le r$.

Theorem 4.4. *Let $[A_{jk}]$ be a positive block matrix such that $(A_{jk} : 1 \le j, k \le n)$ is CUET. Then $[A^t_{jk}]$ is positive.*

Proof. There is a unitary matrix U such that $A_{jk} = UA^t_{jk}U^*$ for $1 \le j, k \le n$. Let $\widetilde{U}$ be the block matrix $[\delta_{jk}U]$, with $\delta_{jk} = 0$ for $j \ne k$ and 1 for $j = k$. Then $\widetilde{U}$ is unitary and $[A^t_{jk}] = \widetilde{U}^*[A_{jk}]\widetilde{U}$. So $[A^t_{jk}]$ is positive. ▯

Construction 4.5. Remark 4.3 and Theorem 4.4 put together tell us how to construct PPT matrices.

Step 1: Let $n \ge 2$ and put $m = \frac{n(n+1)}{2}$. Use Remark 4.3 to construct a CUET m-tuple $(Y_j : 1 \le j \le m)$ with $Y_j \ge 0$ for $1 \le j \le n$ (we may take all Y_j, $1 \le j \le n$, to be zero, for instance). For $1 \le j \le m$, we have $Y_j = QY^t_jQ^*$ and, therefore $Y^*_j = QY^{*t}_jQ^*$. We set $B_{jj} = Y_j$ for $1 \le j \le n$, arrange Y_j for $n+1 \le j \le \frac{n(n+1)}{2}$ as B_{pq}, $1 \le p < q \le n$ and take $B_{qp} = B^*_{pq}$ for $1 \le p < q \le n$. Thus, we obtain a block matrix $B = [B_{jk}]$ which is Hermitian and $\{B_{jk} : 1 \le j, k \le n\}$ is CUET.

Step 2: The set $\{a \in \mathbb{R} : B + a\ I_{n^2} \ge 0\}$ is an interval $[a_0, \infty)$ for some $a_0 \in \mathbb{R}$. We take any a in this interval and set $A = B + a\ I_{n^2}$ i.e., $A_{jk} = B_{jk}$ for $j \ne k$, whereas $A_{jj} = B_{jj} + aI_n$ for $1 \le j, k \le n$. Then $\{A_{jk} : 1 \le j, k \le n\}$ is CUET and $A \ge 0$. So we can apply Theorem 4.4 to conclude that A is a PPT matrix.

Remark 4.6. We can, of course, formulate "unitarily equivalent" versions of various notions in §3 above for subsets $\mathcal{M}$ of M_n. Suitable variants of results in §3 can then be obtained to give new classes of positive matrices with positive partial transpose and thus, PPT states. We do not go into the details here.

5. Quantum Dynamical Semigroups involving Separable and Entangled States. Let $\mathcal{H}$ be a Hilbert space and τ the transpose map on $\mathcal{B}(\mathcal{H})$ with respect to some orthonormal basis for $\mathcal{H}$. Let $*$ or $\dagger$ be the adjoint map on $\mathcal{B}(\mathcal{H})$ that takes x to x^*. Let $\mathcal{X}$ be a linear subspace of $\mathcal{B}(\mathcal{H})$ which is closed under τ as well as $*$. We shall consider C_0-semigroups $(T_t)_{t\ge0}$ as well as T_0-constricted C_0-semigroups $(T_t)_{t\ge0}$ of operators on $\mathcal{X}$ to itself.

We begin with a few examples.

Examples 5.1.

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(i) This is modelled on Størmer's Example 8.13 [86] and is in a foliated form with $\Lambda_{t1} = \begin{bmatrix} 1 & 0 \\ 1-e^{-t} & e^{-t} \end{bmatrix}$, $\Lambda_{t2} = e^{-\frac{t}{2}}\mathcal{I}_{F_2}$ and $\Lambda_t = \Lambda_{t1} \oplus \Lambda_{t2}$ in the notation of item 2.H(i)(b). It is a non-PPT quantum dynamical semigroup.

(ii) If we are interested in separable maps we have to do away with the condition $T_0 = \mathcal{I}d$, which we now do.

This example is modelled on the example of the two spin $\frac{1}{2}$-states given by Horodecki et al [45]. It is in a foliated form with $\Lambda_1^{p,a,b} = \begin{bmatrix} pa^2 & (1-p)b^2 \\ (1-p)a^2 & pb^2 \end{bmatrix}$ and $\Lambda_2^{p,a,b} = \begin{bmatrix} pab & (1-p)ab \\ (1-p)ab & pab \end{bmatrix}$ with $0 \le p \le 1$, $a > 0$, $b > 0$ and $\Lambda^{p,a,b} = \Lambda_1^{p,a,b} \oplus \Lambda_2^{p,a,b}$.

Taking Pauli matrices $\sigma_0 = I_2$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as a basis, the map $\Lambda^{p,a,b}$ has the simple form

$$\begin{pmatrix} \frac{a^2+b^2}{2} & 0 & 0 & \frac{a^2-b^2}{2} \\ 0 & ab & 0 & 0 \\ 0 & 0 & 2(\frac{1}{2}-p)ab & 0 \\ (p-\frac{1}{2})(a^2-b^2) & 0 & 0 & (p-\frac{1}{2})(a^2+b^2) \end{pmatrix}.$$

(a) The map $\wedge^{p,a,b}$ is unital if and only if $a^2 + b^2 = 2$ and either $p = \frac{1}{2}$ or $a^2 = b^2$ $(= 1)$. On the other hand, the map is trace preserving if and only if $a^2 = 1 = b^2$.

