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A. Amparan
Universidad del Pais Vasco, agurtzane.amparan@ehu.es

S. Marcaida
Universidad del Pais Vasco, silvia.marcaida@ehu.es

Ion Zaballa
Universidad del País Vasco, ion.zaballa@ehu.es

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FINITE AND INFINITE STRUCTURES OF RATIONAL MATRICES: A LOCAL APPROACH*

A. AMPARAN†, S. MARCAIDA†, AND I. ZABALLA†

Abstract. The structure of a rational matrix is given by its Smith-McMillan invariants. Some properties of the Smith-McMillan invariants of rational matrices with elements in different principal ideal domains are presented: In the ring of polynomials in one indeterminate (global structure), in the local ring at an irreducible polynomial (local structure), and in the ring of proper rational functions (infinite structure). Furthermore, the change of the finite (global and local) and infinite structures is studied when performing a Möbius transformation on a rational matrix. The results are applied to define an equivalence relation in the set of polynomial matrices, with no restriction on size, for which a complete system of invariants are the finite and infinite elementary divisors.

Key words. Rational matrices, Polynomial matrices, Smith-McMillan form, Finite and infinite structures, Localization, Möbius transformations.

AMS subject classifications. 15A54, 47A56, 15A21, 37P05.

1. Introduction. By a matrix polynomial, we will understand a polynomial in one indeterminate, s, whose coefficients are matrices of the same size with entries in an arbitrary field $\mathbb{F}$. We will identify matrix polynomials and polynomial matrices and we will write $\mathbb{F}[s]^{m\times n}$ to denote the set of matrix polynomials whose coefficients are of size $m\times n$. We will say that $m\times n$ is the size of the matrix polynomials in $\mathbb{F}[s]^{m\times n}$.

There are many equivalence relations defined on sets of matrix polynomials. The primitive one is usually attributed to Smith (H.J.S. Smith, 1848–1928) who obtained the famous canonical form named after him when solving systems of integral linear equations [23]. This equivalence relation, called *equivalence of matrix polynomials* or *unimodular equivalence* (or *Smith equivalence of matrix polynomials* to be distinguished from other equivalences that will be introduced later on) is defined in any set of matrix polynomials of the same size. A complete system of invariants is given by the so-called *invariant factors* that are polynomials and they are also called *invariant polynomials*. When a prime factorization of the invariant factors is computed, the

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†Departamento de Matemática Aplicada y EIO, UPV/EHU, Apdo. Correos 644, Bilbao 48080, Spain (agurtzane.amparan@ehu.es, silvia.marcaida@ehu.es, ion.zaballa@ehu.es). Supported by Dirección General de Investigación, Proyecto de Investigación MTM2013-40960-P, Gobierno Vasco GIC13/IT-710-13 and UPV/EHU UFI11/52.

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obtained powers of irreducible polynomials are called the *elementary divisors* of the matrix polynomial. It is plain that the elementary divisors depend on the field in the sense that if $\mathbb{F}$ is a subfield of $\mathbb{L}$ then the invariant factors may have different prime factorizations over $\mathbb{F}$ and over $\mathbb{L}$.

The Smith equivalence of matrix polynomials can be generalized in two directions. First, for matrices defined on different or more general rings (actually Smith himself worked on matrices with integral entries). In fact, Smith's procedure can be slightly modified to be applied to any matrix over a Principal Ideal Domain (PID) ([17, 18], for example) or, more general, over Elementary Divisor Domains ([7, Th. 1.12.1]).

The Smith equivalence can be generalized also to include matrix polynomials of different sizes. For example, in [20] the authors define the *extended unimodular equivalence* motivated by its implications in the Theory of Linear Control Systems. This is an equivalence relation defined in the set $\mathcal{P}(m,l)$ of $(m+r)\times(l+r)$ matrix polynomials over $\mathbb{F}[s]$ where m and l are fixed integers and r ranges over all integers which are greater than $\max(-m,-l)$. It turns out that two matrix polynomials in $\mathcal{P}(m,l)$ are extended unimodular equivalent if and only if they have the same elementary divisors.

The invariant factors (and the elementary divisors) constitute the *finite structure* of the corresponding matrix polynomial. This terminology is motivated by the fact that when $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ the roots (in $\mathbb{C}$) of the invariant factors are the points of the complex plane where the rank of the matrix polynomial decreases with respect to its normal rank. Thus, the Smith equivalence is perfectly suited to classify the matrix polynomials according to their finite structure. However, two equivalent matrices may have different structure at infinity (see Section 5 for the definition). It turns out that the infinite structure is also important in Control Theory ([2, 3, 6, 10]) and the question of providing an equivalence relation in closing form (i.e., using elementary operations) that classifies the matrix polynomials of possibly different sizes according to their finite and infinite elementary divisors arose very soon. There have been several contributions in this respect ([5, 11, 14, 15, 21, 25]), each one with its own equivalence relation that preserves the finite and/or the infinite elementary divisors. Specifically, the *divisor equivalence* defined in [15] keeps invariant the finite and infinite elementary divisors of all matrices in $\mathcal{P}(m,l)$. In addition, if we define

$$\mathbb{F}_c[s]=\{A(s)\in\mathbb{F}[s]^{n\times n}: c=n\deg(A), n\geq 2\},$$

where $\deg(A)$ is the degree of A (i.e., the degree of the polynomial of greatest degree in $A(s)$), it turns out that two matrices in $\mathbb{F}_c[s]$ are divisor equivalent if and only if they have the same finite and infinite elementary divisors. Finally, in [5] a simple characterization of the extended unimodular equivalence, already implicit in [20], is provided and a new equivalence, called *spectral equivalence*, is introduced and its inter-

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est in Numerical Linear Algebra revealed (specifically, in the study of linearizations or quadrifications of matrix polynomials). Namely, let $A(s), B(s) \in \mathcal{P}(m, l)$ and assume that the sizes of $A(s)$ and $B(s)$ are $(m + r_1) \times (l + r_1)$ and $(m + r_2) \times (l + r_2)$, respectively. Then they are extended unimodular equivalent if and only if $\operatorname{Diag}(A(s), I_a)$ and $\operatorname{Diag}(B(s), I_b)$ are Smith equivalent for any a and b such that $m + r_1 + a = m + r_2 + b$ and $l + r_1 + a = l + r_2 + b$. And $A(s), B(s) \in \mathcal{P}(m, l)$ are spectrally equivalent if $A(s)$ and $B(s)$ on the one hand, and $\operatorname{rev} A(s)$, $\operatorname{rev} B(s)$ on the other hand, are extended unimodular equivalent, where $\operatorname{rev} A(s) = s^{\deg(A)} A(\frac{1}{s})$ and $\operatorname{rev} B(s) = s^{\deg(B)} B(\frac{1}{s})$.

The presence of $\operatorname{rev} A(s)$ is quite natural because, by definition, the structure at infinity of a matrix polynomial $A(s)$ is the structure at 0 of $A(\frac{1}{s})$ (see for example [6, 10] and Section 5). This fact reveals the convenience of studying locally the structure of matrix polynomials and the interest of knowing how the elementary divisors of a matrix polynomial change when submitted to a Möbius transformation. No one of these two problems is new. For example, the $\{s_0\}$-equivalence, defined in [15] following [13], can be used to study the elementary divisors which are powers of $(s - s_0)$. The drawback of this definition is that it is not given in terms of "elementary operations". Theorem A.3.4 in [9] is another example, closer to the ideas to be developed here, where the local structure is considered. On the other hand, the change of the finite and infinite elementary divisors by Möbius transformations on a matrix polynomial has been studied in [16, 27] using a "global" approach in contrast with the local approach to be used here, which allows to extend the study to matrices of rational functions.

The goal of this paper is to formalize the notion of local structure (including the structure at infinity) of rational function matrices. This is done by defining appropriate local rings that are special instances of principal ideal domains. Then, this information is used to study the change of the local invariants when a Möbius transformation is performed on a matrix of rational functions. The key point is that invertible matrices in a given local ring are transformed into invertible matrices of the "transformed" local ring by the Möbius transformation. Finally, the obtained results are applied to define explicitly an equivalence relation in the set of polynomial matrices (of any size, rank and degree) that preserves the finite and infinite elementary divisors.

The paper is structured as follows. In the next section, the Smith canonical form of a matrix with entries from a PID $\mathcal{D}$ is recalled, and it is used to produce the Smith-McMillan form in $\mathcal{D}$ of a matrix with elements in the field of fractions of $\mathcal{D}$. In Sections 3–5, we briefly study the Smith-McMillan forms of a rational matrix in three different principal ideal domains: When the principal ideal domain is the ring of polynomials in one indeterminate, we obtain the global structure of a matrix (Section 3); if the principal ideal domain is a local ring at an irreducible polynomial then we get the local

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structure at that polynomial (Section 4) and if the principal ideal domain is the ring of proper rational functions then the infinite structure of a matrix is obtained (Section 5). This part is finished with a result that stresses the important fact that the usual equivalence of rational matrices (including matrix polynomials) is, in essence, a local property (see [4]). In Section 6, the previous results are used to investigate how the finite (global and local) and infinite structures change when a Möbius transformation is performed on any given rational matrix. Finally, in Section 7, an equivalence is introduced that preserves the finite and infinite elementary divisors of any matrix polynomial. The idea is to appropriately augment the matrices preserving the finite and infinite elementary divisors, to transfer the infinite structure to a finite point by means of a convenient Möbius transformation and to require the two matrices to be globally equivalent.

2. Smith-McMillan form in a principal ideal domain. Let $\mathcal{D}$ be a principal ideal domain. Recall that a principal ideal domain is a commutative ring with no zero divisors such that every ideal is principal, that is, such that every ideal consists of all the multiples of a fixed element. Let $\mathcal{D}^{m\times n}$ be the set of $m\times n$ matrices with entries in $\mathcal{D}$ and $\mathrm{Gl}_m(\mathcal{D})$ the group of units of $\mathcal{D}^{m\times m}$, i.e., the invertible matrices in $\mathcal{D}^{m\times m}$. Using the formula $U^{-1} = \operatorname{Adj} U/\det U$ where $\operatorname{Adj} U$ is the adjugate (transposed matrix of the cofactors) of U, we have that a matrix in $\mathcal{D}^{m\times m}$ is invertible if and only if its determinant is a unit of $\mathcal{D}$.

Two matrices $D_1, D_2 \in \mathcal{D}^{m\times n}$ are *equivalent in* $\mathcal{D}$ if there exist both $U \in \mathrm{Gl}_m(\mathcal{D})$ and $V \in \mathrm{Gl}_n(\mathcal{D})$ so that $D_2 = UD_1V$. This is an equivalence relation. There exists a canonical form for this relation, which is called the Smith normal form (see [17, 18, 26]).

Theorem 2.1 (Smith normal form in $\mathcal{D}$). *Every matrix $D \in \mathcal{D}^{m\times n}$ is equivalent in $\mathcal{D}$ to a matrix*

$$\begin{bmatrix} \mathrm{Diag}(\gamma_1,\ldots,\gamma_r) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank} D$, $\gamma_1,\ldots,\gamma_r$ are nonzero elements of $\mathcal{D}$ (unique up to multiplication by units of $\mathcal{D}$) and $\gamma_1 \mid \cdots \mid \gamma_r$, where $\mid$ stands for divisibility.

The elements $\gamma_1,\ldots,\gamma_r$ are uniquely determined (up to multiplication by units) by D and are called the *invariant factors in* $\mathcal{D}$ of D.

Remark 2.2. It is known that the product $\gamma_1\cdots\gamma_k$ is the kth determinantal divisor of D, that is, the greatest common divisor (unique up to a product by units) of all the minors of order k of D.

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Let $\mathcal{F}$ be the field of fractions of $\mathcal{D}$, that is, $\mathcal{F} = \left\{\frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{D}, \beta \neq 0\right\}$. Let $\mathcal{F}^{m\times n}$ be the set of $m \times n$ matrices with entries in $\mathcal{F}$. Two matrices $F_1, F_2 \in \mathcal{F}^{m\times n}$ are said to be *equivalent in* $\mathcal{D}$ if there exist $U \in \mathrm{Gl}_m(\mathcal{D})$ and $V \in \mathrm{Gl}_n(\mathcal{D})$ such that $F_2 = UF_1V$. This is an equivalence relation and there is a canonical form in $\mathcal{F}^{m\times n}$. To see this we proceed as in [22] or [12]. Let $F \in \mathcal{F}^{m\times n}$ and its entries $f_{ij} = \frac{\alpha_{ij}}{\beta_{ij}}$, $\alpha_{ij}, \beta_{ij} \in \mathcal{D}$, $\beta_{ij} \neq 0$. Let β be the least common multiple (unique up to multiplication by units) of β_{ij}. Write $D = \beta F$. Notice that $D \in \mathcal{D}^{m\times n}$. Bring D to its Smith normal form

$$UDV = \begin{bmatrix} \mathrm{Diag}(\gamma_1, \dots, \gamma_r) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $U \in \mathrm{Gl}_m(\mathcal{D})$, $V \in \mathrm{Gl}_n(\mathcal{D})$, $r = \operatorname{rank} D$, $\gamma_1, \dots, \gamma_r$ are nonzero elements of $\mathcal{D}$ and $\gamma_1 \mid \cdots \mid \gamma_r$. Multiply both sides of (2.1) by $\frac{1}{\beta}$ and cancel the common factors so that $\frac{\gamma_i}{\beta} = \frac{\epsilon_i}{\psi_i}$ and $\gcd(\epsilon_i, \psi_i) =$ unit. We obtain

$$UFV = \begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1}{\psi_1}, \dots, \frac{\epsilon_r}{\psi_r}\right) & 0 \\ 0 & 0 \end{bmatrix}.$$

We have proved

Theorem 2.3 (Smith-McMillan form in $\mathcal{D}$). *Every matrix $F \in \mathcal{F}^{m\times n}$ is equivalent in $\mathcal{D}$ to a matrix*

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1}{\psi_1}, \dots, \frac{\epsilon_r}{\psi_r}\right) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank} F$, $\epsilon_1, \dots, \epsilon_r, \psi_1, \dots, \psi_r$ are nonzero elements of $\mathcal{D}$ (unique up to multiplication by units), ϵ_i, ψ_i are coprime for all $i = 1, \dots, r$, and $\epsilon_1 \mid \cdots \mid \epsilon_r$ while $\psi_r \mid \cdots \mid \psi_1$.

