On $(T,f)$-connections of matrices and generalized inverses of linear operators

Marek Niezgoda
University of Life Sciences in Lublin, marek.niezgoda@up.lublin.pl

Follow this and additional works at: https://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.3111

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
ON \((T, f)\)-CONNECTIONS OF MATRICES AND GENERALIZED INVERSES OF LINEAR OPERATORS

MAREK NIEZGODA

Abstract. In this note, generalized connections \(\sigma_{T,f}\) are investigated, where \(A\sigma_{T,f}B = T_A f_T A - A(B)\) for positive semidefinite matrix \(A\) and hermitian matrix \(B\), and operator monotone function \(f : J \to \mathbb{R}\) on an interval \(J \subset \mathbb{R}\). Here the symbol \(T_A\) denotes a reflexive generalized inverse of a positive bounded linear operator \(T\). The problem of estimating a given generalized connection by other ones is studied. The obtained results are specified for special cases of \(\alpha\)-arithmetic, \(\alpha\)-geometric and \(\alpha\)-harmonic operator means.

Key words. Positive definite matrix, Positive linear map, Operator monotone function, Connection, \(\alpha\)-Arithmetic (\(\alpha\)-Geometric, \(\alpha\)-Harmonic) operator mean, Generalized inverse of linear operator.

AMS subject classifications. 15A45, 47A63, 47A64.

1. Motivation. We begin with some notation. By \(M_n\) we denote the \(C^\ast\)-algebra of \(n \times n\) complex matrices. The symbol \(B(M_n)\) stands for the \(C^\ast\)-algebra of all bounded linear operators from \(M_n\) into \(M_n\). We denote the (real) space of \(n \times n\) Hermitian matrices by \(H_n\). The spectrum of an hermitian matrix \(A \in H_n\) is denoted by \(\text{Sp} (A)\).

For matrices \(X, Y \in M_n\), we write \(Y \leq X\) (resp., \(Y < X\)) if \(X - Y\) is positive semidefinite (resp., positive definite).

We say that a linear map \(\Phi : M_n \to M_k\) is positive if \(0 \leq \Phi(X)\) for \(0 \leq X \in M_n\). If in addition \(0 < \Phi(X)\) for \(0 < X \in M_n\) then we say that \(\Phi\) is strictly positive.

A function \(h : J \to \mathbb{R}\) with an interval \(J \subset \mathbb{R}\) is said to be an operator monotone function, if for all Hermitian matrices \(A\) and \(B\) (of the same order) with spectra in \(J\),

\[ A \leq B \quad \text{implies} \quad h(A) \leq h(B) \]

(see \([3\text{, p. 112}]\)).

Let \(f : J \to \mathbb{R}\) be a continuous function on an interval \(J \subset \mathbb{R}\). The \(f\)-connection of \(n \times n\) positive definite matrices \(A\) and \(B\) such that \(\text{Sp} (A^{-1/2}BA^{-1/2}) \subset J\), is
defined by

\[ A\sigma f B = A^{1/2} f \left( A^{-1/2} B A^{-1/2} \right) A^{1/2} \]  
(1.1)  
(cf. [6, p. 637]).

For \( f(t) \) of the form \( \alpha t + 1 - \alpha, t^\alpha \) and \( (\alpha t + 1 - \alpha)^{-1} \), one obtains the \( \alpha \)-arithmetic mean, \( \alpha \)-geometric mean and \( \alpha \)-harmonic mean, respectively, defined as follows (see (1.2), (1.3) and (1.4)).

For \( \alpha \in [0, 1] \), the \( \alpha \)-arithmetic mean of \( n \times n \) positive definite matrices \( A \) and \( B \) is defined by

\[ A\nabla_\alpha B = (1 - \alpha) A + \alpha B. \]  
(1.2)

For \( \alpha \in [0, 1] \), the \( \alpha \)-geometric mean of \( n \times n \) positive definite matrices \( A \) and \( B \) is defined as follows:

\[ A_\sharp_\alpha B = A^{1/2} (A^{-1/2} B A^{-1/2})^{\alpha} A^{1/2} \]  
(1.3)  
(see [8, 13]).