(b) As noted by Horodecki et al, it is a separable map if and only if $p = \frac{1}{2}$ and, in that case, the matrix becomes $\begin{pmatrix} \frac{a^2+b^2}{2} & 0 & 0 & \frac{a^2-b^2}{2} \\ 0 & ab & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and thus the range is the commutative algebra spanned by σ_0 and σ_1.

(c) Taking $a^2 + b^2 = 1$, $a = \cos\theta$, $b = \sin\theta$, $0 < \theta \le \frac{\pi}{4}$, $u = \sin 2\theta$ motivates the semigroup

$$T_t = \left(\frac{1}{2}\right)^t \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & u^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad t \ge 0.$$

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We note that T_0 is the idempotent

$$\begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) We may consider the variant for $t \geq 0$,

$$S_t = \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-u^2} \\ 0 & u^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

All these take I_2 to itself and also $S_0 = T_0$. Any one of S_t's is (and thus, all are) trace preserving if and only if $u = 1$,

i.e., S_t's are all the same as the projection given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 5.2 (Trichotomy). *Let $(T_t)_{t\geq 0}$ be a T_0 constricted quantum dynamical semigroup. Then one and only one of the following holds.*

(i) *For each $t \geq 0$, T_t is separable.*
(ii) *There exists $t_0 > 0$ such that T_t is entangled for $t < t_0$ but T_t is separable for $t \geq t_0$.*
(iii) *For each $t \geq 0$, T_t is entangled.*

Moreover, (i) holds if and only if T_0 is separable.

Proof. The three conditions are mutually exclusive. Suppose (iii) does not hold. Then the set $S = \{t \geq 0 : T_t \text{ is separable}\} \neq \phi$. If $t \in S$ then for $s > t$, $T_s = T_{s-t}T_t$. By Theorem 2.1, T_s is separable. So S is an interval of the form (t_0, ∞) or $[t_0, \infty)$. By item 2.D(vi) and the condition of strong continuity on $(T_t)_{t\geq 0}$, T_{t_0} is separable. So $S = [t_0, \infty)$. If (i) does not hold, we have $t_0 > 0$. Thus (ii) holds.

Now suppose T_0 is separable. Then the set $S = \{t \geq 0 : T_t \text{ is separable}\}$ contains 0. As seen above, if $S \neq \phi$, then $S = [t_0, \infty)$ for some $t_0 \in [0, \infty)$. So $S = [0, \infty)$ i.e. (i) holds. $\square$

Definition 5.3. The space $\mathcal{X}$ will be said to be *normal* if for $x \in \mathcal{X}$, $\#\{x, x^*, xx^*, x^*x\}\cap$

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$\mathcal{X} \leq 3$. In other words, each $x \in \mathcal{X}$ is either normal or else at most one of xx^* and x^*x is in $\mathcal{X}$.

PROPOSITION 5.4. *Let φ be an idempotent $*$-map on $\mathcal{X}$. If φ is co-Schwarz then the range of φ is normal.*

Proof. Let $\mathcal{Y} = \varphi(\mathcal{X})$. Since φ is a $*$-map, for $x \in \mathcal{Y}$, x^* is in $\mathcal{Y}$. Since $\varphi^2 = \varphi$ we have $\varphi|\mathcal{Y} = \mathcal{I}d_{\mathcal{Y}}$. Let, if possible, there exist $y \in \mathcal{Y}$ with y^*y, $yy^* \in \mathcal{Y}$. Since φ is co-Schwarz, we have $\tau\varphi(y^*y) \geq \tau\varphi(y^*)\tau\varphi(y)$, i.e. $\tau(y^*y) \geq \tau(y^*)\tau(y)$. So $\tau(y)\tau(y^*) \geq \tau(y^*)\tau(y)$. We may interchange the role of y and y^* and get $\tau(y^*)\tau(y) \geq \tau(y)\tau(y^*)$. So $\tau(yy^*) = \tau(y^*y)$. Therefore $yy^* = y^*y$. ∎

REMARK 5.5. Let $(T_t)_{t\geq 0}$ be a T_0-constricted quantum dynamical semigroup.

(i) If the range $\mathcal{R}_0$ of T_0 is normal, then for $t > 0$, the range $\mathcal{R}_t$ of T_t is normal simply because $\mathcal{R}_t \subset \mathcal{R}_0$.
(ii) One can have more Trichotomy results by replacing "separable" by
  (a) PPT, or
  (b) has Schmidt number $\leq r$, or
  (c) has normal range
  and then "entangled" by the corresponding negations like non-PPT, has Schmidt number $> r$ and has non-normal range.
(iii) In fact, the first condition in any such Trichotomy holds if and only if it holds for T_0. By 2.D(viii), it holds if $\mathcal{R}_0$ is contained in an abelian C^* algebra acting on a separable Hilbert space $\mathcal{H}$.
(iv) A non-commutative C^*-algebra is not normal. So for an interesting theory, we can give up the condition (i) of Trichotomy and instead take $T_0 = \mathcal{I}d$.