The irreducible fractions $\frac{\epsilon_1}{\psi_1}, \dots, \frac{\epsilon_r}{\psi_r}$ are uniquely determined (up to multiplication by units) by F and will be called the *invariant fractions in* $\mathcal{D}$ of F.

Remark 2.4. It is important to notice that $\psi_1 = \beta$ (up to multiplication by a unit). Otherwise, any element of D would be divisible by a factor of β and, therefore, β would not be a least common denominator of F.

We will say that the invariant fractions of a matrix $F \in \mathcal{F}^{m\times n}$ in $\mathcal{D}$ provide us with the *structure* of F in that domain. Since the same field $\mathcal{F}$ may be the field of fractions of different principal ideal domains, the same matrix may have different structures according to the principal ideal domain where the equivalence relation is defined. This is the case of the field of rational functions as we will see in the next sections.

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3. Global structure. Let $\mathbb{F}$ be any arbitrary field and $\mathbb{F}[s]$ the ring of polynomials over $\mathbb{F}$. Its units are the nonzero constants and it can be characterized as

$$\mathbb{F}[s] = \{cp(s) : c \text{ is a nonzero constant and } p(s) \text{ is a monic polynomial}\} \cup \{0\}.$$

A monic polynomial is a polynomial whose leading coefficient is 1.

It is well known that $\mathbb{F}[s]$ is a principal ideal domain and that its field of fractions is $\mathbb{F}(s)$, the field of rational functions. Let $\mathbb{F}[s]^{m\times n}$ be the set of $m \times n$ polynomial matrices and $\mathrm{Gl}_m(\mathbb{F}[s])$ the set of unimodular matrices, which are the invertible matrices in $\mathbb{F}[s]^{m\times m}$.

Two polynomial matrices $P_1(s), P_2(s) \in \mathbb{F}[s]^{m\times n}$ are *globally* or *finite equivalent* if they are *equivalent in* $\mathbb{F}[s]$, that is, if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ such that $P_2(s) = U(s)P_1(s)V(s)$.

From Theorem 2.1 we deduce that every matrix $P(s) \in \mathbb{F}[s]^{m\times n}$ is globally or finite equivalent to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}(\gamma_1(s), \ldots, \gamma_r(s)) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank} P(s)$ and $\gamma_1(s), \ldots, \gamma_r(s)$ are nonzero monic polynomials such that $\gamma_1(s) \mid \cdots \mid \gamma_r(s)$.

The polynomials $\gamma_1(s), \ldots, \gamma_r(s)$ are uniquely determined by $P(s)$ and are called the *global* or *finite invariant factors* of $P(s)$ or the *invariant factors in* $\mathbb{F}[s]$ of $P(s)$. If they are decomposed as products of powers of monic irreducible factors over $\mathbb{F}$ (prime decomposition):

$$\begin{aligned}
&\gamma_1(s) = \pi_1(s)^{d_{11}} \cdots \pi_t(s)^{d_{1t}}, \\
&\gamma_2(s) = \pi_1(s)^{d_{21}} \cdots \pi_t(s)^{d_{2t}}, \\
&\vdots \\
&\gamma_r(s) = \pi_1(s)^{d_{r1}} \cdots \pi_t(s)^{d_{rt}},
\end{aligned}$$

where $d_{rj} \geq \cdots \geq d_{1j} \geq 0$, $j = 1, \ldots t$, then the powers $\pi_j(s)^{d_{ij}}$ with $d_{ij} > 0$ are the *finite elementary divisors* of $P(s)$ in $\mathbb{F}[s]$.

The roots of the finite invariant factors (or of the finite elementary divisors) in the algebraic closure of $\mathbb{F}$ are the *finite zeros* of $P(s)$.

We study now the global or finite structure of rational matrices. Let $\mathbb{F}(s)^{m\times n}$ be the set of $m \times n$ rational matrices. Two matrices $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are *globally* or *finite equivalent* if they are *equivalent in* $\mathbb{F}[s]$, that is, if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ such that $R_2(s) = U(s)R_1(s)V(s)$.

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From Theorem 2.3 (see also [22] or [24]) we deduce that every matrix $R(s) \in \mathbb{F}(s)^{m\times n}$ is globally or finite equivalent to a matrix of the form

$$\begin{bmatrix} \operatorname{Diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)},\dots,\frac{\epsilon_r(s)}{\psi_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank} R(s)$, $\epsilon_1(s),\dots,\epsilon_r(s),\psi_1(s),\dots,\psi_r(s)$ are nonzero monic polynomials, $\epsilon_i(s),\psi_i(s)$ are coprime for all $i = 1,\dots,r$, and $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ while $\psi_r(s) \mid \cdots \mid \psi_1(s)$.

The rational functions $\frac{\epsilon_1(s)}{\psi_1(s)},\dots,\frac{\epsilon_r(s)}{\psi_r(s)}$ form a complete set of invariants for the equivalence in $\mathbb{F}[s]$ and are called the *global* or *finite invariant rational functions* of $R(s)$ or the *invariant rational functions in* $\mathbb{F}[s]$ of $R(s)$. The roots of $\epsilon_1(s),\dots,\epsilon_r(s)$ (respectively, $\psi_1(s),\dots,\psi_r(s)$) in the algebraic closure of $\mathbb{F}$ are the *finite zeros* (*finite poles*, respectively) of $R(s)$.

Recall (see Remark 2.4) that $\psi_1(s)$ is the monic least common denominator of the entries in $R(s)$. Hence, $R(s)$ is a polynomial matrix if and only if $\psi_1(s) = 1$; i.e., if and only if it does not have finite poles.

Example 3.1. Let

$$R(s) = \begin{bmatrix} \dfrac{s+1}{s} & -\dfrac{s}{s+1} \\ 1 & \dfrac{1}{(s+1)^2} \end{bmatrix},$$

and let us compute its finite zero and pole structure in $\mathbb{R}$, the field of real numbers. Notice that the monic least common denominator is $s(s+1)^2$ and that the global or finite Smith normal form of

$$P(s) = s(s+1)^2 R(s) = \begin{bmatrix} (s+1)^3 & -s^2(s+1) \\ s(s+1)^2 & s \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 \\ 0 & s(s+1)^3(s^2+1) \end{bmatrix}.$$

In fact, $\gcd((s+1)^3, -s^2(s+1), s(s+1)^2, s) = 1$ and $\det P(s) = s(s+1)^3(s^2+1)$ and so, the finite invariant factors of $P(s)$ are $1, s(s+1)^3(s^2+1)$ and its finite elementary divisors in $\mathbb{R}$ are $s, (s+1)^3, s^2+1$. As a consequence, the finite zeros of $P(s)$ are 0, -1 of multiplicity 3, i and $-i$. Also, the finite invariant rational functions of $R(s)$ are $\frac{1}{s(s+1)^2}, (s+1)(s^2+1)$. Therefore, the finite zeros of $R(s)$ are -1, i and $-i$ while its finite poles are 0 and -1 of multiplicity 2.

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4. Local structure. The following concepts are taken from [4, 26]. A ring with exactly one maximal ideal is called a local ring. If $\mathcal{R}$ is a ring and there exists an ideal $I \neq \mathcal{R}$ such that every $r \in \mathcal{R} \setminus I$ is a unit of $\mathcal{R}$, then $\mathcal{R}$ is a local ring and I its maximal ideal. Let P be a prime ideal of $\mathcal{R}$. Then $A = \mathcal{R} \setminus P$ is a multiplicatively closed set. Construct the quotient ring $A^{-1}\mathcal{R} = \left\{\frac{r}{a} : r \in \mathcal{R}, a \in A\right\}$. Since every element in $A^{-1}\mathcal{R} \setminus A^{-1}P$ is a unit of $A^{-1}\mathcal{R}$, $A^{-1}\mathcal{R}$ is a local ring. It is denoted by $\mathcal{R}_P$.

Notice that the prime ideals of $\mathbb{F}[s]$ are generated by non-constant irreducible polynomials. Let $\pi(s) \in \mathbb{F}[s]$ be such a polynomial. Define $A = \mathbb{F}[s] \setminus (\pi(s))$, where $(\pi(s))$ is the ideal generated by $\pi(s)$. The elements of A are the polynomials relatively prime with $\pi(s)$. As above, denote $A^{-1}\mathbb{F}[s]$ by $\mathbb{F}_\pi(s)$. Thus,

$$\mathbb{F}_\pi(s) = A^{-1}\mathbb{F}[s] = \left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \gcd(q(s), \pi(s)) = \text{constant}\right\}$$

is a local ring. It is called the *local ring of* $\mathbb{F}[s]$ *at* $\pi(s)$.

The units of $\mathbb{F}_\pi(s)$ are the rational functions whose numerators and denominators are both prime with $\pi(s)$. Moreover, for any nonzero $\frac{p(s)}{q(s)} \in \mathbb{F}_\pi(s)$ we can write $p(s) = p_1(s)\pi(s)^d$ with $\gcd(p_1(s), \pi(s)) = \text{constant}$ and d a nonnegative integer. Thus, $\frac{p(s)}{q(s)} = \frac{p_1(s)}{q(s)}\pi(s)^d$ and $\frac{p_1(s)}{q(s)}$ is a unit of $\mathbb{F}_\pi(s)$. Hence, $\mathbb{F}_\pi(s)$ can be characterized as follows:

$$\mathbb{F}_\pi(s) = \left\{u_\pi(s)\pi(s)^d : u_\pi(s) \text{ a unit}, d \in \mathbb{N}\right\} \cup \{0\}.$$

Observe that if $\pi(s)$ is not monic and its leading coefficient is π_0, then $(\pi(s)) = (\frac{1}{\pi_0}\pi(s))$. Therefore, the local ring at $\pi(s)$ and the local ring at $\frac{1}{\pi_0}\pi(s)$ are the same. From now on, we fix $\pi(s)$ a non-constant monic irreducible polynomial.

We can define an Euclidean division in $\mathbb{F}_\pi(s)$: for $u_{\pi 1}(s)\pi(s)^{d_1}, u_{\pi 2}(s)\pi(s)^{d_2} \in \mathbb{F}_\pi(s)$,

$$u_{\pi 2}(s)\pi(s)^{d_2} = q(s)u_{\pi 1}(s)\pi(s)^{d_1} + r(s),$$

where

$$\begin{cases} q(s) = \frac{u_{\pi 2}(s)}{u_{\pi 1}(s)}\pi(s)^{d_2 - d_1}, r(s) = 0 & \text{if} \quad d_2 \geq d_1 \\ q(s) = 0, r(s) = u_{\pi 2}(s)\pi(s)^{d_2} & \text{if} \quad d_2 < d_1. \end{cases}$$

Thus, $\mathbb{F}_\pi(s)$ is an Euclidean domain and so a principal ideal domain. It is easily seen that its field of fractions is $\mathbb{F}(s)$.

The divisibility in $\mathbb{F}_\pi(s)$ is very easy: $u_{\pi 1}(s)\pi(s)^{d_1} \mid u_{\pi 2}(s)\pi(s)^{d_2}$ if and only if $d_1 \leq d_2$. In consequence, two elements of $\mathbb{F}_\pi(s)$ are coprime in $\mathbb{F}_\pi(s)$ if and only if

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at least one of them is a unit of $\mathbb{F}_\pi(s)$. As a convention, we will say that an element $u_\pi(s)\pi(s)^d \in \mathbb{F}_\pi(s)$ is *monic* if $u_\pi(s) = 1$.