For \( \alpha \in [0, 1] \), the \( \alpha \)-harmonic mean of \( n \times n \) positive definite matrices \( A \) and \( B \) is defined by

\[ A\!\!\!\!
\]  
(1.4)

In [1] Ando showed that if \( \Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k \) is a positive linear map and \( A, B \in \mathbb{M}_n \) are positive definite then

\[ \Phi(A^{\sharp_\alpha} B) \leq \Phi(A^{\sharp} B). \]

Aujla and Vasudeva [2] proved that

\[ \Phi(A\sigma f B) \leq \Phi(A)\sigma f \Phi(B) \]

for an operator monotone function \( f : (0, \infty) \rightarrow (0, \infty) \).

A related result is due to Kaur et al. [7].

**Theorem A [7, Theorem 2.1]** Let \( A \) and \( B \) be \( n \times n \) positive definite matrices such that \( 0 < b_1 \leq A \leq a_1 \) and \( 0 < b_2 \leq B \leq a_2 \) for some scalars \( 0 < b_i < a_i \), \( i = 1, 2 \).

If \( \Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k \) is a strictly positive linear map, then for any operator mean \( \sigma \) with the representing function \( f \), the following double inequality holds:

\[ \omega^{1-\alpha} (\Phi(A)^{\sharp_\alpha} \Phi(B)) \leq (\omega \Phi(A))\nabla_\alpha \Phi(B) \leq \frac{\alpha}{\mu} \Phi(A\sigma B), \]
where $\mu = \frac{a_1 b_1 (f(b_2 a_1^{-1}) - f(a_2 b_1^{-1}))}{a_1 b_2 - a_2 b_1}$, $\nu = \frac{a_1 a_2 f(b_2 a_1^{-1}) - b_1 b_2 f(a_2 b_1^{-1})}{a_1 a_2 - b_1 b_2}$, and $\omega = \frac{a_2 b_2}{(1 - \alpha) \mu}$ and $\alpha \in (0, 1)$.

The following result was proved in [11]. The symbol $\circ$ means the composition of maps.

**Theorem B** [11, Theorem 2.7] Let $f_1, f_2, g_1, g_2$ be continuous real functions defined on an interval $J = [m, M] \subset \mathbb{R}_+$. Assume that $g_2 > 0$ and $g_2 \circ g_1^{-1}$ are operator monotone on intervals $J$ and $J' = g_1(J)$, respectively, with invertible $g_1$ and concave $g_2$. Let $A$ and $B$ be $n \times n$ positive definite matrices such that $m A \leq B \leq M A$ with $0 < m < M$.

If $\Phi : M_n \to M_k$ is a strictly positive linear map, $g_1(t) \leq f_1(t)$ and $f_2(t) \leq g_2(t)$ for $t \in J$,

and

$$\max_{t \in J} g_1(t) = \max_{t \in J} f_1(t),$$

then

$$c_{g_2} \Phi(A) \sigma_{f_2} \Phi(B) \leq \Phi(A \sigma_{g_2} B) \leq \Phi(A \sigma_{g_2 \circ g_1^{-1}}(A \sigma_{f_1} B)),$$

where $c_{g_2}$ is defined by

$$a_{g_2} = \frac{g_2(M) - g_2(m)}{M - m}, \quad b_{g_2} = \frac{g_2(M) - m g_2(M)}{M - m} \quad \text{and} \quad c_{g_2} = \min_{t \in J} \frac{a_{g_2} f_2(t) + b_{g_2}}{g_2(t)}.$$

In the present note, we extend Theorems A and B from $f$-connections of type [11] to a class of $(T, f)$-connections of the form

$$A \sigma_{T, f} B = T_A f T_A^{-1}(B)$$

for $A \geq 0$ and $B \in \mathbb{H}_n$, where $T_A$ denotes a positive reflexive generalized inverse of a positive bounded map $T_A$, and in addition the map $T : X \to T_X$ is positive.

**2. Results.** A generalized inverse of a linear map $L : V \to W$ between linear spaces $V$ and $W$ is a linear map $L^{-1} : W \to V$ satisfying $L L^{-1} L = L$. If in addition $L^{-1} L L^{-1} = L^{-1}$ then $L^{-1}$ is called a reflexive generalized inverse of $L$.

By $\text{Ran}(L)$ we denote the range of a linear map $L$.