THEOREM 5.6. *Let $\mathcal{X}$ be a non-commutative C^*-algebra and $(T(t))_{t\geq 0}$ be a quantum dynamical semigroup of completely positive maps. If for some $t_0 > 0$, $T(t_0)^{-1}$ exists and is a Schwarz map, then for each $t > 0$, $T(t)$ is non-PPT.*

Proof. We refer to item 2.G(vii)(a) as for the proof of Theorem 2.4. We use the fact that the product of two Schwarz maps is a Schwarz map. For $0 < t < t_0$, $T(t)^{-1} = T(t_0 - t)(T(t_0))^{-1}$, and therefore, $T(t)^{-1}$ is a Schwarz map. Also for $n \in \mathbb{N}$, $0 < s < t_0$, $t = nt_0 + s$, $T(t)^{-1} = T(s)^{-1}(T(t_0)^{-1})^n$, and therefore, $T(t)^{-1}$ is a Schwarz map. Let, if possible, for some $t > 0$, $T(t)$ be PPT. Then $\tau T(t)$ is completely positive. So $\tau = \tau T(t)(T(t))^{-1}$ is a Schwarz map, which is not so because $\mathcal{X}$ is non-commutative. ∎

We now illustrate results in this section with examples of generalized Choi maps discussed in 2.H above.

EXAMPLE 5.7.

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(i) This may be thought of as continuation of 2.H. We begin by recalling relevant details, which are well-known from the theory of circulant matrices (cf. [4], [8], [32]).

(ii) Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$. Then

$$\begin{aligned} D(\alpha,\beta,\bar{\beta}) &= \alpha I_3 + \beta(E_{12}+E_{23}+E_{31}) + \bar{\beta}(E_{21}+E_{32}+E_{13}) \\ &= \alpha I_3 + \beta L + \bar{\beta} L^*, \end{aligned}$$

where $L = E_{12}+E_{23}+E_{31}$.

We note that $L^2 = L^*$, $LL^* = L^*L = I_3$. So L is a unitary matrix with eigenvalues $1, \omega, \omega^2$ and is expressible as $L = W\mathrm{Diag}(1,\omega,\omega^2)W^*$ with $W = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$.

Here $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, a cube root of unity.

So $D(\alpha,\beta,\bar{\beta}) = W\mathrm{Diag}(\alpha+\beta+\bar{\beta}, \alpha+\beta\omega+\bar{\beta}\omega^2, \alpha+\beta\omega^2+\bar{\beta}\omega)W^*$.

(iii) For $a,b,c \in \mathbb{C}$, $D(a,b,c)^* = D(\bar{a},\bar{c},\bar{b})$ and thus, $D = D(a,b,c)$ is normal. Further, $D_1 = \frac{1}{2}(D+D^*) = D(\mathrm{Re}\, a, \beta, \bar{\beta})$ with $\beta = \frac{1}{2}(b+\bar{c})$, and $D_2 = \frac{1}{2i}(D - D^*) = D(\mathrm{Im}\, a, \gamma, \bar{\gamma})$ with $\gamma = \frac{1}{2i}(b-\bar{c})$. So by (ii),

$$\begin{aligned} D(a,b,c) &= D_1 + iD_2 \\ &= W\mathrm{Diag}(a + (\beta+i\gamma) + (\bar{\beta}+i\bar{\gamma}), a + (\beta+i\gamma)\omega + (\bar{\beta}+i\bar{\gamma})\omega^2, \\ &\quad a + (\beta+i\gamma)\omega^2 + (\bar{\beta}+i\bar{\gamma})\omega)W^* \\ &= W\mathrm{Diag}(a+b+c, a+b\omega+c\omega^2, a+b\omega^2+c\omega)W^*. \end{aligned}$$

(iv) For $n \in \mathbb{N}$, $a,b,c \in \mathbb{C}$

$$(D(a,b,c))^n = W\,\mathrm{Diag}((a+b+c)^n, (a+bw+cw^2)^n, (a+bw^2+cw)^n))W^*,$$

and therefore, for $t \in \mathbb{C}$,

$$e^{tD(a,b,c)} = W\,\mathrm{Diag}(e^{t(a+b+c)}, e^{t(a+bw+cw^2)}, e^{t(a+bw^2+cw)})W^*.$$

We note that all these matrices are in $GL(3,\mathbb{C})$ and $(e^{tD(a,b,c)})^{-1} = e^{-tD(a,b,c)} = e^{tD(-a,-b,-c)}$ for $a,b,c,t \in \mathbb{C}$.

(v) Let $a,b,c,\, t \in \mathbb{C}$.

By (iii) $e^{tD(a,b,c)} = D(\boldsymbol{a}(t), \boldsymbol{b}(t), \boldsymbol{c}(t))$ with $\begin{pmatrix} \boldsymbol{a}(t) \\ \boldsymbol{b}(t) \\ \boldsymbol{c}(t) \end{pmatrix} = \frac{1}{\sqrt{3}} W^* \begin{pmatrix} e^{t(a+b+c)} \\ e^{t(a+b\omega+c\omega^2)} \\ e^{t(a+b\omega^2+c\omega)} \end{pmatrix}$.