Let $\mathbb{F}_\pi(s)^{m\times n}$ be the set of $m \times n$ matrices with entries in $\mathbb{F}_\pi(s)$ and $\mathrm{Gl}_m(\mathbb{F}_\pi(s))$ the set of invertible matrices in $\mathbb{F}_\pi(s)^{m\times m}$. Two rational matrices $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ will be said to be *locally equivalent at* $\pi(s)$ if they are *equivalent in* $\mathbb{F}_\pi(s)$, that is, if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_\pi(s))$ such that $R_2(s) = U(s)R_1(s)V(s)$. This is an equivalence relation. With all the above in mind and Theorem 2.3 we can conclude that every $R(s) \in \mathbb{F}(s)^{m\times n}$ is locally equivalent at $\pi(s)$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{1}{\pi(s)^{-h_1}}, \dots, \frac{1}{\pi(s)^{-h_k}}, 1, \dots, 1, \frac{\pi(s)^{h_u}}{1}, \dots, \frac{\pi(s)^{h_r}}{1}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} R(s)$ and $h_1 \le \cdots \le h_k < 0 = h_{k+1} = \cdots = h_{u-1} < h_u \le \cdots \le h_r$ are integers. The local Smith-McMillan form at $\pi(s)$ of $R(s)$ can be also written as

$$\begin{bmatrix} \mathrm{Diag}(\pi(s)^{h_1}, \dots, \pi(s)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix}.$$

The rational functions $\pi(s)^{h_1}, \dots, \pi(s)^{h_r}$ constitute a complete system of invariants for the local equivalence at $\pi(s)$ and are called the *local invariant rational functions at* $\pi(s)$ of $R(s)$ or the *invariant rational functions in* $\mathbb{F}_\pi(s)$ of $R(s)$. The integers $h_1, \dots, h_r$ are the *invariant orders at* $\pi(s)$ of $R(s)$. The positive integers $h_u, \dots, h_r$ are the orders of the *zeros at* $\pi(s)$ of $R(s)$ while $-h_1, \dots, -h_k$ are the orders of the *poles at* $\pi(s)$ of $R(s)$.

Recall (see Remark 2.4) that the monic least common denominator in $\mathbb{F}_\pi(s)$ of $R(s)$ is $\pi(s)^{-h_1}$ if $h_1 < 0$ or 1 otherwise. Therefore, $R(s) \in \mathbb{F}_\pi(s)^{m\times n}$ if and only if $h_1, \dots, h_r$ are nonnegative integers or, equivalently, if and only if $R(s)$ has no poles at $\pi(s)$.

We show now that for any $j = 1, \dots, r$, $h_1 + \cdots + h_j$ is the smallest integer among the powers of $\pi(s)$ of all minors of order j of $R(s)$. In fact, if $R(s) \in \mathbb{F}_\pi(s)^{m\times n}$, by Remark 2.2, $\pi(s)^{h_1+\cdots+h_j}$ is the monic greatest common divisor of all minors of order j of $R(s)$. Since the minors of $R(s)$ are elements of $\mathbb{F}_\pi(s)$, they can be written as $u_\pi(s)\pi(s)^d$. Therefore the monic greatest common divisor of all minors of order j of $R(s)$ must be of the form $\pi(s)^q$, q being the smallest exponent among the exponents of all minors of order j of $R(s)$. Thus, $\pi(s)^{h_1+\cdots+h_j} = \pi(s)^q$ and $h_1 + \cdots + h_j = q$. If $R(s)$ is not in the local ring then $h_1 < 0$ and $\pi(s)^{-h_1}$ is its monic least common denominator in $\mathbb{F}_\pi(s)$. Put $G(s) = \pi(s)^{-h_1}R(s)$. Notice that $G(s) \in \mathbb{F}_\pi(s)^{m\times n}$ and that its invariant orders at $\pi(s)$ are $h_1 - h_1, \dots, h_r - h_1$. Moreover, if $u_\pi(s)\pi(s)^d$ is a minor of order j of $R(s)$ then $u_\pi(s)\pi(s)^{d-h_1 j}$ is the corresponding minor of $G(s)$.

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Therefore $(h_1 - h_1) + \cdots + (h_j - h_1)$ is the smallest exponent among the powers of $\pi(s)$ of all minors of order j of $G(s)$, which implies that $h_1 + \cdots + h_j$ is the smallest integer among the powers of $\pi(s)$ of all minors of order j of $R(s)$.

According to [4], a property $\mathbb{P}$ of a ring $\mathcal{R}$ is local if the following holds true: $\mathcal{R}$ has $\mathbb{P}$ if and only if $\mathcal{R}_P$ has $\mathbb{P}$ for every prime ideal P of $\mathcal{R}$. We prove in the next theorem that the equivalence of rational matrices is a local property of $\mathbb{F}[s]$. For that we will use the following auxiliary lemma which is a slight extension of a very well-known result [8, p. 142].

LEMMA 4.1. *Let* $R(s) \in \mathbb{F}(s)^{m\times n}$, $\operatorname{rank} R(s) = r$. *Let* $\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}$ *be the global invariant rational functions of* $R(s)$ *such that* $\epsilon_i(s) = \pi_1(s)^{n_{i1}} \cdots \pi_t(s)^{n_{it}}$ *and* $\psi_i(s) = \pi_1(s)^{d_{i1}} \cdots \pi_t(s)^{d_{it}}$ *with* $\pi_1(s), \ldots, \pi_t(s)$ *different monic non-constant irreducible polynomials,* $n_{ij} \geq 0$ *and* $d_{ij} \geq 0$. *Then* $\pi_j(s)^{n_{1j}-d_{1j}}, \ldots, \pi_j(s)^{n_{rj}-d_{rj}}$ *are the local invariant rational functions at* $\pi_j(s)$ *of* $R(s)$ *for* $j = 1, \ldots, t$, *and* $1, \ldots, 1$ *(r times) are the local invariant rational functions at* $\pi(s)$ *of* $R(s)$ *for all monic non-constant irreducible* $\pi(s) \in \mathbb{F}[s]$ *different from* $\pi_1(s), \ldots, \pi_t(s)$.

Conversely, let $\pi_j(s)^{l_{1j}}, \ldots, \pi_j(s)^{l_{rj}}$ *be the local invariant rational functions at* $\pi_j(s)$ *of* $R(s)$ *for* $j = 1, \ldots, t$, *and* $1, \ldots, 1$ *(r times) be the local invariant rational functions at* $\pi(s)$ *of* $R(s)$ *for all monic non-constant irreducible* $\pi(s) \in \mathbb{F}[s]$ *different from* $\pi_1(s), \ldots, \pi_t(s)$. *Let* $\epsilon_i(s) = \pi_1(s)^{n_{i1}} \cdots \pi_t(s)^{n_{it}}$ *and* $\psi_i(s) = \pi_1(s)^{d_{i1}} \cdots \pi_t(s)^{d_{it}}$ *where* $n_{ij} = l_{ij}$ *and* $d_{ij} = 0$ *if* $l_{ij} \geq 0$ *and* $n_{ij} = 0$ *and* $d_{ij} = -l_{ij}$ *otherwise. Then* $\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}$ *are the global invariant rational functions of* $R(s)$.

Proof. Suppose that $\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}$ are the invariant rational functions in $\mathbb{F}[s]$ of $R(s)$. Then there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ such that

$$R(s) = U(s) \begin{bmatrix} \operatorname{Diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$

Put

$$U_j(s) = \prod_{k=1}^{j-1} \begin{bmatrix} \operatorname{Diag}\left(\pi_k(s)^{n_{1k}-d_{1k}}, \ldots, \pi_k(s)^{n_{rk}-d_{rk}}\right) & 0 \\ 0 & I_{m-r} \end{bmatrix}, \ j = 2, \ldots, t+1,$$

$$V_j(s) = \prod_{k=j+1}^{t} \begin{bmatrix} \operatorname{Diag}\left(\pi_k(s)^{n_{1k}-d_{1k}}, \ldots, \pi_k(s)^{n_{rk}-d_{rk}}\right) & 0 \\ 0 & I_{n-r} \end{bmatrix}, \ j = 1, \ldots, t-1,$$

$U_1(s) = I_m$ and $V_t(s) = I_n$. Then for each $j = 1, \ldots, t$, we can write

$$R(s) = U(s)U_j(s) \begin{bmatrix} \operatorname{Diag}\left(\pi_j(s)^{n_{1j}-d_{1j}}, \ldots, \pi_j(s)^{n_{rj}-d_{rj}}\right) & 0 \\ 0 & 0 \end{bmatrix} V_j(s)V(s)$$

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and

$$R(s) = U(s)U_{t+1}(s) \begin{bmatrix} \mathrm{Diag}\,(1,\ldots,1) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$

Notice that $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_\pi(s))$ for all monic non-constant irreducible $\pi(s) \in \mathbb{F}[s]$, $U_{t+1}(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$ for all $\pi(s)$ different from $\pi_1(s),\ldots,\pi_t(s)$, $U_j(s) \in \mathrm{Gl}_m(\mathbb{F}_{\pi_j}(s))$, $V_j(s) \in \mathrm{Gl}_n(\mathbb{F}_{\pi_j}(s))$. Since $\epsilon_i(s)$ and $\psi_i(s)$ are coprime polynomials we must have $n_{ij} = 0$ or $d_{ij} = 0$ for each $i = 1,\ldots,r$ and $j = 1,\ldots,t$. Even more, it follows from $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ and $\psi_r(s) \mid \cdots \mid \psi_1(s)$ that $n_{1j} \le \cdots \le n_{rj}$ and $d_{1j} \ge \cdots \ge d_{rj}$, respectively. Thus, $n_{ij} - d_{ij} \le n_{i+1j} - d_{i+1j}$ for all $i = 1,\ldots,r-1$, $j = 1,\ldots,t$. Thus, $\pi_j(s)^{n_{1j}-d_{1j}},\ldots,\pi_j(s)^{n_{rj}-d_{rj}}$ are the invariant rational functions in $\mathbb{F}_{\pi_j}(s)$ of $R(s)$ for $j = 1,\ldots,t$ and $1,\ldots,1$ are the invariant rational functions in $\mathbb{F}_\pi(s)$ of $R(s)$ for all $\pi(s) \in \mathbb{F}[s]$ different from $\pi_1(s),\ldots,\pi_t(s)$.

Conversely, let $\pi_j(s)^{l_{1j}},\ldots,\pi_j(s)^{l_{rj}}$ be the invariant rational functions in $\mathbb{F}_{\pi_j}(s)$ of $R(s)$ for $j = 1,\ldots,t$, and $1,\ldots,1$ (r times) be the invariant rational functions in $\mathbb{F}_\pi(s)$ of $R(s)$ for all monic non-constant irreducible $\pi(s) \in \mathbb{F}[s]$ different from $\pi_1(s),\ldots,\pi_t(s)$. Suppose that $\frac{\epsilon'_i(s)}{\psi'_i(s)}$, $i = 1,\ldots,r$, are the global invariant rational functions of $R(s)$ such that $\epsilon'_i(s) = \sigma_1(s)^{n'_{i1}}\cdots\sigma_u(s)^{n'_{iu}}$, $\psi'_i(s) = \sigma_1(s)^{d'_{i1}}\cdots\sigma_u(s)^{d'_{iu}}$, with $\sigma_1(s),\ldots,\sigma_u(s)$ different monic non-constant irreducible polynomials, $n'_{ij} \ge 0$ and $d'_{ij} \ge 0$. By the "if" part, $\sigma_k(s)^{n'_{1k}-d'_{1k}},\ldots,\sigma_k(s)^{n'_{rk}-d'_{rk}}$ are the invariant rational functions in $\mathbb{F}_{\sigma_k}(s)$ of $R(s)$ for $k = 1,\ldots,u$, and $1,\ldots,1$ (r times) are the invariant rational functions in $\mathbb{F}_\pi(s)$ of $R(s)$ for all monic non-constant irreducible $\pi(s) \in \mathbb{F}[s]$ different from $\sigma_1(s),\ldots,\sigma_u(s)$. Notice that n'_{ik}, d'_{ik} are non-negative integers such that $n'_{ik} = 0$ or $d'_{ik} = 0$ for each $i = 1,\ldots,r$ and $k = 1,\ldots,u$ and $n'_{ik} - d'_{ik} \le n'_{i+1k} - d'_{i+1k}$ for all $i = 1,\ldots,r-1$, $k = 1,\ldots,u$. By the uniqueness of the local invariant rational functions we have that $u = t$, and reordering if necessary, $\sigma_j(s) = \pi_j(s)$ and $n'_{ij} - d'_{ij} = l_{ij}$. Thus, if $l_{ij} \le 0$ then, on the one hand, $n'_{ij} = 0$ and $d'_{ij} = -l_{ij}$ and, on the other hand, by definition of n_{ij} and d_{ij}, $n_{ij} = 0$ and $d_{ij} = -l_{ij}$. Therefore $n'_{ij} = n_{ij}$ and $d'_{ij} = d_{ij}$. Otherwise, if $l_{ij} \ge 0$ then, reasoning in the same way, we get the same result, that is, $n'_{ij} = n_{ij}$ and $d'_{ij} = d_{ij}$. □

THEOREM 4.2. *Two rational matrices $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are globally or finite equivalent if and only if $R_1(s), R_2(s)$ are locally equivalent at $\pi(s)$ for all monic non-constant irreducible $\pi(s) \in \mathbb{F}[s]$.*

Proof. $R_1(s)$ and $R_2(s)$ are globally equivalent if and only if they have the same finite invariant rational functions. By the previous lemma, they have the same finite invariant rational functions if and only if they have the same local invariant rational functions at $\pi(s)$ for every monic non-constant irreducible $\pi(s) \in \mathbb{F}[s]$, and this is so if and only if $R_1(s)$ and $R_2(s)$ are equivalent in $\mathbb{F}_\pi(s)$ for all monic non-constant

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irreducible $\pi(s) \in \mathbb{F}[s]$. $\square$

Remark 4.3. A direct consequence of the previous lemma is that the finite elementary divisors of a polynomial matrix are its nontrivial local invariant rational functions.