If $L^{-1}$ is a reflexive generalized inverse of $L$, then

(2.1) \quad $L L^{-1}(Y) = Y$ for $Y \in \text{Ran}(L)$,


and

\[ L^- L(X) = X \quad \text{for} \quad X \in \text{Ran}(L^-). \]

It is known that if \( L : H \to H \) is a bounded linear map with a Hilbert space \( H \), then there exists a generalized inverse of \( L \) if and only if \( L \) has closed range \([4]\).

Throughout this note, whenever the symbol \( L^- \) is used, it is assumed that there exists a generalized inverse \( L^- \) of a linear map \( L \).

We denote the Loewner cone of all positive semidefinite \( n \times n \) complex matrices by \( \mathbb{L}_n \), i.e., \( \mathbb{L}_n = \{ X \in \mathbb{H}_n : X \geq 0 \} \).

Let \( T : X \to T_X \) be a map from \( \mathbb{L}_n \) into \( B(\mathbb{H}_n) \) with \( T_X : \mathbb{H}_n \to \mathbb{H}_n \) satisfying \( T_X \) is positive for \( X \geq 0 \).

Let \( f : J \to \mathbb{R} \) be a continuous function on an interval \( J \subset \mathbb{R} \). The \((T, f)\)-connection \( \sigma_{T, f} \) of an \( n \times n \) positive semidefinite matrix \( A \) and an \( n \times n \) hermitian matrix \( B \) such that \( \text{Sp}(T_A B) \subset J \), is defined by

\[
A \sigma_{T, f} B = T_A f T_A^*(B)
\]

(cf. \([6\text{, p. 637}]\)). For some applications of connections of the form \((2.2)\), see \([10, 11, 12]\).

**Example 2.1.** Let \( A \geq 0 \) be positive semidefinite (not necessarily positive definite) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). By spectral decomposition \( A = U \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U^* \) for some unitary matrix \( U \in \mathbb{M}_n \). We consider a generalized inverse \( A^- \) of \( A \) given by \( A^- = U \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)U^* \), where \( \mu_i = \frac{1}{\lambda_i} \) if \( \lambda_i > 0 \), and \( \mu_i = 0 \) if \( \lambda_i = 0 \), \( i = 1, 2, \ldots, n \). Thus, \( A^- \) is positive semidefinite, and \( A^- = (A^-)^{1/2}(A^-)^{1/2} \) with

\[
(A^-)^{1/2} = (A^{1/2})^- = U \text{diag}(\sqrt{\mu_1}, \sqrt{\mu_2}, \ldots, \sqrt{\mu_n})U^*
\]

We are now in a position to set

\[
T_A = A^{1/2}(\cdot)A^{1/2} \quad \text{and} \quad T_A^- = (A^-)^{1/2}(\cdot)(A^-)^{1/2}.
\]

Then \((2.2)\) takes the form (cf. \([11, 12]\))

\[
A \sigma_{T, f} B = A^{1/2} f (A^-)^{1/2} B (A^-)^{1/2} A^{1/2} \quad \text{for} \quad B \in \mathbb{H}_n.
\]

**Example 2.2.** (Cf. \([12\text{, Example 3.5}]\).) By \( E \) we denote the \( n \times n \) matrix of ones.
Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite matrix with $0 \leq a_{ij} < 2$, $i, j = 1, 2, \ldots, n$, and such that $0 \leq A \leq E$. We define 

$$T_A(X) = A \odot X = (a_{ij}x_{ij}) \text{ for } X = (x_{ij}) \in \mathbb{M}_n,$$

where $\odot$ stands for the Schur product of $n \times n$ matrices.

By Schur Product Theorem (see [5, Theorem 5.2.1]), the map $T_A$ is positive, i.e., $X \geq 0$ implies $T_A(X) \geq 0$.

We consider the generalized inverse of $T_A$ defined by

$$T_A^{-1}(Y) = A^{-1} \odot Y \text{ for } Y = (y_{ij}) \in \mathbb{M}_n,$$

where $A^{-1} = (c_{ij})$ with $c_{ij} = \frac{1}{a_{ij}}$ if $a_{ij} \neq 0$, and $c_{ij} = 0$ if $a_{ij} = 0$.