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Therefore,

$$\begin{aligned}
\boldsymbol{a}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + e^{t(a+b\omega+c\omega^2)} + e^{t(a+b\omega^2+c\omega)}\right],\\
\boldsymbol{b}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + \omega^2\, e^{t(a+b\omega+c\omega^2)} + \omega\ e^{t(a+b\omega^2+c\omega)}\right] \quad \text{and}\\
\boldsymbol{c}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + \omega\ e^{t(a+b\omega+c\omega^2)} + \omega^2\ e^{t(a+b\omega^2+c\omega)}\right].
\end{aligned}$$

We set $\boldsymbol{d}(t) = e^{td}$. We note that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are all entire functions and $\boldsymbol{a}(0) = 1 = \boldsymbol{d}(0)$ whereas $\boldsymbol{b}(0) = 0 = \boldsymbol{c}(0)$. Further, for a, b, c, d, t all real, $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, $\boldsymbol{d}(t)$ are all real.

(vi) Let $a, b, c, d \in \mathbb{C}$. For $t \in \mathbb{C}$,

$$\begin{aligned}
\boldsymbol{a}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + (a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)}\\
&\quad + (a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}],\\
\boldsymbol{b}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + \omega^2\,(a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)}\\
&\quad + \omega\,(a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}],\\
\boldsymbol{c}'(t) &= \frac{1}{3}[(a+b+c)e^{t(a+b+c)} + \omega\,(a+b\omega+c\omega^2)e^{t(a+b\omega+c\omega^2)}\\
&\quad + \omega^2\,(a+b\omega^2+c\omega)e^{t(a+b\omega^2+c\omega)}] \qquad \text{and}\\
\boldsymbol{d}'(t) &= d\boldsymbol{e}^{td}.
\end{aligned}$$

In particular, $\boldsymbol{a}'(0) = a$, $\boldsymbol{b}'(0) = b$, $\boldsymbol{c}'(0) = c$, and $\boldsymbol{d}'(0) = d$. As a consequence, if $\boldsymbol{a}(t_n)$ (respectively $\boldsymbol{b}(t_n)$, $\boldsymbol{c}(t_n)$) are all real for a real sequence (t_n) convergent to zero then a (respectively b, c) is real.
Thus in view of the last line of (v) we may say that $\boldsymbol{a}(t_n)$, $\boldsymbol{b}(t_n)$, $\boldsymbol{c}(t_n)$ are all real for a real sequence (t_n) convergent to zero if and only if a, b, c are all real if and only $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$ are all real for all real t. A similar statement holds for the function $\boldsymbol{d}$ as well.

(vii) Let $a, b, c, d \in \mathbb{C}$ and set $\rho = \rho[a, b, c, d]$. Then by (v) above, for $t \in \mathbb{C}$, $\boldsymbol{\rho}(t) = e^{t\rho}$ coincides with $\rho[\boldsymbol{a}(t), \boldsymbol{b}(t), \boldsymbol{c}(t), \boldsymbol{d}(t)]$. We first note that in view of (iv) above, each $\boldsymbol{\rho}(t)$ is a bijective map on M_3 to itself. Further, for $b = 0 = c$, $\boldsymbol{a}(t) = e^{at}$ and $\boldsymbol{b}(t) = 0 = \boldsymbol{c}(t)$, so that

$$\boldsymbol{\rho}(t) = e^{ta}\mathcal{I}d_{D_n} \oplus e^{td}\mathcal{I}d_{F_n} \text{ for } t \in \mathbb{C}.$$

(a) Items (i) and (vi) may be combined to give: $\boldsymbol{\rho}(t_n)$ are all $*$-maps for a real sequence (t_n) convergent to 0 if and only if a, b, c, d are all real if and only if $\boldsymbol{\rho}(t)$ are all $*$-maps for all real t.
From now onwards we consider only real a, b, c, d, t.

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(b) By 2.H(ii)(c) and (v) above, for any $t \neq 0$, $\boldsymbol{\rho}(t)$ is unital if and only if $a+b+c=0$ and in that case all $\boldsymbol{\rho}(t)$ are unital as t varies in $\mathbb{R}$. Similar statements hold with unital replaced by trace-preserving.

(viii) Let a, b, c, d be real. Set $u = \frac{1}{2}(b+c)$, $v = \frac{1}{2}(b-c)$. Then

$$\begin{aligned} a+b+c &= a+2u, \\ a+b\omega+c\omega^2 &= a-u+i\sqrt{3}v, \\ a+b\omega^2+c\omega &= a-u-i\sqrt{3}v. \end{aligned}$$