Example 4.4. Let $R(s)$ be the matrix of Example 3.1 and let us compute its local structure. Its global invariant rational functions are $\frac{1}{s(s+1)^2}, (s+1)(s^2+1)$. It follows from the previous lemma that its local invariant rational functions at s are $\dfrac{1}{s}, 1$, those at $s+1$ are $\dfrac{1}{(s+1)^2}, s+1$, and those at s^2+1 are $1, s^2+1$. At any other non-constant monic irreducible polynomial the local invariant rational functions are $1, 1$.

5. Infinite structure. The material presented in this section summarizes and places in the context of local rings some of the main ideas in, for example, [24] or [6] about the equivalence of rational matrices at infinity.

Recall that a rational function $r(s)$ has a pole (zero) at infinity if $r\left(\frac{1}{s}\right)$ has a pole (zero) at 0. Following this idea, we can define the local ring at infinity as the set of rational functions, $r(s)$, such that $r\left(\frac{1}{s}\right)$ does not have 0 as a pole, that is, $\mathbb{F}_\infty(s) = \left\{r(s) \in \mathbb{F}(s) : r\left(\frac{1}{s}\right) \in \mathbb{F}_s(s)\right\}$. If $r(s) = \frac{p(s)}{q(s)}$ with $p(s) = a_t s^t + a_{t+1}s^{t+1} + \cdots + a_p s^p,\ a_p \neq 0,\ q(s) = b_r s^r + b_{r+1}s^{r+1} + \cdots + b_q s^q,\ b_q \neq 0,\ p = \deg(p(s)),\ q = \deg(q(s))$, then

$$r\left(\frac{1}{s}\right) = \frac{\frac{a_t}{s^t} + \frac{a_{t+1}}{s^{t+1}} + \cdots + \frac{a_p}{s^p}}{\frac{b_r}{s^r} + \frac{b_{r+1}}{s^{r+1}} + \cdots + \frac{b_q}{s^q}} = \frac{a_t s^{p-t} + a_{t+1}s^{p-t-1} + \cdots + a_p}{b_r s^{q-r} + b_{r+1}s^{q-r-1} + \cdots + b_q} s^{q-p} = \frac{f(s)}{g(s)} s^{q-p}.$$

As $\mathbb{F}_s(s) = \left\{\frac{f(s)}{g(s)} s^d : f(0) \neq 0,\ g(0) \neq 0 \text{ and } d \in \mathbb{N}\right\} \cup \{0\}$, it follows that

$$\mathbb{F}_\infty(s) = \left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \deg(q(s)) \geq \deg(p(s))\right\}.$$

This set is the ring of proper rational functions, usually denoted by $\mathbb{F}_{pr}(s)$. The units in this ring are the rational functions whose numerators and denominators have the same degree. They are called biproper rational functions. In fact, this ring is local and the set of the strictly proper rational functions, i.e. $\frac{p(s)}{q(s)} \in \mathbb{F}_{pr}(s)$ such that $\deg(q(s)) > \deg(p(s))$, is its maximal ideal since any element outside this ideal is biproper.

Fix an arbitrary $a \in \mathbb{F}$. Notice that any nonzero element $\frac{p(s)}{q(s)} \in \mathbb{F}_{pr}(s)$ can be written as $\frac{p(s)}{q(s)} = \frac{p(s)}{q(s)} \frac{(s-a)^{\deg(q(s))-\deg(p(s))}}{(s-a)^{\deg(q(s))-\deg(p(s))}} = \frac{u_{pr}(s)}{(s-a)^{\deg(q(s))-\deg(p(s))}}$ with $u_{pr}(s) =$

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$\frac{p(s)}{q(s)}(s-a)^{\deg(q(s))-\deg(p(s))}$ a unit in $\mathbb{F}_{pr}(s)$. Therefore,

$$\mathbb{F}_{pr}(s) = \left\{ \frac{u_{pr}(s)}{(s-a)^d} : u_{pr}(s) \text{ a unit, } d \in \mathbb{N} \right\} \cup \{0\}.$$

Now the properties of the local rings analyzed in Section 4 apply to this local ring by substituting $\pi(s)$ by $\frac{1}{(s-a)}$. For example, $\mathbb{F}_{pr}(s)$ is an Euclidean domain with the Euclidean division defined as follows:

$$u_{pr2}(s)\frac{1}{(s-a)^{d_2}} = q(s)u_{pr1}(s)\frac{1}{(s-a)^{d_1}} + r(s),$$

where

$$\begin{cases} q(s) = \frac{u_{pr2}(s)}{u_{pr1}(s)}\frac{1}{(s-a)^{d_2-d_1}}, r(s) = 0 & \text{if} \quad d_2 \geq d_1 \\ q(s) = 0, r(s) = u_{pr2}(s)\frac{1}{(s-a)^{d_2}} & \text{if} \quad d_2 < d_1. \end{cases}$$

Also, $u_{pr1}(s)\frac{1}{(s-a)^{d_1}} | u_{pr2}(s)\frac{1}{(s-a)^{d_2}}$ if and only if $d_1 \leq d_2$ and two proper rational functions are coprime in $\mathbb{F}_{pr}(s)$ if and only if at least one of them is a unit of $\mathbb{F}_{pr}(s)$. As for the local rings at $\pi(s)$, we will say that a proper rational function $u_{pr}(s)\frac{1}{(s-a)^d}$ is monic if $u_{pr}(s) = 1$.

Again $\mathbb{F}_{pr}(s)$ is a principal ideal domain and its field of fractions is $\mathbb{F}(s)$.

Let $\mathbb{F}_{pr}(s)^{m\times n}$ be the set of $m \times n$ proper matrices (i.e., with entries in $\mathbb{F}_{pr}(s)$) and $\mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ the set of biproper matrices. These are the invertible matrices in $\mathbb{F}_{pr}(s)^{m\times m}$. Two rational matrices $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are said to be *equivalent at infinity* if they are *equivalent in* $\mathbb{F}_{pr}(s)$, that is, if there exist matrices $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_{pr}(s))$ such that $R_2(s) = U(s)R_1(s)V(s)$. This is again an equivalence relation and every $R(s) \in \mathbb{F}(s)^{m\times n}$ is equivalent at infinity to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{1}{\frac{1}{(s-a)^{-q_1}}}, \ldots, \frac{1}{\frac{1}{(s-a)^{-q_k}}}, 1, \ldots, 1, \frac{\frac{1}{(s-a)^{q_u}}}{1}, \ldots, \frac{\frac{1}{(s-a)^{q_r}}}{1} \right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} R(s)$ and $q_1 \leq \cdots \leq q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \leq \cdots \leq q_r$ are integers. The Smith-McMillan form at infinity can be written also as

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{1}{(s-a)^{q_1}}, \ldots, \frac{1}{(s-a)^{q_r}} \right) & 0 \\ 0 & 0 \end{bmatrix}.$$

The rational functions $\frac{1}{(s-a)^{q_1}}, \ldots, \frac{1}{(s-a)^{q_r}}$ are uniquely determined by $R(s)$ and the chosen scalar a. They are called the *invariant rational functions at infinity* of

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$R(s)$ with respect to a or the *invariant rational functions in* $\mathbb{F}_{pr}(s)$ of $R(s)$ with respect to a. The integers $q_1, \ldots, q_r$ are called the *invariant orders at infinity* of $R(s)$. The scalar a can be changed by any other scalar $b \in \mathbb{F}$ and the invariant orders will remain the same because $\frac{(s-a)}{(s-b)} \frac{1}{(s-a)} = \frac{1}{(s-b)}$ and $\frac{(s-a)}{(s-b)}$ is biproper. Thus, the invariant orders at infinity form a complete system of invariants for the equivalence at infinity in $\mathbb{F}(s)^{m\times n}$. They determine the zeros and poles of $R(s)$ at infinity. Namely, $R(s)$ has $r-u+1$ *zeros at infinity* each one of order $q_u, \ldots, q_r$ and k *poles at infinity* each one of order $-q_1, \ldots, -q_k$.

It is a usual practice to choose $a = 0$ (see for instance [24] or [6]) but using an arbitrary a is more convenient for our developments in the following sections. It is also worth-remarking that the invariant orders at infinity in these references are the opposite to the ones used here. The notation that we are using aims to stress the parallelism between the structures at any prime polynomial $\pi(s)$ and the structure at infinity for any rational matrix.

Recall (see Remark 2.4) that the monic least common denominator in $\mathbb{F}_{pr}(s)$ of $R(s)$ is $\frac{1}{(s-a)^{-q_1}}$ if $q_1 < 0$ or 1 otherwise. Therefore $R(s) \in \mathbb{F}_{pr}(s)^{m\times n}$ if and only if $q_1, \ldots, q_r$ are nonnegative integers, that is, if and only if $R(s)$ has no poles at infinity. Therefore, non-constant polynomial matrices always have poles at infinity and they may also have zeros at infinity.

As for the local rings of Section 4, $q_1 + \cdots + q_j$ is the smallest integer among the exponents of $\frac{1}{s-a}$ of all minors of order j of $R(s)$. Bearing in mind that each rational function $\frac{p(s)}{q(s)}$ can be written as

$$\frac{p(s)}{q(s)} = \frac{u_{pr}(s)}{(s-a)^{\deg(q(s))-\deg(p(s))}},$$

we conclude that $q_1 + \cdots + q_j$ is the smallest integer among the differences of the degrees of the denominators and numerators of all minors of order j of $R(s)$. In particular, if $R(s)$ is a polynomial matrix all denominators are of degree 0 so that the smallest difference of the denominators and numerators of the entries of $R(s)$ is $-\deg R(s)$. In other words $-q_1 = \deg R(s)$.

Example 5.1. Let

$$P(s) = \begin{bmatrix} s & (s+1)^2 & s-1 \\ 0 & 1 & 0 \\ 1 & 0 & s^2 \end{bmatrix},$$

an let us compute its invariant orders at infinity. Notice that $\deg P(s) = 2$. Thus, $q_1 = -2$. Now, the greatest degree of the 2×2 minors is 4. Then $q_1 + q_2 = -4$ and $q_2 = -2$. Finally, $\det P(s) = s^3 - s + 1$ and so $q_1 + q_2 + q_3 = -3$. This means that

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$q_3 = 1$. Hence, $P(s)$ has one zero at infinity of order 1 and two poles at infinity of order 2.

6. Möbius transformations. In this section, we aim to show how the finite and infinite structure of the rational matrices is modified by a Möbius transformation. The local rings and their units studied in the previous sections will help us in this task. The basic idea is to show that the Möbius transformations preserve the property of being local and transform units into units.

Formally, a Möbius transformation is a homomorphism on the field of rational functions:

$$\begin{array}{rccc} g & : & \mathbb{F}(s) & \longrightarrow & \mathbb{F}(s) \\ & & r(s) & \to & r\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right), \qquad \alpha\delta - \beta\gamma \neq 0. \end{array}$$

So given $u(s), v(s) \in \mathbb{F}(s)$

(i) if $r(s) = u(s) + v(s)$ then $r\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) = u\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) + v\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$, and

(ii) if $r(s) = u(s)v(s)$ then $r\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) = u\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)v\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$.

The homomorphism g extends to the corresponding matrix sets:

$$\begin{array}{rccc} \widetilde{g} & : & \mathbb{F}(s)^{m\times n} & \longrightarrow & \mathbb{F}(s)^{m\times n} \\ & & R(s) = [r_{ij}(s)] & \to & R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) = \left[\ r_{ij}\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)\ \right]. \end{array}$$

It is worth-remarking that a Möbius transformation has been already used to define the infinite structure of rational matrices.

6.1. Translations and dilations. Consider first the particular case $\gamma = 0$. The Möbius transformation reduces to $m(s) = cs + d$ with $c = \frac{\alpha}{\delta} \neq 0$ and $d = \frac{\beta}{\delta}$.

If $\lambda(s)$ is a monic polynomial with $\deg(\lambda(s)) = p$ it is plain that $\lambda(cs + d)$ is a polynomial of degree p with leading coefficient c^p and $\widetilde{\lambda}(s) := \frac{\lambda(cs+d)}{c^p} \in \mathbb{F}[s]$ is monic. Also, if $\lambda_1(s), \lambda_2(s) \in \mathbb{F}[s]$ and $\lambda_1(s)|\lambda_2(s)$ then $\lambda_1(cs + d)|\lambda_2(cs + d)$. Moreover, if $\lambda_1(s), \lambda_2(s)$ are coprime then $\lambda_1(cs + d), \lambda_2(cs + d)$ are coprime as well because if $\lambda_1(cs + d) = \mu(s)\nu_1(s)$, $\lambda_2(cs + d) = \mu(s)\nu_2(s)$ with $\mu(s), \nu_1(s), \nu_2(s)$ non-constant polynomials such that $\nu_1(s), \nu_2(s)$ are coprime then $\lambda_i(s) = \mu(\frac{s-d}{c})\nu_i(\frac{s-d}{c})$ for $i = 1, 2$. We have that if $U(s)$ is a unimodular matrix then $U(cs + d)$ is unimodular too. An immediate consequence of this fact is the following result.