Under the hypothesis that $A = (a_{ij})$ with $0 \leq a_{ij} < 2$, $i, j = 1, 2, \ldots, n$, we have

$$A^{-1} = E + (E - A) + (E - A)^2 + (E - A)^3 + \cdots,$$

the convergent Schur-power series (see [5] pp. 449–450). Here the $m$-th Schur-power of $E - A$ is defined by

$$(E - A)^{[m]} = (E - A) \odot \cdots \odot (E - A), \quad m = 1, 2, \ldots,$$

with $(E - A)^{[0]} = E$.

It is evident by Schur Product Theorem that $(E - A)^{[m]} \geq 0$, $m = 0, 1, 2, 3, \ldots$, because $E - A \geq 0$. Therefore,

$$A^{-1} \geq 0$$

(see [5] Theorem 6.3.5)). So, the map $T_A^{-1}$ is positive by (2.3) and (2.4).

In summary, in this situation $(T, f)$-connection is given by

$$A(\sigma T, f)B = A \odot f \left( A^{-1} \odot B \right) \text{ for } B \in \mathbb{H}_n.$$
In the forthcoming theorem, we extend [9, Corollary 3.4] from the classical map $X \to T_X(\cdot) = X^{1/2}(\cdot)X^{1/2}$, $X > 0$, with the inverse $T_X^{-1}(\cdot) = X^{-1/2}(\cdot)X^{-1/2}$, to an arbitrary positive map $X \to T_X$ with a reflexive generalized inverse $T_X^{-1}$ of $T_X$.

**Theorem 2.3.** Assume that

(i) for any $X \geq 0$, $T_X : \mathbb{H}_n \to \mathbb{H}_n$ is a bounded linear operator, and $T_X^- : \mathbb{H}_n \to \mathbb{H}_n$ is a reflexive generalized inverse of $T_X$, satisfying the following conditions:

\begin{align}
(2.6) & \quad T_X \text{ and } T_X^- \text{ are positive,} \\
(2.7) & \quad T_X(I_n) = X, \\
(2.8) & \quad I_n \in \text{Ran}(T_X^-),
\end{align}

(ii) $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a strictly positive linear map,

(iii) $A$ is an $n \times n$ positive semidefinite matrix and $B$ is an $n \times n$ Hermitian matrix with $\text{Sp}(T_A B) \subset J = [m, M]$, $m < M$, such that

\begin{align}
(2.9) & \quad B \in \text{Ran} T_A \text{ and } \Phi(B) \in \text{Ran} T_{\Phi(A)}.
\end{align}

If $g : J \to \mathbb{R}_+$ is a continuous concave operator monotone function, then

\[ c_g \Phi(A) \sigma_{T,g} \Phi(B) \leq \Phi(A \sigma_{T,g} B), \]

where $a_g$, $b_g$ and $c_g$ are defined by (2.5).

**Proof.** The proof of inequality (2.10) is based on the proof of [6, Theorem 1] (cf. also [9, Corollary 3.4]).

Since $g : J \to (0, \infty)$ is concave, it is not hard to verify that

\begin{align}
(2.10) & \quad a_g t + b_g \leq g(t) \text{ for } t \in J.
\end{align}

It follows from (2.5) that

\[ c_g \geq \frac{a_g t + b_g}{g(t)} \text{ for } t \in J. \]

However, $g$ is positive, so $a_g t + b_g > 0$ for $t \in J$. Thus, $c_g$ is positive. Therefore,

\begin{align}
(2.11) & \quad g(t) \leq \frac{a_g}{c_g} t + \frac{b_g}{c_g}.
\end{align}

In consequence, (2.10) and (2.11) give

\begin{align}
(2.12) & \quad a_g t + b_g \leq g(t) \leq \frac{a_g}{c_g} t + \frac{b_g}{c_g} \text{ for } t \in [m, M],
\end{align}
which is the required estimation.