Let $t \in \mathbb{R}$. Then

$$\begin{aligned} \boldsymbol{a}(t) &= \frac{1}{3}\left[e^{t(a+b+c)} + e^{t(a+b\omega+c\omega^2)} + e^{t(a+b\omega^2+c\omega)}\right] \\ &= \frac{1}{3}\left[e^{t(a+2u)} + e^{t(a-u+i\sqrt{3}v)} + e^{(a-u-i\sqrt{3}v)}\right] \\ &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt)\right], \\ \boldsymbol{b}(t) &= \frac{1}{3}\left[e^{t(a+2u)} + e^{t(a-u+i\sqrt{3}v)-\frac{2}{3}\pi i} + e^{t(a-u-i\sqrt{3}v)+\frac{2}{3}\pi i}\right] \\ &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt - \frac{2}{3}\pi)\right], \\ \boldsymbol{c}(t) &= \frac{1}{3}e^{t(a-u)}\left[e^{3tu} + 2\cos(\sqrt{3}vt + \frac{2}{3}\pi)\right], \\ \boldsymbol{d}(t) &= e^{td} > 0. \end{aligned}$$

We recall from (v) above that $\boldsymbol{a}(0) = 1$, $\boldsymbol{b}(0) = 0 = c(0)$, $\boldsymbol{d}(0) = 1$.
If $\boldsymbol{\rho}$ is a positive map, then by 2.H(ii)(c) $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, are ≥ 0.
We begin by finding out when $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, are ≥ 0 and then go on to find conditions under which $\boldsymbol{\rho}(t)$ is completely positive, PPT, separable etc.

(ix) We can argue as in (vi) above and have that if $\boldsymbol{b}(t_n)$ (respectively $\boldsymbol{c}(t_n)$) are all non-negative for a sequence (t_n) in $(0, \infty)$ convergent to zero, then b (respectively c) is ≥ 0. So from now onwards we take $b, c \geq 0$.

(x) Let $b = 0 = c$. Then $\boldsymbol{a}(t) = e^{ta}$ and $\boldsymbol{d}(t) = e^{td} > 0$ for all $t \in \mathbb{R}$ whereas $\boldsymbol{b}(t) = 0 = \boldsymbol{c}(t)$ for all $t \in \mathbb{R}$. By 2.H(iii)(c) no $\boldsymbol{\rho}(t)$ is completely copositive. By 2.H(iii)(b), $\boldsymbol{\rho}(t)$ is completely positive if and only if $e^{ta} \geq e^{td}$ if and only if $ta \geq td$.

(a) For $a = d, \{\boldsymbol{\rho}(t) : t \in \mathbb{R}\} \equiv \{e^{td} Id_{M_3} : t \in \mathbb{R}\}$ is a group of completely positive maps that are all non-PPT, which illustrates Theorem 5.6. For $a = d = 0$ it is the trivial group $\{Id_{M_3}\}$ for $t \in \mathbb{R}$.

(b) The family $\{\boldsymbol{\rho}(t) : t \geq 0\}$ is a quantum dynamical semigroup if and only if $a \geq d$ and all the maps are non-PPT and therefore, entangled. This illustrates the condition (iii) of the Trichotomy in Theorem 5.2 and Remark 5.5(ii) (a).

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(c) Let $a = 0 > d$ and $t > 0$. Then by (vii)(b) above each $\boldsymbol{\rho}(t)$ is unital and trace-preserving; in particular, each $\boldsymbol{\rho}(t)$ is a quantum channel. By Theorem 2.6(ii)(e), for $0 \le t < -\frac{(\log 2)}{d}$. $\boldsymbol{\rho}(t)$ has Schmidt number 3, and is, therefore, not partially entanglement braking.

(d) It follows from Remark 2.8(i)(a) that for $a = d$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 1 for all $t \in \mathbb{R}$ and, on the other hand, for $a > d$, $t > 0$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 3.

(xi) Let $(b, c) \neq (0, 0)$, b, $c \geq 0$. We refer to (viii) above.
Then $u > 0$, $u \geq |v|$. So for $t < 0$, $e^{3tu} < 1$. Also $2\cos(\sqrt{3}vt + \frac{2}{3}\pi)$ assumes value -1 for some $t < 0$ and thus $\boldsymbol{b}(t) < 0$. Similar conclusions hold for $\boldsymbol{c}(t)$. So we consider only the case $t \geq 0$. As already noted in (v) $\boldsymbol{a}(0) = 1$, $\boldsymbol{b}(0) = 0$, $\boldsymbol{c}(0) = 0$.

(a) In case $b = c$, i.e., $v = 0$, we immediately have for $t > 0$,

$$\boldsymbol{a}(t) = \frac{1}{3}e^{t(a-u)}\left[e^{3ut} + 2\right] > 0,$$
$$\boldsymbol{b}(t) = \frac{1}{3}e^{t(a-u)}\left[e^{3ut} - 1\right] > 0,$$
$$\boldsymbol{c}(t) = \frac{1}{3}e^{t(a-u)}\left[e^{3ut} - 1\right] = \boldsymbol{b}(t) > 0.$$

For the general case some computations are needed.

