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THE LAPLACIAN QUADRATIC FORM AND EDGE CONNECTIVITY OF A GRAPH

WILLIAM WATKINS†

Abstract. Let $G$ be a simple connected graph with associated positive semidefinite integral quadratic form $Q(x) = \sum (x(i) - x(j))^2$, where the sum is taken over all edges $ij$ of $G$. It is showed that the minimum positive value of $Q(x)$ for $x \in \mathbb{Z}^n$ equals the edge connectivity of $G$. By restricting $Q(x)$ to $x \in \mathbb{Z}^n - 1 \times \{0\}$, the quadratic form becomes positive definite. It is also showed that the number of minimal disconnecting sets of edges of $G$ equals twice the number of vectors $x \in \mathbb{Z}^{n-1} \times \{0\}$ for which the form $Q$ attains its minimum positive value.

Key words. Graph, Laplacian matrix, Edge connectivity, Integral quadratic form.

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1. Statement of results. Let $G$ be a simple connected graph (no loops or multiple edges). The vertex set for $G$ is $V(G) = \{1, 2, \ldots, n\}$ and the edge set is denoted by $E(G)$. The Laplacian quadratic form associated with $G$ is defined by:

$$Q(x) = \sum_{ij \in E(G)} (x(i) - x(j))^2,$$

for $x = (x(1), \ldots, x(n)) \in \mathbb{Z}^n$. The matrix for this quadratic form is the Laplacian matrix $L(G)$ for the graph. See [2] for a survey of results about the Laplacian matrix.

A set of $t$ edges $E = \{ij_1, \ldots, ij_t\}$ of $G$ disconnects $G$ if the graph $G' = G - E$, obtained by removing these edges from $G$, is not connected. And the edge connectivity of $G$ is the fewest number of edges that disconnect $G$. We call such a set of edges a minimal disconnecting set of edges of $G$.

Theorem 1.1. Let $G$ be a simple connected graph. Then the least positive value of $Q(x)$ for $x \in \mathbb{Z}^n$ equals the edge connectivity of $G$.

Let $k$ be the common value of the least positive value of $Q(x)$ and the edge connectivity of $G$. The next theorem compares the number of minimal disconnecting sets of edges

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of $G$ with the number of integral vectors $x = (x(1), x(2), \ldots, x(n-1), 0)$ for which $Q(x) = k$.

Theorem 1.2. Let $G$ be a simple connected graph and let $k$ be the edge connectivity of $G$. Then the number of vectors $x \in \mathbb{Z}^n$ with $x(n) = 0$ such that $Q(x) = k$ is twice the number of minimal disconnecting sets of edges of $G$.

The restriction of the vectors $x \in \mathbb{Z}^n$ to those with $x(n) = 0$ is necessary because $Q(x)$ is not positive definite. Indeed its null space is spanned by the all-ones vector $e = (1, 1, \ldots, 1)$ and so if $Q(x) = k$ then $Q(x + ze) = k$ for every integer $z$. Thus, there are infinitely many vectors $y$ in $\mathbb{Z}^n$ for which $Q(y) = k$. But the restriction of the quadratic form to $\mathbb{Z} = \{x \in \mathbb{Z}^n : x(n) = 0\}$ is positive definite, which implies that there are only finitely many vectors $y \in \mathbb{Z}$ such that $Q(y) = k$. Furthermore, the positive integers represented by $Q$ over $\mathbb{Z}^n$ are the same as those represented by $Q$ over $\mathbb{Z}$ because $Q(x) = Q(y)$ for $y = x - x(n)e \in \mathbb{Z}$.

Before proceeding to the proofs, we insert a few remarks about the relationship between the quadratic form $Q$ and its restriction to $\mathbb{Z}$. If we view the restriction as a quadratic form over $(x(1), x(2), \ldots, x(n-1)) \in \mathbb{Z}^{n-1}$, then its matrix is the principal sub matrix of the Laplacian $L(G)$ in rows and columns $1, 2, \ldots, n-1$. The famous matrix tree theorem of Kirchhoff [1, 2] states that the determinant of every $(n-1) \times (n-1)$ sub matrix of $L(G)$ equals plus or minus the number of spanning trees of $G$. In addition, all of the $(n-1) \times (n-1)$ principal sub matrices of $L(G)$ are congruent to each other by a unimodular matrix [3, 4]. So there is nothing special about restricting $Q$ to vectors with $x(n) = 0$. Indeed, if we restrict $Q$ by taking $x \in \mathbb{Z}^n$ with $x(i) = 0$ for some other vertex $i$ instead of $x(n) = 0$, all of the resulting quadratic forms are equivalent to each other.