We now denote
\[ l_1(t) = a_g t + b_g \quad \text{and} \quad l_2(t) = \frac{a_g}{c_g} t + \frac{b_g}{c_g} \quad \text{for} \quad t \in J = [m, M]. \]

Consider \( C = \Phi(A) \) and \( D = \Phi(B) \). Because \( \text{Sp}(T_A B) \subset [m, M] \), we establish 
\[ mI_n \leq T_A B \leq MI_n, \] and next \( mTA I_n \leq T_AT_A^* B \leq MTA I_n \), because \( T_A \) is positive.
In addition, \( B \in \text{Ran} T_A \) and \( A = T_A(I_n) \) (see (2.1) and (2.7)). Consequently, 
\[ mA \leq B \leq MA \] (see (2.1)). From this \( m\Phi(A) \leq \Phi(B) \leq M\Phi(A) \), i.e., 
\( mC \leq D \leq MC \).
But \( T_C \) is positive, so \( mT_C C \leq T_C C \leq MT_C C \). Furthermore, \( T_C C = I_n \), because \( T_C I_n = C \) (see (2.7)) and hence \( I_n = T_C T_C I_n = T_C C \) by \( I_n \in \text{Ran} T_C \) (see (2.8)).
Thus, we derive \( mI_n \leq T_C D \leq MI_n \) and \( \text{Sp}(T_C D) \subset [m, M] \).

Since \( \text{Sp}(T_C^{-1} D) \subset [m, M] \) and \( g(t) \leq l_2(t) \) for \( t \in [m, M] \) (see (2.12)), we find that
\[ g(T_C^{-1} D) \leq l_2(T_C^{-1} D). \]
Now, by making use of positivity of \( T_C \) we obtain
\[ T_C g(T_C^{-1} D) \leq T_C l_2(T_C^{-1} D). \]
But \( l_2(t) = \frac{1}{c_g} l_1(t) \) for \( t \in [m, M] \). So, we have
\[ c_g T_C g(T_C^{-1} D) \leq T_C l_1(T_C^{-1} D). \]
Therefore,
\[ c_g C_{T,g} D \leq C_{T,l_1} D, \]
i.e.,
\[ c_g \Phi(A) \sigma_{T,g} \Phi(B) \leq \Phi(A) \sigma_{T,l_1} \Phi(B). \]
(2.13)

Simultaneously, it is clear that
\[ C_{T,l_1} D = T_C l_1 T_C D = T_C (a_g T_C^{-1} D + b_g I_n) = a_g T_C T_C^{-1} D + b_g T_C I_n. \]
However, \( T_C(I_n) = C \) and \( D \in \text{Ran} T_C \) (see (2.7) and (2.10)). Hence, \( T_C T_C^{-1} (D) = D \) (see (2.1)). Finally, from (2.14) we obtain
\[ C_{T,l_1} D = a_g D + b_g C, \]
which means
\[ \Phi(A) \sigma_{T,l_1} \Phi(B) = a_g \Phi(B) + b_g \Phi(A) = \Phi(a_g B + b_g A). \]
By virtue of (2.13) and (2.16), we get
\[ c_g \Phi(A) \sigma_{T,g} \Phi(B) \leq \Phi(a_g B + b_g A). \]

On the other hand, since \( \text{Sp}(T_A B) \subset [m, M] \) and \( l_1(t) \leq g(t) \) for \( t \in [m, M] \) (see (2.12)), we have
\[ l_1(T_A^{-1} B) \leq g(T_A^{-1} B), \]
and further
\[ T_A l_1(T_A^{-1} B) \leq T_A g(T_A^{-1} B). \]
Therefore,
\[ A \sigma_{T,l_1} B \leq A \sigma_{T,g} B. \]
Hence,
\[ (2.17) \quad \Phi(A \sigma_{T,l_1} B) \leq \Phi(A \sigma_{T,g} B). \]

Moreover, it follows that
\[ A \sigma_{T,l_1} B = T_A(a_g T_A^{-1} B + b_g I_n) = a_g B + b_g A, \]
since \( B \in \text{Ran} T_A, \ T_A T_A^{-1}(B) = B \) and \( T_A(I_n) = A \) (see (2.9), (2.11) and (2.17)).

So, in light of (2.17) we see that
\[ (2.18) \quad \Phi(a_g B + b_g A) \leq \Phi(A \sigma_{T,g} B). \]

In summary, combining (2.13), (2.16) and (2.18) leads to
\[ c_g \Phi(A) \sigma_{T,g} \Phi(B) \leq \Phi(A) \sigma_{T,l_1} \Phi(B) = \Phi(a_g B + b_g A) \leq \Phi(A \sigma_{T,g} B). \]

**Remark 2.4.** In the case \( T_A(\cdot) = A^{1/2}(\cdot) A^{1/2} \) with \( A > 0 \), Theorem 2.3 reduces to [6, Theorem 1], cf. also [9, Corollary 3.4].