(b) Let $\alpha = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$. Set $f_\alpha(t) = e^{3ut} + 2\cos(\sqrt{3}vt + \alpha)$, $t \in \mathbb{R}$. Then f_α is infinitely differentiable and $f_\alpha(0) = 1 + 2\cos\alpha \geq 0$. Further, for $t \in \mathbb{R}$, $f'_\alpha(t) = 3ue^{3ut} - 2\sqrt{3}v\sin(\sqrt{3}vt + \alpha)$. Therefore, for $t \in \mathbb{R}$, $f''_\alpha(t) = (3u)^2e^{3ut} - 2(\sqrt{3}v)^2\cos(\sqrt{3}vt + \alpha) \geq 9u^2e^{3ut} - 6v^2 = 9u^2(e^{3ut} - 1) + (9u^2 - 6v^2)$. So for $t > 0$, $f''_\alpha(t) > 0$.
As a consequence f'_α is strictly increasing on $[0, \infty)$. Now $f'_\alpha(0) = 3u - 2\sqrt{3}v\sin\alpha$, which is $3u$, $3u - 3v$, $3u + 3v$ respectively for $\alpha = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$ respectively i.e. $3u$, $3c$, $3b$ respectively. But $3u$, $3b$, $3c$ are all > 0. So $f'_\alpha(t) > 0$ for $t > 0$. Therefore, f_α is strictly increasing on $[0, \infty)$. Consequently $f_\alpha(t) > 0$ for $t > 0$ and hence $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$ are all > 0 for $t > 0$.

(c) Now $\boldsymbol{a}(t) \geq \boldsymbol{d}(t)$ if and only if $\frac{1}{3}e^{t(a-u)}\left[e^{3ut} + 2\cos(\sqrt{3}vt)\right] \geq e^{td}$ if and only if $e^{-ut}\left[e^{3ut} + 2\cos(\sqrt{3}vt)\right] \geq 3e^{t(d-a)}$. Set $w = a - d$. The condition $\boldsymbol{a}(t) \geq \boldsymbol{d}(t)$ is equivalent to

$$e^{-ut}\left[e^{3ut} + 2\cos(v\sqrt{3}t)\right] \geq 3e^{-wt}.$$

Let $g(t) = e^{2ut} + 2e^{-ut}\cos(v\sqrt{3}t) - 3e^{-wt}$, $t \in \mathbb{R}$. Then g is infinitely differ-

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entiable and $g(0) = 0$. Also for $t \in \mathbb{R}$,

$$\begin{aligned}
g'(t) &= 2ue^{2ut} + 2e^{-ut}\left(-u\cos(v\sqrt{3}t) - v\sqrt{3}\sin(v\sqrt{3}t)\right) + 3we^{-wt} \\
&= 2u\left(e^{2ut} - e^{-ut}\cos(v\sqrt{3}t)\right) - 2v\sqrt{3}e^{-ut}\sin(v\sqrt{3}t) + 3we^{-wt} \\
&= 2u\left[(e^{2ut} - e^{-ut}) + 2e^{-ut}\sin^2(\frac{v\sqrt{3}}{2}t)\right] - 2v\sqrt{3}e^{-ut}\sin(v\sqrt{3}t) + 3we^{-wt}.
\end{aligned}$$

In particular, $g'(0) = 3w$. So if $g(t_n) \geq 0$ for a sequence (t_n) in $(0, \infty)$ with t_n convergent to 0 then $g'(0) \geq 0$, i.e., $w \geq 0$. Now assume $w \geq 0$. Then for $t > 0$, using $|\sin t| \leq |t|$ for all t,

$$\begin{aligned}
g'(t) &\geq 2u(e^{2ut} - e^{-ut}) - 2v\sqrt{3}e^{-ut}(v\sqrt{3}t) \\
&= 2e^{-ut}\left[u(e^{3ut} - 1) - 3v^2t\right].
\end{aligned}$$

Let $h(t) = u(e^{3ut} - 1) - 3v^2t$, $t \in \mathbb{R}$. Then h is infinitely differentiable and $h(0) = 0$. Also for $t > 0$,

$$\begin{aligned}
h'(t) &= u.3ue^{3ut} - 3v^2 \\
&= 3u^2(e^{3ut} - 1) + 3(u^2 - v^2) > 0\,.
\end{aligned}$$

So $h(t) > 0$ for $t > 0$. As a consequence, $g'(t) > 0$ for $t > 0$. This gives $g(t) > 0$ for all $t > 0$. Thus $g(t_n) \geq 0$ for a sequence (t_n) in $(0, \infty)$ with t_n convergent to zero if and only if $w \geq 0$ if and only if $g(t) > 0$ for all $t > 0$. Hence $\boldsymbol{\rho}(t_n)$ are all completely positive maps for a sequence (t_n) in $(0, \infty)$ convergent to 0 if and only if $w \geq 0$ if and only if $\boldsymbol{\rho}(t)$ are all completely positive maps for all $t > 0$.

(d) Moreover, Remark 2.8 then gives that for $a \geq d$, $t > 0$, the Choi matrix $C_{\boldsymbol{\rho}(t)}$ has rank 9. Consider this case together with $a + b + c = 0$, i.e., $a = -2u$. We refer to Theorem 2.6((ii)(e) and Theorem 2.7(ii)(e) with a, b, c, d there replaced by $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, $\boldsymbol{d}(t)$ respectively. The inequalities $\boldsymbol{d}(t) \leq \boldsymbol{a}(t) < \boldsymbol{a}(t) + 2(\boldsymbol{b}(t) + \boldsymbol{c}(t)) < 2\boldsymbol{d}(t)$ are satisfied for t in the non-empty interval $[0, -(\log 3 - \log 2)/d]$, for sure. So, for all such t, in view of (vii)(b) above each $\boldsymbol{\rho}(t)$ is a quantum channel that is not partially entanglement breaking and is non-PPT as well.