Proposition 6.1. *Let $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ and let $c, d \in \mathbb{F}$ be arbitrary scalars with $c \neq 0$. $R_1(s)$ and $R_2(s)$ are globally equivalent if and only if $R_1(cs + d)$ and $R_2(cs + d)$ are globally equivalent.*

Proposition 6.2. *Let $R(s) \in \mathbb{F}(s)^{m\times n}$ be a matrix of rank r and let $c, d \in \mathbb{F}$*

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be arbitrary scalars with $c \neq 0$. *Let* $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ *and* $\psi_r(s) \mid \cdots \mid \psi_1(s)$ *be non-zero monic polynomials such that* $\gcd(\epsilon_i(s), \psi_i(s)) = 1$, $i = 1, \ldots, r$. *Define* $\widetilde{\epsilon}_i(s) = \frac{\epsilon_i(cs+d)}{c^{n_i}}$ *and* $\widetilde{\psi}_i(s) = \frac{\psi_i(cs+d)}{c^{d_i}}$ *with* $n_i = \deg(\epsilon_i(s))$ *and* $d_i = \deg(\psi_i(s))$ *for* $i = 1, \ldots, r$. *Then* $\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}$ *are the finite invariant rational functions of* $R(s)$ *if and only if* $\frac{\widetilde{\epsilon}_1(s)}{\widetilde{\psi}_1(s)}, \ldots, \frac{\widetilde{\epsilon}_r(s)}{\widetilde{\psi}_r(s)}$ *are the finite invariant rational functions of* $R(cs+d)$.

Proof. Assume that $\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}$ are the finite invariant rational functions of $R(s)$. Then there exist unimodular matrices $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ such that

$$R(s) = U(s) \begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$

Therefore

$$R(cs+d) = U(cs+d) \begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1(cs+d)}{\psi_1(cs+d)}, \ldots, \frac{\epsilon_r(cs+d)}{\psi_r(cs+d)}\right) & 0 \\ 0 & 0 \end{bmatrix} V(cs+d).$$

Put $W = \mathrm{Diag}(c^{n_1-d_1}, \ldots, c^{n_r-d_r}, 1, \ldots, 1)$. Notice that $W, U(cs+d) \in \mathrm{Gl}_m(\mathbb{F}[s])$ and $V(cs+d) \in \mathrm{Gl}_n(\mathbb{F}[s])$. Thus,

$$R(cs+d) = U(cs+d)W \begin{bmatrix} \mathrm{Diag}\left(\frac{\widetilde{\epsilon}_1(s)}{\widetilde{\psi}_1(s)}, \ldots, \frac{\widetilde{\epsilon}_r(s)}{\widetilde{\psi}_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix} V(cs+d).$$

Since $\widetilde{\epsilon}_1(s), \ldots, \widetilde{\epsilon}_r(s), \widetilde{\psi}_1(s), \ldots, \widetilde{\psi}_r(s)$ are nonzero monic polynomials, $\widetilde{\epsilon}_i(s), \widetilde{\psi}_i(s)$ are coprime for each i, $\widetilde{\epsilon}_1(s) \mid \cdots \mid \widetilde{\epsilon}_r(s)$ and $\widetilde{\psi}_r(s) \mid \cdots \mid \widetilde{\psi}_1(s)$, it follows that the finite invariant rational functions of $R(cs+d)$ are $\frac{\widetilde{\epsilon}_1(s)}{\widetilde{\psi}_1(s)}, \ldots, \frac{\widetilde{\epsilon}_r(s)}{\widetilde{\psi}_r(s)}$.

The converse can be proved along the same lines taking into account that for any non-zero monic polynomial $r(s)$ of degree d, $\widetilde{r}(s) = \frac{1}{c^d} r(cs+d)$ if and only if $r(s) = c^d \widetilde{r}\left(\frac{s-d}{c}\right)$ $(c \neq 0)$. $\square$

Regarding the local Smith-McMillan form, we have that if $\pi(s) \in \mathbb{F}[s]$ is a non-constant irreducible polynomial then $\pi(cs+d)$ is irreducible as well because if $\pi(cs+d) = \mu(s)\nu(s)$, with $\mu(s), \nu(s) \in \mathbb{F}[s]$, then $\pi(s) = \mu\left(\frac{s-d}{c}\right)\nu\left(\frac{s-d}{c}\right)$. We have already seen that if $\pi(s)$ is a monic irreducible polynomial with $\deg(\pi(s)) = p$ then $\widetilde{\pi}(s) = \frac{\pi(cs+d)}{c^p}$ is a monic irreducible polynomial with the same degree. Now let $\frac{p(s)}{q(s)} \in \mathbb{F}_\pi(s)$. Since $q(s)$ and $\pi(s)$ are coprime, $q(cs+d)$ and $\pi(cs+d)$ are also coprime and we conclude that $\frac{p(cs+d)}{q(cs+d)} \in \mathbb{F}_{\widetilde{\pi}}(s)$. A consequence of this fact is that $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$ if and only if $U(cs+d) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}}(s))$.

Proposition 6.3. *Let* $R_1(s), R_2(s) \in \mathbb{F}(s)^{m \times n}$ *and let* $c, d \in \mathbb{F}$ *be arbitrary scalars with* $c \neq 0$. *Let* $\pi(s)$ *be a monic non-constant irreducible polynomial of degree*

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p. $R_1(s)$ and $R_2(s)$ are locally equivalent at $\pi(s)$ if and only if $R_1(cs+d)$ and $R_2(cs+d)$ are locally equivalent at $\widetilde{\pi}(s)=\frac{\pi(cs+d)}{c^p}$.

Proposition 6.4. *Let $R(s)\in\mathbb{F}(s)^{m\times n}$ be a matrix of rank r and let $c,d\in\mathbb{F}$ be arbitrary scalars with $c\neq 0$. Let $\pi(s)\in\mathbb{F}[s]$ be a monic non-constant irreducible polynomial of degree p and let $h_1\leq\cdots\leq h_r$ be integers. Then $\pi(s)^{h_1},\ldots,\pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\widetilde{\pi}(s)^{h_1},\ldots,\widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s)=\frac{1}{c^p}\pi(cs+d)$ of $R(cs+d)$.*

Proof. Suppose that $\pi(s)^{h_1},\ldots,\pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$. Then there exist $U(s)\in\mathrm{Gl}_m(\mathbb{F}_\pi(s))$ and $V(s)\in\mathrm{Gl}_n(\mathbb{F}_\pi(s))$ such that

$$R(s)=U(s)\begin{bmatrix}\mathrm{Diag}(\pi(s)^{h_1},\ldots,\pi(s)^{h_r}) & 0\\ 0 & 0\end{bmatrix}V(s).$$

Thus,

$$\begin{aligned}R(cs+d) &= U(cs+d)\begin{bmatrix}\mathrm{Diag}(\pi(cs+d)^{h_1},\ldots,\pi(cs+d)^{h_r}) & 0\\ 0 & 0\end{bmatrix}V(cs+d)\\ &= U(cs+d)W\begin{bmatrix}\mathrm{Diag}(\widetilde{\pi}(s)^{h_1},\ldots,\widetilde{\pi}(s)^{h_r}) & 0\\ 0 & 0\end{bmatrix}V(cs+d)\end{aligned}$$

with $W=\mathrm{Diag}(c^{ph_1},\ldots,c^{ph_r},1,\ldots,1)\in\mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}}(s))$, $U(cs+d)\in\mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}}(s))$ and $V(cs+d)\in\mathrm{Gl}_n(\mathbb{F}_{\widetilde{\pi}}(s))$. Since $h_1\leq\cdots\leq h_r$, it follows that $\widetilde{\pi}(s)^{h_1},\ldots,\widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s)=\frac{1}{c^p}\pi(cs+d)$ of $R(cs+d)$.

The converse can be proved as in Proposition 6.2. ∎

If $\frac{p(s)}{q(s)}$ is a rational function then it is plain that $\deg(p(s))=\deg(p(cs+d))$ and $\deg(q(s))=\deg(q(cs+d))$. So if $B(s)$ is a biproper matrix then $B(cs+d)$ is also biproper. The following proposition is easily proved.

Proposition 6.5. *Let $R_1(s),R_2(s)\in\mathbb{F}(s)^{m\times n}$ be matrices of rank r and let $c,d\in\mathbb{F}$ be arbitrary scalars with $c\neq 0$. $R_1(s)$ and $R_2(s)$ are equivalent at infinity if and only if $R_1(cs+d)$ and $R_2(cs+d)$ are equivalent at infinity.*

Proposition 6.6. *Let $R(s)\in\mathbb{F}(s)^{m\times n}$ and let $c,d\in\mathbb{F}$ be arbitrary scalars with $c\neq 0$. Then $R(s)$ and $R(cs+d)$ have the same invariant orders at infinity.*

Proof. Let $q_1\leq\cdots\leq q_r$ be the invariant orders at infinity of $R(s)$. Then there are biproper matrices $U(s)\in\mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ and $V(s)\in\mathrm{Gl}_n(\mathbb{F}_{pr}(s))$ such that

$$R(s)=U(s)\begin{bmatrix}\mathrm{Diag}\left(\frac{1}{s^{q_1}},\ldots,\frac{1}{s^{q_r}}\right) & 0\\ 0 & 0\end{bmatrix}V(s).$$

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Thus,

$$R(cs+d) = U(cs+d)\begin{bmatrix} \operatorname{Diag}\left(\frac{1}{(cs+d)^{q_1}},\dots,\frac{1}{(cs+d)^{q_r}}\right) & 0 \\ 0 & 0 \end{bmatrix} V(cs+d).$$

Since $U(cs+d)$ and $V(cs+d)$ are biproper matrices, it is clear that the invariant orders at infinity of $R(cs+d)$ are $q_1,\dots,q_r$. And conversely. $\square$

Corollary 6.7. *Let $R(s) \in \mathbb{F}(s)^{m\times n}$ and let $c,d \in \mathbb{F}$ be arbitrary scalars with $c \neq 0$. Then $R(s)$ and $R(cs+d)$ are equivalent at infinity.*

6.2. Möbius transformations that modify the structure at infinity. We deal now with the case $\gamma \neq 0$. It should be noticed, first of all, that in this case, taking into account that $\alpha\delta - \beta\gamma \neq 0$, one has

$$m(s) := \frac{\alpha s+\beta}{\gamma s+\delta} = \frac{\beta\gamma-\alpha\delta}{\gamma^2}\left(\frac{\alpha\gamma}{\beta\gamma-\alpha\delta} + \frac{1}{s+\frac{\delta}{\gamma}}\right). \tag{6.1}$$

Henceforth, a general Möbius transformation is the composition of the following four more elementary transformations: For $i = 1,2,3,4$ let $t_i : \mathbb{F}(s) \longrightarrow \mathbb{F}(s)$ be defined by:

(i) $t_1(s) = s + \frac{\delta}{\gamma}$,
(ii) $t_2(s) = \frac{1}{s}$,
(iii) $t_3(s) = \frac{\alpha\gamma}{\beta\gamma-\alpha\delta} + s$,
(iv) $t_4(s) = \frac{\beta\gamma-\alpha\delta}{\gamma^2}s$.

Then $m = t_4 \circ t_3 \circ t_2 \circ t_1$.

Transformations t_1 and t_3 are translations and t_4 is a dilation. The behavior of the finite (global and local) and infinite structures of any rational matrix with respect to these types of transformations have been analyzed in the previous section. Then it is only necessary to focus on transformations of type t_2. Nevertheless we will work with a little more general transformations. This will allow us more flexibility in order to make the proof of Proposition 6.12 easier. Specifically, let a and b be arbitrary elements of $\mathbb{F}$ and define

$$f(s) = a + \frac{1}{s-b}.$$

This is a Möbius transformation that includes t_2 as a particular case and taking $a = \frac{\alpha\gamma}{\beta\gamma-\alpha\delta}$ and $b = \frac{-\delta}{\gamma}$, we have $m = t_4 \circ f$. Its inverse function is of the same form: $f^{-1}(s) = b + \frac{1}{s-a}$. These transformations were analyzed in [1, Sec. 4.2]. In what follows the ideas developed there will be freely used.

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Any polynomial $p(s)$ can be written as linear combinations of powers of $s-a$

$$p(s) = a_{d_p}(s-a)^{d_p} + a_{d_p-1}(s-a)^{d_p-1} + \cdots + a_0, \quad a_{d_p} \neq 0.$$

Then

$$p\left(f(s)\right) = a_{d_p}\frac{1}{(s-b)^{d_p}} + a_{d_p-1}\frac{1}{(s-b)^{d_p-1}} + \cdots + a_0 =$$

$$\frac{a_{d_p} + a_{d_p-1}(s-b) + \cdots + a_0(s-b)^{d_p}}{(s-b)^{d_p}}$$

and $(s-b)^{d_p}p(f(s))$ is a polynomial written as linear combinations of powers of $(s-b)$. Denote by $\operatorname{rev}_f p(s)$ the reversal of $p(s)$ with respect to f:

$$\operatorname{rev}_f p(s) := (s-b)^{d_p} p\left(f(s)\right) = a_0(s-b)^{d_p} + \cdots + a_{d_p-1}(s-b) + a_{d_p}.$$

Notice that $p(s) = (s-a)^{d_p} \operatorname{rev}_f p(f^{-1}(s))$, but this polynomial is not always the reversal with respect to f^{-1} of $\operatorname{rev}_f p(s)$. Specifically, the following lemma gives some basic properties of this reversal polynomial.