We should also note that the Laplacian matrices $L(G_1), L(G_2)$ are congruent by a unimodular matrix if and only if the graphs $G_1, G_2$ are cycle isomorphic [3, 4]. Thus, every invariant for unimodular congruence is shared by all graphs in the same cycle-isomorphism class.

2. Proofs. Let $G$ be a simple connected graph, $k$ be the edge connectivity of $G$, and $m$ be the minimum positive integer represented by $Q$. The general outline for the proofs is to show that $m = k$ and that if $Q(x) = m$ for $x \in \mathbb{Z}$ then all the coordinates of $x$ are either in $\{0, 1\}$ or all are in $\{0, -1\}$. Then we establish a bijection between the minimal disconnecting sets of edges of $G$ and the vectors $x \in \{0, 1\}^{n-1} \times \{0\}$ with $Q(x) = m$. This will prove Theorem 1.2 because if $Q(x) = m$ for some $x \in \mathbb{Z}$ then
$Q(-x) = m$ as well. Thus, every pair of vectors $\pm x$ with $Q(x) = m$ corresponds to a minimal disconnecting set of edges of $G$.

2.1. A lemma from graph theory. We need the following lemma about connected graphs:

**Lemma 2.1.** Let $G$ be a simple connected graph and $E = \{i_1j_1, \ldots, i_kj_k\}$ be a minimal disconnecting set of edges of $G$. Then the graph $G' = G - E$ obtained by removing the edges in $E$ has exactly two connected components.

**Proof.** Since $E$ disconnects $G$, $G'$ has at least two components. Suppose it has more than two components. The vertices $i_k, j_k$ are in just one or two of the components leaving a third component whose vertices do not include either $i_k$ or $j_k$. It follows that this third component is still a component of the subgraph $G'' = G - \{i_1j_1, \ldots, i_{k-1}j_{k-1}\}$. Thus, $\{i_1j_1, \ldots, i_{k-1}j_{k-1}\}$ disconnects $G$, which contradicts the minimality of $k$. \(\square\)

2.2. Notation. We use the following notation: For a positive integer $l$, let

\[ \mathcal{X}(l) = \{ x \in \{0,1\}^{n-1} \times \{0\} : Q(x) = l \}, \]

\[ \mathcal{E}(l) = \{ E \subseteq E(G) : E \text{ disconnects } G \text{ and } |E| = l \}. \]

Of course, $\mathcal{X}(l)$ is empty if $l < m$ and $\mathcal{E}(l)$ is empty if $l < k$. Later we will show that $\mathcal{X}(m)$ is not empty. That is, there is a $(0,1)$ vector $x$ with $Q(x) = m$.

For each $x \in \{0,1\}^{n-1} \times \{0\}$, partition the vertices of $G$ into two sets:

\[ V_0(x) = \{ i \in \{1,2,\ldots,n\} : x(i) = 0 \}, \]

\[ V_1(x) = \{ i \in \{1,2,\ldots,n\} : x(i) = 1 \}, \]

and the edges of $G$ into three sets:

\[ E_0(x) = \{ ij \in E(G) : x(i) = x(j) = 0 \}, \]

\[ E_1(x) = \{ ij \in E(G) : x(i) = x(j) = 1 \}, \]

\[ E_{01}(x) = \{ ij \in E(G) : x(i) = 0 \text{ and } x(j) = 1, \text{ or } x(i) = 1 \text{ and } x(j) = 0 \}. \]

One thing is already clear: If $x \in \{0,1\}^{n-1} \times \{0\}$ then

\[ |E_{01}(x)| = Q(x). \] (2.1)

Since $E_0(x)$, $E_1(x)$, $E_{01}(x)$ partition the edges of $G$, the sum $\sum (x(i) - x(j))^2$ over all edges $ij$ of $G$ equals the sum of three sums: Over edges in $E_0(x)$, edges in $E_1(x)$ and edges in $E_{01}(x)$. The first and second sums are zero and the third sum equals $|E_{01}(x)|$. 