**Remark 2.5.** It is evident that Theorem 2.3 simplifies if \( T_X \) is invertible. In fact, then the condition (2.9) is automatically fulfilled, and therefore, it can be dropped off. For the same reason, condition (2.8) can be deleted.

**Remark 2.6.** In Theorem 2.3 condition (i) can be assumed to hold for \( X = A \) and \( X = \Phi(A) \), only.
The next result is an extension of [7, Theorem 2.1].

**Theorem 2.7.** Under the assumptions (i)–(iii) of Theorem 2.3 for $X$, $T_X$, $\Phi$, $A$ and $B$, let $f_1, f_2, g_1, g_2$ be continuous real functions defined on an interval $J = [m, M]$, $m < M$. Suppose that $g_1$ is invertible on $J$, $g_2$ is positive concave and operator monotone on $J$, and $g_2 \circ g_1^{-1}$ is operator monotone on $J' = g_1(J)$, and

\begin{align}
(2.19) & \quad g_1(t) \leq f_1(t) \quad \text{and} \quad f_2(t) \leq g_2(t) \quad \text{for} \quad t \in J, \\
(2.20) & \quad \max_{t \in J} g_1(t) = \max_{t \in J} f_1(t), \\
(2.21) & \quad f_1 \text{Ran} (T^-_A) \subset \text{Ran} (T^-_A) \quad \text{and} \quad g_1 \text{Ran} (T^-_A) \subset \text{Ran} (T^-_A).
\end{align}

Then

\begin{align}
(2.22) & \quad c_{g_2} \Phi(A)\sigma_{T,f_2} \Phi(B) \leq \Phi(A\sigma_{T,g_2} B) \leq \Phi(A\sigma_{T,g_2 \circ g_1^{-1}}(A\sigma_{T,f_1} B)),
\end{align}

where $c_{g_2}$ is defined by (2.5) with $g = g_2$.

**Proof.** As in the proof of Theorem 2.3, we obtain $mA \leq B \leq MA$, and further $m\Phi(A) \leq \Phi(B) \leq M\Phi(A)$ by the positivity of $\Phi$. Hence, by the positivity of $T^-_{\Phi(A)}$, we establish

\[ mT^-_{\Phi(A)} \Phi(A) \leq T^-_{\Phi(A)} \Phi(B) \leq MT^-_{\Phi(A)} \Phi(A). \]

Now, from (2.7)–(2.8) we deduce that

\[ mI_n \leq T^-_{\Phi(A)} \Phi(B) \leq MI_n. \]

Therefore, $\text{Sp} (T^-_{\Phi(A)} \Phi(B)) \subset [m, M]$.

In light of the second inequality of (2.19), we have

\[ f_2 T^-_{\Phi(A)} \Phi(B) \leq g_2 T^-_{\Phi(A)} \Phi(B), \]

and next, by the positivity of $T_{\Phi(A)}$,

\[ T_{\Phi(A)} f_2 T^-_{\Phi(A)} \Phi(B) \leq T_{\Phi(A)} g_2 T^-_{\Phi(A)} \Phi(B). \]

That is,

\begin{align}
(2.23) & \quad \Phi(A)\sigma_{T,f_2} \Phi(B) \leq \Phi(A)\sigma_{T,g_2} \Phi(B).
\end{align}

It follows from Theorem 2.3 applied to the function $g = g_2$ that

\[ c_{g_2} \Phi(A)\sigma_{T,g_2} \Phi(B) \leq \Phi(A\sigma_{T,g_2} B). \]
For this reason, (2.23) implies

\[ c_{g_2} \Phi(A)\sigma_{T, f_2} \Phi(B) \leq \Phi(A\sigma_{T, g_2} B). \]  

This proves the left-hand side inequality of (2.22).