Lemma 6.8. *Let $p(s), q(s) \in \mathbb{F}[s]$ and $f(s) = a + \frac{1}{s-b}$ with $a, b \in \mathbb{F}$ arbitrary elements.*

1. $\operatorname{rev}_f p(s) \in \mathbb{F}[s]$ *and* $\deg(\operatorname{rev}_f p(s)) \leq \deg(p(s))$.
2. $\deg(\operatorname{rev}_f p(s)) = \deg(p(s))$ *if and only if* $\gcd(p(s), s-a) = 1$.
3. $\gcd(\operatorname{rev}_f p(s), s-b) = 1$.
4. *If $p(s)$ is a non-constant monic irreducible polynomial and $p(s) \neq s-a$ then $\operatorname{rev}_f p(s)$ is a non-constant irreducible polynomial.*
5. *If $p(s)$ and $q(s)$ are coprime then $\operatorname{rev}_f p(s)$ and $\operatorname{rev}_f q(s)$ are coprime.*

Proof. The proofs of items 1-3 are straightforward. We focus on items 4 and 5. As $p(s)$ is a non-constant monic irreducible polynomial different from $s-a$, by item 2, $\deg(\operatorname{rev}_f p(s)) = \deg(p(s))$. Assume that $d_p = \deg(\operatorname{rev}_f p(s)) = \deg(p(s))$ and $\operatorname{rev}_f p(s) = \mu(s)\nu(s)$. If $\deg(\mu(s)) = d_1$, $\deg(\nu(s)) = d_2$ then $d_p = d_1 + d_2$ and

$$p(s) = (s-a)^{d_p} \operatorname{rev}_f p(f^{-1}(s)) = (s-a)^{d_1}\mu(f^{-1}(s))(s-a)^{d_2}\nu(f^{-1}(s))$$

where $(s-a)^{d_1}\mu(f^{-1}(s))$ and $(s-a)^{d_2}\nu(f^{-1}(s))$ are polynomials. This is a contradiction unless $(s-a)^{d_1}\mu(f^{-1}(s))$ or $(s-a)^{d_2}\nu(f^{-1}(s))$ are units. Let us see that this implies that $\mu(s)$ or $\nu(s)$ are constant polynomials. Assume that $(s-a)^{d_1}\mu(f^{-1}(s))$ is a nonzero constant c. Then $\frac{\mu(s)}{(s-b)^{d_1}} = c$ and $\mu(s) = c(s-b)^{d_1}$. This implies, by item 3 and taking into account that $\mu(s) | \operatorname{rev}_f p(s)$, that $d_1 = 0$ and so $\mu(s)$ is constant.

To prove item 5, assume that $\operatorname{rev}_f p(s) = \alpha(s)\beta_1(s)$ and $\operatorname{rev}_f q(s) = \alpha(s)\beta_2(s)$ with $\alpha(s)$, $\beta_1(s)$, $\beta_2(s)$ polynomials such that $\beta_1(s)$, $\beta_2(s)$ are coprime. Let $d_p =$

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$\deg(p(s))$, $d_q = \deg(q(s))$, $d = \deg(\alpha(s))$, $b_1 = \deg(\beta_1(s))$ and $b_2 = \deg(\beta_2(s))$. By item 1, we have $d_p \geq d + b_1$ and $d_q \geq d + b_2$. Thus,

$$p(s) = (s-a)^d \alpha(f^{-1}(s))(s-a)^{d_p - d}\beta_1(f^{-1}(s)),$$

$$q(s) = (s-a)^d \alpha(f^{-1}(s))(s-a)^{d_q - d}\beta_2(f^{-1}(s)).$$

Since $p(s)$ and $q(s)$ are coprime, $(s-a)^d\alpha(f^{-1}(s)) \in \mathbb{F}[s]$ is a unit, and, as in the proof of item 4, $\alpha(s)$ is a constant. $\square$

Given a non-constant monic irreducible polynomial $\pi(s) \in \mathbb{F}[s]$, $\pi(s) \neq s - a$, $\mathrm{rev}_f\, \pi(s)$ is a non-constant irreducible polynomial as well, but it may not be monic. We denote by $\widetilde{\pi}_f(s)$ the monic reversal polynomial: $\widetilde{\pi}_f(s) = \frac{1}{\pi_0} \mathrm{rev}_f\, \pi(s)$, $\pi_0 = \pi(a)$. Recall that (Section 4) $\mathbb{F}_{\widetilde{\pi}_f}(s) = \mathbb{F}_{\mathrm{rev}_f\, \pi}(s)$. With this notation, the following result is a particular case of [1, Lemma 4.6].

Lemma 6.9. *Let $a, b \in \mathbb{F}$ be arbitrary elements and $f(s) = a + \frac{1}{s-b}$.*

1. *Let $\pi(s) \neq s - a$ be a non-constant monic irreducible polynomial, and let $\widetilde{\pi}_f(s)$ be its monic reversal with respect to f. If $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$ then $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}_f}(s))$.*
2. *If $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{s-a}(s))$ then $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$.*
3. *If $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ then $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{s-b}(s))$.*

Proof. Let $p(s), q(s) \in \mathbb{F}[s]$ with $\deg(p(s)) = d_p$, $\deg(q(s)) = d_q$. Then

$$\frac{p(f(s))}{q(f(s))} = \frac{\mathrm{rev}_f\, p(s)}{\mathrm{rev}_f\, q(s)}(s-b)^{d_q - d_p}.$$

If $\frac{p(s)}{q(s)} \in \mathbb{F}_\pi(s)$ then $\gcd(q(s), \pi(s)) = 1$. Using Lemma 6.8, we have that $\mathrm{rev}_f\, q(s)$ and $\mathrm{rev}_f\, \pi(s)$ are coprime and $\gcd(\mathrm{rev}_f\, \pi(s), s-b) = 1$. Thus, $\frac{p(f(s))}{q(f(s))} \in \mathbb{F}_{\mathrm{rev}_f\, \pi}(s) = \mathbb{F}_{\widetilde{\pi}_f}(s)$. In consequence $U(f(s)) \in \mathbb{F}_{\widetilde{\pi}_f}(s)^{m\times m}$. Furthermore, by hypothesis $\det U(s) = \frac{\hat{p}(s)}{\hat{q}(s)}$ is a unit of $\mathbb{F}_\pi(s)$. In addition, since the Möbius transformations are homomorphisms on the field of rational functions, $\det(U(f(s))) = \frac{\hat{p}(f(s))}{\hat{q}(f(s))}$ and so it is a unit of $\mathbb{F}_{\widetilde{\pi}_f}(s)$. In conclusion, $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}_f}(s))$.

Items 2 and 3 are proved in Lemma 4.6 of [1]. $\square$

Using Lemma 6.9 the proof of the following Proposition is immediate.

Proposition 6.10. *Let $R_1(s)$, $R_2(s) \in \mathbb{F}(s)^{m\times n}$. Let $a, b \in \mathbb{F}$ be arbitrary scalars and $f(s) = a + \frac{1}{s-b}$.*

1. *Let $\pi(s) \neq s - a$ be a non-constant monic irreducible polynomial, and let $\widetilde{\pi}_f(s)$ be its monic reversal with respect to f. $R_1(s)$, $R_2(s)$ are locally equivalent at $\pi(s)$ if and only if $R_1(f(s))$, $R_2(f(s))$ are locally equivalent at $\widetilde{\pi}_f(s)$.*

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2. *$R_1(s)$ and $R_2(s)$ are locally equivalent at $s-a$ if and only if $R_1(f(s))$, $R_2(f(s))$ are equivalent at infinity.*
3. *$R_1(s)$ and $R_2(s)$ are equivalent at infinity if and only if $R_1(f(s))$, $R_2(f(s))$ are locally equivalent at $s-b$.*

The following result describes the change of the finite and infinite structures of a rational matrix under the action of transformation f.

PROPOSITION 6.11. *Let $R(s) \in \mathbb{F}(s)^{m\times n}$ of rank r. Let $a, b \in \mathbb{F}$ be arbitrary scalars and $f(s) = a + \frac{1}{s-b}$. Let $h_1 \leq \cdots \leq h_r$ be integers.*

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1. *Let $\pi(s) \neq s-a$ be a non-constant monic irreducible polynomial, and let $\widetilde{\pi}_f(s)$ be its monic reversal polynomial with respect to f. The functions $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\widetilde{\pi}_f(s)^{h_1}, \ldots, \widetilde{\pi}_f(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}_f(s)$ of $R(f(s))$.*
2. *$(s-a)^{h_1}, \ldots, (s-a)^{h_r}$ are the local invariant rational functions at $s-a$ of $R(s)$ if and only if $h_1, \ldots, h_r$ are the invariant orders at infinity of $R(f(s))$.*
3. *$h_1, \ldots, h_r$ are the invariant orders at infinity of $R(s)$ if and only if $(s-b)^{h_1}, \ldots, (s-b)^{h_r}$ are the local invariant rational functions at $s-b$ of $R(f(s))$.*

Proof.

1. Set $p = \deg(\pi(s))$ and let $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ be the local invariant rational functions at $\pi(s)$ of $R(s)$. There exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_\pi(s))$ such that
$$R(s) = U(s) \begin{bmatrix} \mathrm{Diag}(\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$
Recall that $\widetilde{\pi}_f(s) = \frac{1}{\pi_0} rev_f \pi(s) = \frac{1}{\pi_0}(s-b)^p \pi(f(s))$, $\pi_0 = \pi(a)$. Thus, $\pi(f(s))^{h_i} = \left(\frac{\pi_0}{(s-b)^p}\right)^{h_i} \widetilde{\pi}_f(s)^{h_i}$. Now
$$R(f(s)) \quad = \quad U(f(s)) \begin{bmatrix} \mathrm{Diag}(\pi(f(s))^{h_1}, \ldots, \pi(f(s))^{h_r}) & 0 \\ 0 & 0 \end{bmatrix} V(f(s)) \quad =$$
$$U(f(s)) \begin{bmatrix} \mathrm{Diag}\left(\left(\frac{\pi_0}{(s-b)^p}\right)^{h_1} \widetilde{\pi}_f(s)^{h_1}, \ldots, \left(\frac{\pi_0}{(s-b)^p}\right)^{h_r} \widetilde{\pi}_f(s)^{h_r}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s))$$
$$= \quad U(f(s))W(s) \begin{bmatrix} \mathrm{Diag}\left(\widetilde{\pi}_f(s)^{h_1}, \ldots, \widetilde{\pi}_f(s)^{h_r}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s))$$
with $W(s) = \mathrm{Diag}\left(\left(\frac{\pi_0}{(s-b)^p}\right)^{h_1}, \ldots, \left(\frac{\pi_\ell}{(s-b)^p}\right)^{h_r}, 1, \ldots, 1\right)$.

On the one hand, by Lemma 6.9, $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}_f}(s))$ and $V(f(s)) \in \mathrm{Gl}_n(\mathbb{F}_{\widetilde{\pi}_f}(s))$. On the other hand, $W(s) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}_f}(s))$ because $\gcd(\widetilde{\pi}_f(s), s-b) = 1$ (see Lemma 6.8). Therefore, $U(f(s))W(s) \in \mathrm{Gl}_m(\mathbb{F}_{\widetilde{\pi}_f}(s))$. In conclusion, $\widetilde{\pi}_f(s)^{h_1}, \ldots, \widetilde{\pi}_f(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}_f(s)$ of $R(f(s))$, as desired.

The proof of the converse is the same taking into account that $R(f(f^{-1}(s))) = R(s)$ and $\mathrm{rev}_{f^{-1}} \mathrm{rev}_f \pi(s) = \pi(s)$ because $\pi(s)$ is a non-constant irreducible polynomial and $\pi(s) \neq s-a$.

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2. If $(s-a)^{h_1}$,..., $(s-a)^{h_r}$ are the invariant rational functions at $s-a$ of $R(s)$ then there exist matrices $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{s-a}(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_{s-a}(s))$ such that

$$R(s) = U(s) \begin{bmatrix} \mathrm{Diag}((s-a)^{h_1}, \dots, (s-a)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$

Thus,

$$R(f(s)) \quad = \quad U(f(s)) \begin{bmatrix} \mathrm{Diag}\left(\frac{1}{(s-b)^{h_1}}, \dots, \frac{1}{(s-b)^{h_r}}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s)).$$

By Lemma 6.9, $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ and $V(f(s)) \in \mathrm{Gl}_n(\mathbb{F}_{pr}(s))$. Hence, $h_1 \leq \cdots \leq h_r$, are the invariant orders at infinity of $R(f(s))$.

Since $R(f(f^{-1}(s))) = R(s)$, $U(f(f^{-1}(s))) = U(s)$, $V(f(f^{-1}(s))) = V(s)$ and $\frac{1}{(f^{-1}(s)-b)^{h_i}} = (s-a)^{h_i}$, we can reverse the arguments to prove the converse.

3. Let $h_1 \leq \cdots \leq h_r$ be the invariant orders at infinity of $R(s)$. There exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_{pr}(s))$ such that

$$R(s) = U(s) \begin{bmatrix} \mathrm{Diag}\left(\frac{1}{(s-a)^{h_1}}, \dots, \frac{1}{(s-a)^{h_r}}\right) & 0 \\ 0 & 0 \end{bmatrix} V(s).$$

Changing s by $f(s)$ we get

$$R(f(s)) = U(f(s)) \begin{bmatrix} \mathrm{Diag}\left((s-b)^{h_1}, \dots, (s-b)^{h_r}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s))$$

with $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{s-b}(s))$, $V(f(s)) \in \mathrm{Gl}_n(\mathbb{F}_{s-b}(s))$ (see Lemma 6.9). Since $h_1 \leq \cdots \leq h_r$, $(s-b)^{h_1}, \dots, (s-b)^{h_r}$ are the local invariant rational functions at $s-b$ of $R(f(s))$.