2.3. The map \( \theta : \mathcal{E}(k) \to \mathcal{X}(k) \). Let \( k \) be the edge connectivity of \( G \) and let \( E \in \mathcal{E}(k) \) be a minimal disconnecting set of edges of \( G \). By Lemma 2.1, the subgraph \( G' = G - E \) has two connected components, \( H_0, H_1 \). To be definite we take \( H_0 \) to be the component containing vertex \( n \). Define \( x_E \in \{0,1\}^{n-1} \times \{0\} \) by

\[
x_E(i) = \begin{cases} 
0, & \text{if } i \text{ is a vertex of } H_0, \\
1, & \text{if } i \text{ is a vertex of } H_1. 
\end{cases}
\]

The edges of \( G \) are partitioned by the edges of \( H_0 \), the edges of \( H_1 \), and \( E \). Thus, \( Q(x_E) = |E| = k \). So, \( x_E \in \mathcal{X}(k) \) and the function \( E \to x_E \) maps \( \mathcal{E}(k) \) into \( \mathcal{X}(k) \). It follows from the minimality of \( E \) that \( m \leq k \).

2.4. \( \mathcal{X}(m) \) is not empty. Again let \( m \) be the minimum positive integer represented by \( Q \), say \( Q(x) = m \) for some \( x \in \mathcal{Z} \). Define a zero-one vector \( y \) by \( y(i) = 0 \) whenever \( x(i) \) is even and \( y(i) = 1 \) whenever \( x(i) \) is odd. Since \( x(n) = 0 \) is even, \( y(n) = 0 \). Now \( y \neq 0 \) because if all the coordinates of \( x \) are even, then \( x/2 \in \mathcal{Z} \) and \( Q(x/2) = m/4 \), which contradicts the minimality of \( m \). Clearly, \( Q(y) \leq Q(x) = m \). Since \( y \neq 0 \) and \( m \) is minimal we have \( Q(y) = m \). That is \( y \in \mathcal{X}(m) \), which shows that \( \mathcal{X}(m) \) is not empty.

2.5. \( m = k \). Let \( y \) be any vector in \( \mathcal{X}(m) \). Then \( E_{01}(y) \) is a disconnecting set of edges of \( G \) and (by Equation (2.1)) \( |E_{01}(y)| = Q(y) = m \). From the minimality of \( k \), we have \( k \leq m \). Therefore, \( k = m \) and Theorem 1.1 is proved.

From here on we use \( k \) to denote both the minimum positive value of \( Q(x) \) and the edge connectivity of \( G \).

2.6. \( \theta : \mathcal{E}(k) \to \mathcal{X}(k), x \to x_E \) is one-to-one. Let \( E, F \) be disconnecting sets of edges in \( \mathcal{E}(k) \) with \( x_E = x_F \). Then

\[
G - E = H_0 + H_1, \\
G - F = K_0 + K_1,
\]

where \( H_0, H_1 \) are the components of \( G - E \), \( K_0, K_1 \) are the components of \( G - F \), and \( n \) is a vertex in \( H_0 \) and \( K_0 \). Since \( x_E = x_F \) we have \( i \in V(H_0) \) if and only if \( i \in V(K_0) \). Thus, \( V(H_0) = V(K_0) \). The edges of \( H_0 \) are just the edges \( ij \) of \( G \) with \( i, j \in V(H_0) \). It follows that \( E(H_0) = E(K_0) \). Similarly \( E(H_1) = E(K_1) \). The edges of \( G \) are partitioned in two ways

\[
E(G) = E(H_0) \cup E(H_1) \cup E, \\
E(G) = E(K_0) \cup E(K_1) \cup F.
\]

Thus, \( E = F \).
2.7. **θ : E(k) → X(k), x ↦ x_E is onto.** Let \( x \in X(k) \). We must show that there exists \( E \in E(k) \) such that \( x = x_E \). The obvious, and correct, candidate is \( E = E_{00}(x) \).

Let \( H_i(x) \) be the subgraph of \( G \) with vertices \( V_i(x) \) and edges \( E_i(x) \) for \( i = 1, 2 \). Clearly, \( H_0(x), H_1(x) \) are the components of \( G' = G - E \). So \( x_E(ij) = 0 \) if and only if \( i \) is a vertex of \( H_0(x) \). Also \( x(i) = 0 \) if and only if \( i \in V_0(x) = V(H_0) \). So \( x_E = x \) and \( θ \) maps \( E(k) \) onto \( X(k) \).

We have proved that \(|E(k)| = |X(k)|\).