Furthermore, we find that

\[ A\sigma_{T, g_2} B = A\sigma_{T, h \circ g_1} B = A\sigma_{T, h}(A\sigma_{T, g_1} B), \]  

where \( h = g_2 \circ g_1^{-1} \) and \( \circ \) means composition. Indeed, by (2.21) we get \( g_1 T_A^- B \in \text{Ran} \, T_A^- \). So, we have

\[ g_1 T_A^- B = T_A^- T_A g_1 T_A^- B. \]  

Hence,

\[ A\sigma_{T, h \circ g_1} B = T_A h g_1 T_A^- B = T_A h T_A^- T_A g_1 T_A^- B = T_A h T_A^- (A\sigma_{T, g_1} B) = A\sigma_{T, h}(A\sigma_{T, g_1} B), \]  

which yields (2.25).

On the other hand, from the first inequality of (2.19) we obtain

\[ g_1 T_A^- B \leq f_1 T_A^- B, \]  

and further

\[ T_A g_1 T_A^- B \leq T_A f_1 T_A^- B \]  

by the positivity of \( T_A \). Thus, we have

\[ A\sigma_{T, g_1} B \leq A\sigma_{T, f_1} B. \]  

From (2.19) we see that

\[ \min_{t \in J} g_1(t) \leq \min_{t \in J} f_1(t). \]  

This together with (2.20) implies

\[ f_1(J) \subset g_1(J). \]  

We now introduce

\[ Z_0 = T_A^- (A\sigma_{T, g_1} B) \quad \text{and} \quad W_0 = T_A^- (A\sigma_{T, f_1} B). \]
Clearly, by (2.26) and (2.21),
\[ Z_0 = g_1 T_A (B) \quad \text{and} \quad W_0 = f_1 T_A (B). \]
Then Sp (Z_0) \subset g_1 (J) and Sp (W_0) \subset f_1 (J) \subset g_1 (J), because Sp (T_A B) \subset J.

So, from (2.27) and (2.28), we obtain
\[ A_{\sigma T, h} (A_{\sigma T, f_1} B) \leq A_{\sigma T, h} (A_{\sigma T, g_1} B), \]
because h is operator monotone on J' = g_1 (J), and T_A and T_A^- are positive.

From (2.25) and (2.29), it follows that
\[ \Phi (A_{\sigma T, g_2} B) \leq \Phi (A_{\sigma T, g_2} g_1^{-1} (A_{\sigma T, f_1} B)). \]
Now, according to (2.24) and (2.30), we infer that (2.22) is satisfied.

The discussion of inequality (2.22) for the cases f_1 = f_2 and g_1 = g_2 with T_A = A^{-1/2} (.) A^{-1/2}, A > 0, can be found in [11].

We now consider the case g_1 = g_2 of (2.22) for arbitrary T_A.

**Corollary 2.8.** Under the assumptions (i)–(iii) of Theorem 2.3 for X, T_X, \Phi, A and B, let f_1, f_2, g be continuous real functions defined on an interval J = [m, M], m < M. Suppose that g is invertible positive concave and operator monotone on J, and
\[ f_2 (t) \leq g(t) \leq f_1 (t) \quad \text{for} \quad t \in J, \]
\[ \max_{t \in J} g(t) = \max_{t \in J} f_1(t), \]
\[ f_1 \text{Ran } (T_A^-) \subset \text{Ran } (T_A^-) \quad \text{and} \quad g \text{Ran } (T_A^-) \subset \text{Ran } (T_A^-). \]

Then
\[ c_g \Phi (A_{\sigma T, f_2} \Phi (B) \leq \Phi (A_{\sigma T, g} B) \leq \Phi (A_{\sigma T, f_1} B), \]
where c_g is defined by (2.5).

**Proof.** It is enough to apply Theorem 2.7 with g_1 = g_2 = g and g_2 \circ g_1^{-1} = id and A_{\sigma T, id} (A_{\sigma T, f_1} B) = A_{\sigma T, f_1} B. \qed

Some concrete versions of inequalities (2.22) of Theorem 2.7 which depend on the form of g_1 \circ g_2^{-1} are included in Corollary 2.9. By making use of affine, power and inverse-affine functions we obtain arithmetic, geometric and harmonic operator means of A and A_{\sigma T, f_1} B, respectively, on the right-hand side of (2.22).