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(e) Suppose $w \geq 0$; then

$$
\begin{aligned}
h(t) &= \boldsymbol{b}(t)\boldsymbol{c}(t) - \boldsymbol{d}(t)^2 \\
&= \frac{1}{9}e^{2t(a-u)}\left[(e^{3ut} - \cos(\sqrt{3}vt))^2 - 3\sin^2(\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2t(a-u)}\left[e^{6ut} - 2e^{3ut}\cos(\sqrt{3}vt) + \cos^2(\sqrt{3}vt) - 3\sin^2(\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2t(a-u)}\left[e^{6ut} - 2e^{3ut}\cos(\sqrt{3}vt) - 1 + 2\cos(2\sqrt{3}vt)\right] - e^{2dt} \\
&= \frac{1}{9}e^{2ta}\left[e^{4ut} - 2e^{ut}\cos(\sqrt{3}vt) - e^{-2ut} + 2e^{-2ut}\cos(2\sqrt{3}vt) - 9e^{-2wt}\right] \\
&= \frac{1}{9}e^{2ta}\, g(t),
\end{aligned}
$$

where $g(t) = e^{4ut} - e^{-2ut} - 9e^{-2wt} - 2e^{ut}\cos(\sqrt{3}vt) + 2e^{-2ut}\cos(2\sqrt{3}vt), t \in \mathbb{R}$. We note that g is infinitely differentiable on $\mathbb{R}$ and $g(0) = -9$.
Since $u > 0$, $g(t) \to \infty$ as $t \to \infty$. So there is an $s_0 \in (0, \infty)$ satisfying $g(t) > 0$ for $t > s_0$. This, in turn, gives that $h(t) > 0$ for $t > s_0$. By 2.H(iii)(c) $\boldsymbol{\rho}(t)$ is PPT for $t > s_0$. As $T_0 = Id$ is not PPT, an application of the Trichotomy result as envisaged in Remark 5.5(ii) (a) immediately gives that there exists a unique $t_0 \in (0, \infty)$ such that for $0 \leq t < t_0$, $\boldsymbol{\rho}(t)$ is not PPT but for $t \geq t_0$, $\boldsymbol{\rho}(t)$ is PPT. This, in view of 2.H(iii)(c), entails that there exists a $t_0 \in (0, \infty)$ satisfying, $h(t) < 0$ for $0 \leq t < t_0$ and $h(t) \geq 0$ for $t \geq t_0$.
We now proceed to refine this observation.

(f) For $t \in \mathbb{R}$,

$$
\begin{aligned}
g'(t) = 4ue^{4ut} &+ 2ue^{-2ut} + 18we^{-2wt} - 2e^{ut}\left(u\cos(\sqrt{3}vt) - \sqrt{3}v\sin(\sqrt{3}vt)\right) \\
&+ 2e^{-2ut}\left(-2u\cos(2\sqrt{3}vt) - 2\sqrt{3}v\sin(2\sqrt{3}vt)\right) \\
= 2u&\left[2e^{4ut} + e^{-2ut} - e^{ut} - 2e^{-2ut}\right] + 18we^{-2wt} \\
&+ 4ue^{ut}\sin^2(\frac{\sqrt{3}}{2}vt) + 2\sqrt{3}ve^{ut}\sin(\sqrt{3}vt) \\
&+ 8ue^{-2ut}\sin^2(\sqrt{3}vt) - 4\sqrt{3}ve^{-2ut}\sin(2\sqrt{3}vt).
\end{aligned}
$$

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We note that $g'(0) = 18w \geq 0$. Now for $t \geq 0$,

$$\begin{aligned}
g'(t) &\geq 2u\left[2e^{4ut} - e^{ut} - e^{-2ut}\right] - 2\sqrt{3}ve^{ut}(\sqrt{3}vt) - 4\sqrt{3}ve^{-2ut}(2\sqrt{3}vt)\\
&= 2\left[u(2e^{4ut} - e^{ut} - e^{-2ut}) - 3v^2e^{ut}t - 4\times 3v^2e^{-2ut}t\right]\\
&= 2e^{-2ut}\left[u(2e^{6ut} - e^{3ut} - 1) - 3v^2e^{3ut}t - 12v^2t\right]\\
&= 2e^{-2ut}\left[u(e^{3ut} - 1)e^{3ut} + u(e^{6ut} - 1) - 3v^2te^{3ut} - 12v^2t\right]\\
&\geq 2e^{-2ut}\left[3u^2te^{3ut} + 6u^2t - 3v^2te^{3ut} - 12v^2t\right]\\
&= 2e^{-2ut}\left[\frac{3}{2}u^2te^{3ut} + (u^2 - 2v^2)\left(\frac{3}{2}te^{3ut} + 6t\right)\right].
\end{aligned}$$

Because $u > 0$, we have for $t > 0$, $g'(t) > 0$ in case $u^2 \geq 2v^2$.