2.8. **If \( x \in X \) and \( Q(x) = k \) then \( x \in X(k) \) or \( -x \in X(k) \).** In this section, we show that the only vectors \( x \in X \) for which \( Q \) achieves the minimum positive value \( k \) are those all of whose coordinates are in \( \{0, 1\} \) or all are in \( \{0, -1\} \).

Suppose \( x \in X \) and \( Q(x) = k \). Define a vector \( y \in \{0, 1\}^{n-1} \times \{0\} \) by

\[
y(i) = \begin{cases} 
0 & \text{if } x(i) \text{ is even}, \\
1 & \text{if } x(i) \text{ is odd}.
\end{cases}
\]

Arguing as in Section 2.5, we get \( y \neq 0 \). Now partition the edges of \( G \) into three sets, \( E_0(y), E_1(y), \) and \( E_{01}(y) \). It is clear that \((y(i) - y(j))^2 \leq (x(i) - x(j))^2 \), for all \( i, j \).

Therefore, we have the following inequalities for the sums:

\[
0 = \sum_{ij \in E_0(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_0(y)} (x(i) - x(j))^2 \\
0 = \sum_{ij \in E_1(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_1(y)} (x(i) - x(j))^2 \\
k = \sum_{ij \in E_{01}(y)} (y(i) - y(j))^2 \leq \sum_{ij \in E_{01}(y)} (x(i) - x(j))^2.
\]

But \( Q(x) \), which is the sum of the three sums above on the right, equals \( k \). Therefore, \( Q(y) = k \) and \( y \in X(k) \). In addition, we have equality for each of the three inequalities. This shows that \( x(i) = x(j) \) for all \( ij \in E_0(y) \), \( x(i) = x(j) \) for all \( ij \in E_1(y) \), and \(|x(i) - x(j)| = 1 \) for all \( ij \in E_{01}(y) \).

We now show that there is an integer \( a \) such that \( x(i) = a \) for all \( i \in V_0(y) \) and an integer \( b \) such that \( x(i) = b \) for all \( i \in V_1(y) \). The set of edges \( E_{01}(y) \) disconnects \( G \) and it is a minimal disconnecting set (\(|E_{01}(y)| = k\)). Lemma 2.4 applies so \( G' = G - E_{01}(y) = H_0 + H_1 \) where \( H_0, H_1 \) are the connected components of \( G' \) and \( n \) is a vertex of \( H_0 \). It is clear that \( V(H_i) = V_i(y) \) and \( E(H_i) = E_i(y) \) for \( i = 1, 2 \).

Because \( H_0 \) is connected, there is a path joining any two vertices in \( H_0 \). But \( x(i) = x(j) \) for any edge \( ij \) in \( E_0(y) = E(H_0) \). It follows that there is an integer \( a \) such that \( x(i) = a \) for all \( i \in V(H_0) = V_0(y) \). Likewise there is an integer \( b \) such that \( x(i) = b \).
for all \( i \in V(H_1) = V_1(y) \). Now \( x(n) = 0 \) and \( n \in V(H_0) \), so \( a = 0 \). There is at least one edge \( ij \) in \( E_{01}(y) \) or else \( G \) is not connected. By adjusting the notation we may suppose that \( i \) is a vertex in \( H_0 \) and \( j \) a vertex in \( H_1 \) for this edge in \( E_{01}(y) \). Therefore, \( 1 = |x(i) - x(j)| = |0 - b| = 1 \). It follows that \( b = \pm 1 \) and therefore either \( x \in \mathcal{X}(k) \) or \( -x \in \mathcal{X}(k) \).

2.9. Conclusion. The preceding arguments show that for every \( x \in \mathcal{Z} \) with \( Q(x) = k \), either \( x \in \mathcal{X}(k) \) or \( -x \in \mathcal{X}(k) \). And that the number of minimal disconnecting sets for \( G \) equals the number of \( x \in \mathcal{X}(k) \) for which \( Q(x) = k \). Thus, the number of vectors \( x \) in \( \mathcal{Z} \) such that \( Q(x) = k \) is twice the number of minimal disconnecting sets of edges of \( G \). The proof of Theorem 1.2 is complete.

2.10. A combinatorial observation. The author wishes to thank the referee for this observation: If the vertices of a connected graph \( G \) are colored with two colors, 0 and 1, then the number of two-colored edges is at least the edge connectivity of \( G \) with equality if and only if the set of two-colored edges is a minimal disconnecting set of edges, \( E \). Indeed, the number of two-colored edges is just \( E_{01}(x_E) \) for the 0, 1 coloring vector \( x_E \).

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