The converse is proved as in the previous items. $\square$

We come back now to the general Möbius transformations (6.1) with $\gamma \neq 0$. Recall that they are obtained by composing transformations of the form

$$f(s) = a + \frac{1}{s-b}$$

and dilations

$$t(s) = cs,$$

with $a = \frac{\alpha\gamma}{\beta\gamma - \alpha\delta}$, $b = -\frac{\delta}{\gamma}$ and $c = \frac{\beta\gamma-\alpha\delta}{\gamma^2}$.

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By putting together the results in this and the previous section, we can characterize the change of the finite and infinite structures by a general Möbius transformation on any rational matrix provided that they have the same size.

PROPOSITION 6.12. *Let $R_1(s)$, $R_2(s) \in \mathbb{F}(s)^{m\times n}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ be arbitrary scalars such that $\gamma \neq 0$ and $\beta\gamma - \alpha\delta \neq 0$.*

1. *Let $\pi(s) \neq s - \frac{\alpha}{\gamma}$ be a non-constant monic irreducible polynomial of degree p and $\widetilde{\pi}(s) = \frac{1}{\pi_0}\left(s + \frac{\delta}{\gamma}\right)^p \pi\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$, where $\pi_0 = \pi\left(\frac{\alpha}{\gamma}\right)$. $R_1(s)$ and $R_2(s)$ are locally equivalent at $\pi(s)$ if and only if $R_1\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ and $R_2\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ are locally equivalent at $\widetilde{\pi}(s)$.*
2. *$R_1(s)$ and $R_2(s)$ are locally equivalent at $s - \frac{\alpha}{\gamma}$ if and only if $R_1\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ and $R_2\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ are equivalent at infinity.*
3. *$R_1(s)$ and $R_2(s)$ are equivalent at infinity if and only if $R_1\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ and $R_2\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ are locally equivalent at $s + \frac{\delta}{\gamma}$.*

Proof. Let $a = \frac{\alpha\gamma}{\beta\gamma-\alpha\delta}$, $b = -\frac{\delta}{\gamma}$ and $c = \frac{\beta\gamma-\alpha\delta}{\gamma^2} \neq 0$ and $f(s) = a + \frac{1}{s-b}$.

1. By Proposition 6.3, $R_1(s)$ and $R_2(s)$ are locally equivalent at $\pi(s)$ if and only if $T_1(s) := R_1(cs)$ and $T_2(s) := R_2(cs)$ are locally equivalent at $\nu(s) = \frac{1}{c^p}\pi(cs)$. And by Proposition 6.10, $T_1(s)$ and $T_2(s)$ are locally equivalent at $\nu(s)$ if and only if $T_1(f(s)) = R_1(cf(s))$ and $T_2(f(s)) = R_2(cf(s))$ are locally equivalent at $\widetilde{\nu}_f(s) = \frac{1}{\nu_0}\operatorname{rev}_f \nu(s)$ ($\nu_0 = \nu(a)$) provided that $\nu(s) \neq s - a$.

   On the one hand, $\nu(s)$ and $\pi(s)$ have the same degree and so $\nu(s) = s - a$ if and only if $\pi(s) = s - ac = s - \frac{\alpha}{\gamma}$. And on the other hand,

   $$\widetilde{\nu}_f(s) = \frac{1}{\nu_0}\operatorname{rev}_f \nu(s) = \frac{1}{\nu(a)}(s-b)^p \nu(f(s)) = \frac{1}{\nu(a)c^p}(s-b)^p \pi(cf(s)).$$

   But, $\nu(a)c^p = \pi(ca) = \pi\left(\frac{\alpha}{\gamma}\right) = \pi_0$, $b = -\frac{\delta}{\gamma}$ and $cf(s) = \frac{\alpha s+\beta}{\gamma s+\delta}$. Thus,

   $$\widetilde{\nu}_f(s) = \frac{1}{\pi_0}\left(s + \frac{\delta}{\gamma}\right)^p \pi\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) = \widetilde{\pi}(s).$$

   In conclusion, provided that $\pi(s) \neq s - \frac{\alpha}{\gamma}$, $R_1(s)$ and $R_2(s)$ are locally equivalent at $\pi(s)$ if and only if $R_1\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ and $R_2\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ are locally equivalent at $\widetilde{\pi}(s)$, as claimed.

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2. By Proposition 6.3, $R_1(s)$ and $R_2(s)$ are locally equivalent at $\pi(s) = s - \frac{\alpha}{\gamma}$ if and only if $T_1(s) := R_1(cs)$ and $T_2(s) := R_2(cs)$ are locally equivalent at

$$\widetilde{\pi}(s) = \frac{\pi(cs)}{c} = s - \frac{\alpha}{c\gamma} = s - a.$$

By Proposition 6.10, $T_1(s)$ and $T_2(s)$ are locally equivalent at $s-a$ if and only if $T_1(f(s)) = R_1(cf(s))$ and $T_2(f(s)) = R_2(cf(s))$ are equivalent at infinity.

3. This item is proved very similarly. By Proposition 6.5, $R_1(s)$ and $R_2(s)$ are equivalent at infinity if and only if $T_1(s) = R_1(cs)$ and $T_2(s) = R_2(cs)$ are equivalent at infinity. And, by Proposition 6.10, this is true if and only if $T_1(f(s)) = R_1(cf(s))$ and $T_2(f(s)) = R_2(cf(s))$ are equivalent at $s - b = s + \frac{\delta}{\gamma}$. □

PROPOSITION 6.13. *Let $R(s) \in \mathbb{F}(s)^{m\times n}$ of rank r. Let $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ be arbitrary scalars such that $\gamma \neq 0$ and $\beta\gamma - \alpha\delta \neq 0$. Let also $h_1 \leq \cdots \leq h_r$ be integers.*

1. *Let $\pi(s) \neq s - \frac{\alpha}{\gamma}$ be a non-constant monic irreducible polynomial of degree p and $\widetilde{\pi}(s) = \frac{1}{\pi_0}\left(s + \frac{\delta}{\gamma}\right)^p \pi\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$, where $\pi_0 = \pi\left(\frac{\alpha}{\gamma}\right)$. Then $\pi(s)^{h_1}$, $\dots$, $\pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\widetilde{\pi}(s)^{h_1}, \dots, \widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s)$ of $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$.*
2. *$(s - \frac{\alpha}{\gamma})^{h_1}, \dots, (s - \frac{\alpha}{\gamma})^{h_r}$ are the local invariant rational functions at $s - \frac{\alpha}{\gamma}$ of $R(s)$ if and only if $h_1, \cdots, h_r$ are the invariant orders at infinity of $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$.*
3. *$h_1, \cdots, h_r$ are the invariant orders at infinity of $R(s)$ if and only if $\left(s + \frac{\delta}{\gamma}\right)^{h_1}$, $\dots$, $\left(s + \frac{\delta}{\gamma}\right)^{h_r}$ are the local invariant rational functions at $s+\frac{\delta}{\gamma}$ of $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$.*

Proof. The proof is a direct consequence of Propositions 6.12, 6.11, 6.4 and 6.6. We only prove item 1. Items 2 and 3 are proved similarly.

We will use the same notation as in the proof of Proposition 6.12. In particular, $a = \frac{\alpha\gamma}{\beta\gamma-\alpha\delta}$, $b = -\frac{\delta}{\gamma}$, $c = \frac{\beta\gamma-\alpha\delta}{\gamma^2}$, $\nu(s) = \frac{1}{c^p}\pi(cs)$ and $\widetilde{\nu}_f(s) = \frac{1}{\nu_0}\operatorname{rev}_f \nu(s)$ with $\nu_0 = \nu(a)$ and $\nu(s) \neq s - a$ when $\pi(s) \neq s - \frac{\alpha}{\gamma}$. We recall also that $\widetilde{\nu}_f(s) = \widetilde{\pi}(s)$.

Now, by Proposition 6.4, $\pi(s)^{h_1}, \dots, \pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\nu(s)^{h_1}, \dots, \nu(s)^{h_r}$ are the local invariant rational functions at $\nu(s)$ of $R(cs)$. And by Proposition 6.11, $\nu(s)^{h_1}, \dots, \nu(s)^{h_r}$ are the local invariant rational functions at $\nu(s)$ of $R(cs)$ if and only if $\widetilde{\nu}_f(s)^{h_1}, \dots, \widetilde{\nu}_f(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\nu}_f(s)$ of $R(cf(s))$. That is to say, $\widetilde{\pi}(s)^{h_1}, \dots, \widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s)$ of $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$, as desired. □

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6.3. The case of matrix polynomials. Matrix polynomials (or polynomial matrices) can be seen as matrices of rational functions and so, the results of the previous sections hold for them. However, the notion of infinite elementary divisor (or elementary divisor at infinity) is particularly attached to matrix polynomials.

Consider the transformation of the form $f(s) = a + \frac{1}{s-b}$ with $a = b = 0$. This particular transformation is used over matrix polynomials to define their infinite elementary divisors as follows: Let $P(s) \in \mathbb{F}[s]^{m\times n}$, $d = \deg(P(s))$. The *infinite elementary divisors* of $P(s)$ are defined as the finite elementary divisors which are powers of s of the polynomial matrix $s^d P\left(\frac{1}{s}\right)$. In other words, the infinite elementary divisors of $P(s)$ are the non-trivial local invariant rational functions at s of the reversal of $P(s)$ with respect to $f(s) = \frac{1}{s}$. It is usual to use $\operatorname{rev} P(s)$ (instead of $\operatorname{rev}_f P(s)$) to denote the reversal of $P(s)$ with respect to $f(s) = \frac{1}{s}$.

Recall that in Section 5, two important properties about the infinite structure of matrix polynomials were pointed out: All non-constant matrix polynomials have poles at infinity and if $q_1 \leq \cdots \leq q_r$ are the invariant orders at infinity of a matrix polynomial, then $-q_1$ is its degree. Therefore $P(\frac{1}{s})$ necessarily has poles at s. Actually, by Proposition 6.13, $s^{q_1}, \ldots, s^{q_r}$ are the local invariant rational functions of $P(\frac{1}{s})$ at s. Hence, bearing in mind that $d = -q_1$ is the degree of $P(s)$, we conclude that s^0, $s^{q_2-q_1}$,..., $s^{q_r-q_1}$ are the local invariant rational functions at s of $\operatorname{rev} P(s) = s^d P(\frac{1}{s})$; i.e., the infinite elementary divisors of $P(s)$ are the monomials in s^0, $s^{q_2-q_1}$,..., $s^{q_r-q_1}$ whose exponents are not zero. This proves the following proposition that generalizes [24, Corollary 4.41], where the same result was obtained for regular matrix polynomials. By notational convenience, we will include elementary divisors with exponents equal to zero in this and the next propositions.

PROPOSITION 6.14. *Let $P(s) \in \mathbb{F}[s]^{m\times n}$, $\operatorname{rank} P(s) = r$ and $\deg(P(s)) = d$. Let $s^{e_1}, \ldots, s^{e_r}$ be its infinite elementary divisors (including exponents equal to zero) and let $q_1, \ldots, q_r$ be its invariant orders at infinity. Then*

$$e_i = d + q_i = q_i - q_1, \quad i = 1, \ldots, r.$$

REMARK 6.15. It should be noticed that, unlike for the finite elementary divisors, the number of infinite elementary divisors (with non-zero exponents) of any matrix polynomial is at most $r - 1$, r being its rank.

We deal now with the problem of the relationship between the finite and infinite elementary divisors of two matrix polynomials obtained from each other by a Möbius transformation. The following result was proved in [27] for nonsingular polynomial matrices using a completely different technique (see also [16, 19]).

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PROPOSITION 6.16. *Let $R(s) \in \mathbb{F}[s]^{m\times n}$ be a matrix polynomial of rank r and degree d. Let $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ be arbitrary scalars such that $\beta\gamma - \alpha\delta \neq 0$. Let also $0 \leq h_1 \leq \cdots \leq h_r$ be nonnegative integers.*

(i) Assume that $\gamma \neq 0$. Let $\widetilde{R}(s) = \left(s + \frac{\delta}{\gamma}\right)^d R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$.

1. *Let $\pi(s) \neq s - \frac{\alpha}{\gamma}$ be any non-constant monic irreducible polynomial of degree p and $\widetilde{\pi}(s) = \frac{1}{\pi_0}\left(s + \frac{\delta}{\gamma}\right)^p \pi\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$, where $\pi_0 = \pi\left(\frac{\alpha}{\gamma}\right)$. Then $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\widetilde{\pi}(s)^{h_1}, \ldots, \widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s)$ of $\widetilde{R}(s)$.*
2. *$(s-\frac{\alpha}{\gamma})^{h_1}, \ldots, (s-\frac{\alpha}{\gamma})^{h_r}$ are the local invariant rational functions at $s-\frac{\alpha}{\gamma}$ of $R(s)$ if and only if $d - h_1$ is the degree of $\widetilde{R}(s)$ and $s^0, s^{h_2-h_1}, \ldots, s^{h_r-h_1}$ are its infinite elementary divisors (including exponents equal to 0).*
3. *$s^{h_1}, \ldots, s^{h_r}$ are the infinite elementary divisors of $R(s)$ ($h_1 = 0$ and including possible exponents equal to 0) if and only if $\left(s + \frac{\delta}{\gamma}\right)^{h_1}, \ldots, \left(s + \frac{\delta}{\gamma}\right)^{h_r}$ are the local invariant rational functions at $s + \frac{\delta}{\gamma}$ of $\widetilde{R}(s)$.*

(ii) If $\gamma = 0$, the infinite elementary divisors of $R(s)$ and $R(\alpha s + \beta)$ are the same and for any monic non-constant irreducible polynomial of degree p, $\pi(s) \in \mathbb{F}[s]$, $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ are the local invariant rational functions at $\pi(s)$ of $R(s)$ if and only if $\widetilde{\pi}(s)^{h_1}, \ldots, \widetilde{\pi}(s)^{h_r}$ are the local invariant rational functions at $\widetilde{\pi}(s) = \frac{1}{\alpha^p}\pi(\alpha s + \beta)$ of $R(\alpha s + \beta)$.