One can obtain $g'(t) > 0$ for $t > 0$ for less restricted cases but we prefer to confine our attention to this simple case and go on with the case $u \geq \sqrt{2}|v|$. Then g is strictly increasing on $[0, \infty)$. So there exists a unique $t_0 \in (0, \infty)$ such that $g(t_0) = 0$, $g(t) < 0$ for $0 \leq t < t_0$ and $g(t) > 0$ for $t_0 < t < \infty$. As a consequence, there exists a unique $t_0 \in (0, \infty)$ such that $h(t_0) = 0$, $h(t) < 0$ for $0 \leq t < t_0$ and $h(t) > 0$ for $t_0 < t < \infty$. So by 2.H(iii)(c), $\boldsymbol{\rho}(t)$ is not completely co-positive for $t < t_0$ but is completely co-positive for $t \geq t_0$.

(g) Hence for $(b, c) \neq (0, 0)$, $a \geq d$, $b + c \geq \sqrt{2}|b - c|$, there exists a unique $t_0 \in (0, \infty)$ that satifies
(α) for $0 \leq t < t_0$, $\boldsymbol{\rho}(t)$ is not PPT, and
(β) for $t \geq t_0$, $\boldsymbol{\rho}(t)$ is PPT.

This illustrates the condition (ii) of Trichotomy in Remark 5.5(ii)(a) in a concrete manner.

(xii) Let $\boldsymbol{\tau}(t) = e^{t\tau[a,b,c,d]}$, $t \geq 0$. Then $\boldsymbol{\tau}(t) = D(\boldsymbol{a}(t), \boldsymbol{b}(t), \boldsymbol{c}(t)) \oplus (\cosh(td)I_{F_n} + \sinh(td)\tau_{F_n})$. Its Choi matrix in expanded form is

$$C_{\boldsymbol{\tau}(t)} = \begin{bmatrix}
\boldsymbol{a}(t) & 0 & 0 & 0 & \cosh(td) & 0 & 0 & 0 & \cosh(td)\\
0 & \boldsymbol{c}(t) & 0 & \sinh(td) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \boldsymbol{b}(t) & 0 & 0 & 0 & \sinh(td) & 0 & 0\\
0 & \sinh(td) & 0 & \boldsymbol{b}(t) & 0 & 0 & 0 & 0 & 0\\
\cosh(td) & 0 & 0 & 0 & \boldsymbol{a}(t) & 0 & 0 & 0 & \cosh(td)\\
0 & 0 & 0 & 0 & 0 & \boldsymbol{c}(t) & 0 & \sinh(td) & 0\\
0 & 0 & \sinh(td) & 0 & 0 & 0 & \boldsymbol{c}(t) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \sinh(td) & 0 & \boldsymbol{b}(t) & 0\\
\cosh(td) & 0 & 0 & 0 & \cosh(td) & 0 & 0 & 0 & \boldsymbol{a}(t)
\end{bmatrix}.$$

It has trace $\mu(t) = 3\,(\boldsymbol{a}(t)+\boldsymbol{b}(t)+\boldsymbol{c}(t)) > 0$. Further, for $t > 0$, $C_{\boldsymbol{\tau}(t)}$ is a positive matrix if and only if $\boldsymbol{a}(t) \geq \cosh(td)$, $\boldsymbol{b}(t) \geq 0$, $\boldsymbol{c}(t) \geq 0$ and $\boldsymbol{b}(t)\boldsymbol{c}(t) \geq \sinh^2(td)$. Computations of the type done in this example give that this happens for all $t \geq 0$ if $\frac{2}{3}b = \frac{2}{3}c \geq a \geq |d|$; and, in fact, for less restricted cases as well.

(xiii) **An interesting event.** Around the time of acceptance of the paper, the author visited The Institute of Quantum Computation at Waterloo University, Canada

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from October 25 to 30, 2015. During a discussion, Vern Paulsen told her about the PPT Square conjecture of Matthias Christandl which can be found as (Problem G). In Banff International Research Station workshop: Operator structures in quantum information theory (2012). Available at http://www.birs.ca/workshops/2012/12w5084/report12w5084.pdf. She thought her Example 5.7 would settle the conjecture in the negative. She got in touch with Matthias Christandl and an intense discussion took place. This led to addition of (ii)(e) in Theorem 2.6, deletion of Schmidt number part in earlier Remark 2.9 and a consequent change in part (xi)(d) and (xii) above. It turned out that the example does not affect the conjecture so far.

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Appendix.

Happy 60th Birthday Ravindra B. Bapat

Spreading his special knowledge rays,
bright and colourful in a line,
And concentrating on inherent arrays,
creates Linear models of many a kind.
Looking so sober and systematic,
but goes for a random walk on a tree.
Makes me wonder on a change like this,
but soon he comes back with a glee.
Hands full of Laplacian rays,
beautiful fans with a glow,
Wheels moving with different speeds,
displaying wonderful life flow!
Felicitations to you, Ravindra Bapat,
best wishes for laurels many more,
Happy long life to you and your family,
and each one on ICLAA-14 floor!

Abstract of the actual expository talk at the International Conference on Linear Algebra and its Applications-2014 at Manipal University, Mangalore, India.

The passage from two to three and three to infinity
in classic to quantum channels.

The situation changes drastically for matrices, maps on matrix algebras and applications to Quantum Information theory when we go from order two to three or three to infinity. Examples include maximally entangled bases and Quantum Birkhoff Theory.