**Corollary 2.9.** Under the assumptions of Theorem 2.7 with A, B > 0:
If \( g_2 \circ g_1^{-1} \) is an affine function, i.e., \( g_2 \circ g_1^{-1}(s) = as + b \) for \( s \in g_1(J) \), \( a > 0 \), then the right-hand side inequality of (2.22) reduces to

\[
(2.31) \quad \Phi(A \sigma_{T,g_2} B) \leq a \Phi(A \sigma_{T,f_1} B) + b \Phi(A).
\]

If \( g_2 \circ g_1^{-1} \) is a power function, i.e., \( g_2 \circ g_1^{-1}(s) = s^\alpha \) for \( s \in g_1(J) \subset \mathbb{R}_+ \), \( \alpha \in [0,1] \), then the right-hand side inequality of (2.22) reduces to

\[
(2.32) \quad \Phi(A \sigma_{T,g_2} B) \leq \Phi(T_A(T_A^{-1}(A \sigma_{T,f_1} B))^\alpha).
\]

If \( g_2 \circ g_1^{-1} \) is an inverse function of the form \( g_2 \circ g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1} \) for \( s \in g_1(J) \subset \mathbb{R}_+ \), \( \alpha \in [0,1] \), then the right-hand side inequality of (2.22) reduces to

\[
(2.33) \quad \Phi(A \sigma_{T,g_2} B) \leq \Phi([(1 - \alpha)A^{-1} + \alpha(A \sigma_{T,f_1} B)^{-1}]^{-1}).
\]

Proof. (I) Since \( a > 0 \), the function \( g_2 \circ g_1^{-1}(s) = as + b \) is operator monotone (see [3, p. 113]). Moreover,

\[
A \sigma_{T,h}(A \sigma_{T,f_1} B) = T_A h T_A^{-1}(A \sigma_{T,f_1} B).
\]

Simultaneously,

\[
A \sigma_{T,f_1} B = T_A f_1 T_A^{-1} B \in \text{Ran}(T_A).
\]

So, for \( h(s) = as + b \), by (2.1) we have

\[
A \sigma_{T,h}(A \sigma_{T,f_1} B) = T_A(a T_A^{-1}(A \sigma_{T,f_1} B) + b I_n)
\]

\[
= a T_A T_A^{-1}(A \sigma_{T,f_1} B) + b T_A(I_n) = a A \sigma_{T,f_1} B + b A.
\]

Now, to see (2.31) it is enough to employ (2.22).

(II) The function \( g_2 \circ g_1^{-1}(s) = s^\alpha \) with \( \alpha \in [0,1] \) is operator monotone (see [3, p. 115]). So, to prove (2.32), it is sufficient to apply (2.22).

(III) Inequality (2.33) follows from (2.22) applied to the operator monotone function \( g_2 \circ g_1^{-1}(s) = (as^{-1} + 1 - \alpha)^{-1} \) with \( \alpha \in [0,1] \) (see [3, p. 114]). \( \Box \)

Theorem 2.7 simplifies if in addition \( T_X \) is invertible.

Corollary 2.10. Assume that

(i) for any \( X \geq 0 \), \( T_X : \mathbb{H}_n \rightarrow \mathbb{H}_n \) is an invertible bounded positive linear operator with positive inverse \( T_X^{-1} \) such that \( T_X(I_n) = X \);

(ii) \( \Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k \) is a strictly positive linear map,
(iii) $A$ and $B$ are $n \times n$ positive definite matrices with $\text{Sp}(T^{-1}_A B) \subset J = [m, M]$, $0 < m < M$.

Let $f_1, f_2, g_1, g_2$ be continuous real functions defined on $J$ satisfying conditions \((2.19)\) and \((2.20)\). Suppose that $g_1$ is invertible on $J$, $g_2$ is positive concave and operator monotone on $J$, and $g_2 \circ g_1^{-1}$ is operator monotone on interval $J' = g_1(J)$.

Then inequality \((2.22)\) is satisfied with \((2.7)\) for $g = g_2$.

**Proof.** With $T^{-1}_X = T^{-1}_X$ conditions \((2.8)\) and \((2.21)\) hold automatically. Now it is sufficient to use Theorem \((2.7)\) \(\blacksquare\)

**Acknowledgment.** The author would like to thank an anonymous referee for his/her useful suggestions that improved the presentation of the paper.

**REFERENCES**