Proof. (ii) follows from Propositions 6.6 and 6.4.

In order to prove item 1 of (i), notice that $\gcd\left(\widetilde{\pi}(s), s + \frac{\delta}{\gamma}\right) = 1$ for any non-constant irreducible polynomial $\pi(s) \neq s - \frac{\alpha}{\gamma}$. Hence, $\widetilde{R}(s) = \left(s + \frac{\delta}{\gamma}\right)^d R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ and $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$ are locally equivalent at $\widetilde{\pi}(s)$; i.e., they have the same local invariant rational functions at $\widetilde{\pi}(s)$. The result follows from item 1 of Proposition 6.13.

Now, by item 2 of Proposition 6.13, $(s-\frac{\alpha}{\gamma})^{h_1}, \ldots, (s-\frac{\alpha}{\gamma})^{h_r}$ are the local invariant rational functions at $s - \frac{\alpha}{\gamma}$ of $R(s)$ if and only if $h_1, \ldots, h_r$ are the invariant orders at infinity of $R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$. This is so if and only if $h_1 - d, \ldots, h_r - d$ are the invariant orders at infinity of $\widetilde{R}(s) = \left(s + \frac{\delta}{\gamma}\right)^d R\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right)$. Since $\widetilde{R}(s)$ is a matrix polynomial, its degree is minus the smallest invariant order at infinity: $d - h_1$. Now, using Proposition 6.14, we conclude that $e_i = d - h_1 + h_i - d = h_i - h_1$ are the exponents of the infinite elementary divisors of $\widetilde{R}(s)$ (including exponents equal to 0).

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Finally, by Proposition 6.14, s^{h_1}, s^{h_2}, …, s^{h_r} are the infinite elementary divisors of $R(s)$ ($h_1 = 0$ and including possible exponents equal to 0) if and only if $h_1 - d, \dots,$ $h_r - d$ are the invariant orders at infinity of $R(s)$. By item 3 of Proposition 6.13, this is true if and only if $\left(s + \frac{\delta}{\gamma}\right)^{h_1 - d}, \dots, \left(s + \frac{\delta}{\gamma}\right)^{h_r - d}$ are the local invariant rational functions at $s + \frac{\delta}{\gamma}$ of $R\left(\frac{\alpha s + \beta}{\gamma s + \delta}\right)$. An this is equivalent to $\left(s + \frac{\delta}{\gamma}\right)^{h_r}, \dots, \left(s + \frac{\delta}{\gamma}\right)^{h_1}$ being the local invariant rational functions at $s + \frac{\delta}{\gamma}$ of $\widetilde{R}(s)$. $\square$

7. Equivalence that preserves the finite and infinite elementary divisors. In the previous section, and for notational convenience, we have included polynomials equal to 1 in the list of infinite elementary divisors. By definition, the finite and infinite elementary divisors are polynomials different from 1. This will be the setting from now on.

We deal now with the problem of defining an equivalence relation in the set of all matrix polynomials that preserves the finite and infinite elementary divisors. Notice, first of all, that prescribing the infinite structure of a matrix polynomial (i.e., its invariant orders at infinity) determines its degree but this information is removed when the infinite elementary divisors are prescribed. Similarly, two matrix polynomials may have the same elementary divisors with respect to every monic non-constant irreducible polynomial and they may not be globally equivalent because they may differ in rank. In other words, neither the infinite elementary divisors are enough to determine completely an equivalence class of matrix polynomials at infinity, nor the finite elementary divisors are enough to fix a class of globally equivalent matrix polynomials. However the additional information needed is very small: The finite elementary divisors, the rank and the size form a complete system of invariants for the global equivalence of matrix polynomials. Similarly, the infinite elementary divisors, the rank, the size and degree form a complete system of invariants for the equivalence at infinity of matrix polynomials. As a conclusion, if $P_1(s)$ and $P_2(s)$ have the same finite elementary divisors then by expanding them with 1's and 0's we can produce matrices that are globally equivalent (same rank, size and invariant factors). And by multiplying them by appropriate powers of s and expanding them with 1's and 0's we can obtain matrix polynomials that are equivalent at infinity. These are the basic ideas in order to give the following definition (compare with [5, Definition 3.2]).

DEFINITION 7.1. Let two matrix polynomials (of any sizes) $P_1(s)$ and $P_2(s)$ be given. $P_1(s)$ and $P_2(s)$ will be said to be *spectrally equivalent* if there are nonnegative integers ℓ_1, ℓ_2, v_1, v_2, t_1 and t_2, such that the following two conditions hold:

(i) $\begin{bmatrix} I_{\ell_1} & 0 & 0 \\ 0 & P_1(s) & 0 \\ 0 & 0 & 0_{v_1 \times t_1} \end{bmatrix}$ and $\begin{bmatrix} I_{\ell_2} & 0 & 0 \\ 0 & P_2(s) & 0 \\ 0 & 0 & 0_{v_2 \times t_2} \end{bmatrix}$ are globally equivalent,

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and

(ii) $\begin{bmatrix} I_{\ell_1} & 0 & 0 \\ 0 & \operatorname{rev} P_1(s) & 0 \\ 0 & 0 & 0_{v_1\times t_1} \end{bmatrix}$ and $\begin{bmatrix} I_{\ell_2} & 0 & 0 \\ 0 & \operatorname{rev} P_2(s) & 0 \\ 0 & 0 & 0_{v_2\times t_2} \end{bmatrix}$ are locally equivalent at s.

It is not difficult to see that this is an equivalence relation (the only property that requires some care is the transitivity).

The difference between this definition and [5, Definition 3.2] is the set of matrices where the equivalence is defined and that condition (ii) in Definition 3.2 of [5] provides redundant information. Being a slight extension of the equivalence defined in [5] we will maintain the same name.

We prove now that the finite and infinite elementary divisors form a complete system of invariants for the spectral equivalence.

Theorem 7.2. *Let $P_1(s)$ and $P_2(s)$ be two arbitrary polynomial matrices. Then $P_1(s)$ and $P_2(s)$ are spectrally equivalent if and only if they have the same finite and infinite elementary divisors.*

Proof. Put

$$Q_1(s) = \begin{bmatrix} I_{\ell_1} & 0 & 0 \\ 0 & P_1(s) & 0 \\ 0 & 0 & 0_{v_1\times t_1} \end{bmatrix}, \quad Q_2(s) = \begin{bmatrix} I_{\ell_2} & 0 & 0 \\ 0 & P_2(s) & 0 \\ 0 & 0 & 0_{v_2\times t_2} \end{bmatrix}$$

and

$$\widetilde{Q}_1(s) = \begin{bmatrix} I_{\ell_1} & 0 & 0 \\ 0 & \operatorname{rev} P_1(s) & 0 \\ 0 & 0 & 0_{v_1\times t_1} \end{bmatrix}, \quad \widetilde{Q}_2(s) = \begin{bmatrix} I_{\ell_2} & 0 & 0 \\ 0 & \operatorname{rev} P_2(s) & 0 \\ 0 & 0 & 0_{v_2\times t_2} \end{bmatrix}.$$

Notice that all these matrices are of the same size.

It is plain that the finite elementary divisors of $Q_i(s)$ are those of $P_i(s)$, $i = 1, 2$. And the local invariant rational functions at s of $\widetilde{Q}_i(s)$ are, apart from rational functions equal to 1, those of $\operatorname{rev} P_i(s)$, $i = 1, 2$. Now, if $Q_1(s)$ and $Q_2(s)$ are globally equivalent then they have the same finite elementary divisors and so $P_1(s)$ and $P_2(s)$ have the same finite elementary divisors.

Similarly, suppose now that $\widetilde{Q}_1(s), \widetilde{Q}_2(s)$ are locally equivalent at s and let $r_1 = \operatorname{rank} P_1(s) = \operatorname{rank} \operatorname{rev} P_1(s)$, $r_2 = \operatorname{rank} P_2(s) = \operatorname{rank} \operatorname{rev} P_2(s)$. Then $\operatorname{rank} \widetilde{Q}_i(s) = \ell_i + r_i$, $i = 1, 2$. Let the local invariant rational functions at s of $\operatorname{rev} P_1(s)$ be $s^{h_1} \mid \cdots \mid s^{h_{r_1}}$ and let $s^{g_1} \mid \cdots \mid s^{g_{r_2}}$ be those of $\operatorname{rev} P_2(s)$. Then $1, \ldots, 1 (\ell_1 \text{ times}), s^{h_1}, \ldots, s^{h_{r_1}}$ and $1, \ldots, 1 (\ell_2 \text{ times}), s^{g_1}, \ldots, s^{g_{r_2}}$ are the invariant rational functions at s of $\widetilde{Q}_1(s)$ and

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$\widetilde{Q}_2(s)$, respectively. Then $\ell_1 + r_1 = \ell_2 + r_2$ and the two sequences of polynomials $1, \ldots, 1(\ell_1 \text{ times}), s^{h_1}, \ldots, s^{h_{r_1}}$ and $1, \ldots, 1(\ell_2 \text{ times}), s^{g_1}, \ldots, s^{g_{r_2}}$ coincide. Let $e_1 \leq \cdots \leq e_u$ be the common non-zero exponents in these sequences; i.e.,

$$\{e_1, \ldots, e_u\} := \{h_i \neq 0, 1 \leq i \leq r_1\} = \{g_i \neq 0 : 1 \leq i \leq r_2\}.$$

Thus, $1, \ldots, 1(r_1 - u \text{ times}), s^{e_1}, \ldots, s^{e_u}$ and $1, \ldots, 1(r_2 - u \text{ times}), s^{e_1}, \ldots, s^{e_u}$ are the local invariant rational functions at s of $\operatorname{rev} P_1(s)$ and $\operatorname{rev} P_2(s)$, respectively. It follows from item 3 of (i) of Proposition 6.16 with $\alpha = \delta = 0$ and $\beta = \gamma = 1$ that $s^{e_1}, \ldots, s^{e_u}$ are the infinite elementary divisors of $P_1(s)$ and $P_2(s)$.

Conversely, assume that $P_1(s) \in \mathbb{F}[s]^{m_1 \times n_1}$ and $P_2(s) \in \mathbb{F}[s]^{m_2 \times n_2}$ have the same finite and infinite elementary divisors. Let $r_i = \operatorname{rank} P_i(s)$, $i = 1, 2$ and let ℓ_i, v_i, t_i, $i = 1, 2$, non-negative integers such that $\ell_1 + r_1 = \ell_2 + r_2$, $\ell_1 + m_1 + v_1 = \ell_2 + m_2 + v_2$ and $\ell_1 + n_1 + t_1 = \ell_2 + n_2 + t_2$ and set $m := \ell_1 + m_1 + v_1 = \ell_2 + m_2 + v_2$ and $n := \ell_1 + n_1 + t_1 = \ell_2 + n_2 + t_2$. Then $Q_1(s)$, $Q_2(s)$, $\widetilde{Q}_1(s)$, $\widetilde{Q}_2(s) \in \mathbb{F}[s]^{m \times n}$ and $Q_1(s)$, $Q_2(s)$ have the same finite elementary divisors. Therefore, these matrices are globally equivalent.

In order to prove that $\widetilde{Q}_1(s)$, $\widetilde{Q}_2(s)$ are locally equivalent at s we only have to reverse the proof of the "if" part. Specifically, let $s^{e_1}, \ldots, s^{e_u}$ be the common infinite elementary divisors of $P_1(s)$ and $P_2(s)$. By item (i).3 of Proposition 6.16 with $\alpha = \delta = 0$ and $\beta = \gamma = 1$, it follows that $1, \ldots, 1(r_1 - u \text{ times}), s^{e_1}, \ldots, s^{e_u}$ and $1, \ldots, 1(r_2 - u \text{ times})$, $s^{e_1}, \ldots, s^{e_u}$ are the local invariant rational functions at s of $\operatorname{rev} P_1(s)$ and $\operatorname{rev} P_2(s)$, respectively. Thus, $1, \ldots, 1(\ell_1 + r_1 - u \text{ times}), s^{e_1}, \ldots, s^{e_u}$ are the local invariant rational functions at s of $\widetilde{Q}_1(s)$ and $1, \ldots, 1(\ell_2 + r_2 - u \text{ times}), s^{e_1}, \ldots, s^{e_u}$ are those of $\widetilde{Q}_2(s)$.

Since $\ell_1 + r_1 = \ell_2 + r_2$, we conclude that $\widetilde{Q}_1(s)$ and $\widetilde{Q}_2(s)$ have the same local invariant rational functions at s. Therefore, they are locally equivalent at s. $\square$

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