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SEMILINEAR PRESERVERS OF THE IMMANANTS IN THE SET OF DOUBLY STOCHASTIC MATRICES*

M. ANTÓNIA DUFFNER† AND ROSÁRIO FERNANDES‡

Abstract. Let S_n denote the symmetric group of degree n and M_n denote the set of all n-by-n matrices over the complex field, $\mathbb{C}$. Let $\chi : S_n \to \mathbb{C}$ be an irreducible character of degree greater than 1 of S_n. The immanant $d_\chi : M_n \to \mathbb{C}$ associated with χ is defined by

$$d_\chi(X) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$

Let Ω_n be the set of all n-by-n doubly stochastic matrices, that is, matrices with nonnegative real entries and each row and column sum is one. We say that a map T from Ω_n into Ω_n

- is semilinear if $T(\lambda S_1 + (1-\lambda)S_2) = \lambda T(S_1) + (1-\lambda)T(S_2)$ for all $S_1,\ S_2 \in \Omega_n$ and for all real number λ such that $0 \leq \lambda \leq 1$;
- preserves d_χ if $d_\chi(T(S)) = d_\chi(S)$ for all $S \in \Omega_n$.

We characterize the semilinear surjective maps T from Ω_n into Ω_n that preserve d_χ, when the degree of χ is greater than one.

Key words. Immanants, Linear preserver problems, Doubly stochastic matrices.

AMS subject classifications. 15A69, 15A60, 15A42, 15A45, 15A04, 47B49.

1. Introduction.

Let M_n denote the set of all n-by-n matrices over the complex field, $\mathbb{C}$. We denote by I the identity in M_n. Let S_n be the symmetric group of degree n. We denote by id the identity in S_n. Let $\chi : S_n \to \mathbb{C}$ be an irreducible character of S_n with degree greater than 1 (note that if the degree of χ is one then χ is the sign character or the principal character). The immanant d_χ is defined by

$$d_\chi(X) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$

If the degree of the character χ is one, then d_χ is the determinant or the permanent. We denote the permanent by per,

$$per(X) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} X_{j\sigma(j)}, \quad X = [X_{jk}] \in M_n.$$

Let Ω_n denote the set of all n-by-n doubly stochastic matrices, that is, matrices with nonnegative real entries and each row and column sum is one. Ω_n is a convex polyhedron in the euclidean n^2-space whose vertices are the n-by-n permutation matrices, [2].

Definition 1.1. Let T be a map from Ω_n into Ω_n. We say that T

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- is a semilinear map if

$$T(\lambda S_1 + (1-\lambda)S_2) = \lambda T(S_1) + (1-\lambda)T(S_2)$$

  for all $S_1, S_2 \in \Omega_n$ and for all real number λ such that $0 \leq \lambda \leq 1$;
- preserves d_χ if $d_\chi(T(S)) = d_\chi(S)$ for all $S \in \Omega_n$.

The behavior of the permanent on Ω_n has been studied extensively. In [9], the linear maps T from Ω_n into Ω_n which preserve the permanent are characterized, and in [4], those that verify $T(\Omega_n) = \Omega_n$. In this paper, we characterize the semilinear surjective maps T from Ω_n into Ω_n that preserve d_χ, where the character χ has degree greater than one.

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a partition of n of length r, that is, a sequence of positive integers which are assumed to be nonincreasing and with sum equal to n, [2, 3]. Each partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n is related to a Young diagram, denoted by $[\alpha]$, which consists of r left justified rows of boxes, where the number of boxes in the ith row is α_i. The irreducible characters of S_n are in a bijective correspondence with the ordered partitions of n, [1]. We identify the irreducible character χ with the partition that corresponds to χ, or with the Young diagram $[\chi]$ associated with χ.

Denote by $P(\sigma)$ the permutation matrix associated with $\sigma \in S_n$, that is,

$$P(\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

We denote by S^T the transpose of the matrix S. Recall that $(P(\sigma))^T = P(\sigma^{-1})$.

The main result of this paper is the following theorem.

Theorem 1.2. *Let χ be an irreducible character of S_n of degree greater than one. Let T be a semilinear surjective map from Ω_n into Ω_n. The map T preserves d_χ if and only if there are $\sigma, \alpha \in S_n$, with $\chi(\sigma) = \chi(id)$, such that one of the following conditions must hold:*

(1) $T(S) = P(\sigma)P(\alpha)SP(\alpha^{-1})$ for all $S \in \Omega_n$.
(2) $T(S) = P(\sigma)P(\alpha)S^T P(\alpha^{-1})$ for all $S \in \Omega_n$.

Moreover, if $\chi \neq [2, 2]$, then $P(\sigma) = I$.

In Section 2, we shall present some preliminary definitions and propositions about the immanant of a matrix $S \in \Omega_n$. To characterize the semilinear surjective maps T from Ω_n into Ω_n that preserve d_χ, we will consider several steps. So, in Section 3, we will prove that T must be injective. In Section 4, we will prove that the image by T of a permutation matrix is a permutation matrix. Finally, in Section 5, we will present the proof of the main result.

2. Preliminares. Let χ be an irreducible character of S_n. The boundary of the diagram $[\chi]$ is the set of boxes whose right edge, bottom edge, or bottom right vertex belong to the geometric boundary of the diagram. We will denote by p the number of boundary boxes of $[\chi]$. Note that if χ is an irreducible character of S_n of degree grater than 1 then $p \geq 3$.

A set of successive boundary boxes whose deletion leads to another Young diagram is called a regular boundary part. The number of vertical steps of a regular boundary part is equal to the number of rows involved minus one.

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The Murnaghan-Nakayama Rule is important to calculate the value of $\chi(\sigma)$, for $\sigma \in S_n$. For more details, see for example [1].

Proposition 2.1. (Murnaghan-Nakayama Rule) *Let the disjoint cycles of $\sigma \in S_n$ have lengths $a_1, \ldots, a_q$ in any order. Determine all ways in which the diagram $[\chi]$ can be reduced to 0 by successively omitting regular boundary parts of lengths $a_1, \ldots, a_q$. Let the boundary parts occurring in the sth way contain k_s vertical steps altogether. Then $\chi(\sigma) = \sum_s (-1)^{k_s}$.*

In what follows, we will use this rule, namely, to state the following facts:

- If σ is a cycle of length equal to p then $\chi(\sigma) \neq 0$.
- If σ is a cycle of length greater than p then $\chi(\sigma) = 0$.
- If χ is a single hook, that is, an irreducible character $\chi = [\chi_1, \ldots, \chi_r]$ of S_n such that $\chi_2 = \cdots = \chi_r = 1$, and σ is the product of disjoint cycles of length greater than one, $\sigma_1, \ldots, \sigma_h$, with $h \geq 2$, and there is an integer i, such that $1 \leq i \leq h$ with the length of σ_i greater than $\max\{\chi_1 - 1,\ r-1\}$ then $\chi(\sigma) = 0$.
- If χ is a single hook and σ is the product of two disjoint cycles of length greater than one, σ_1, σ_2, with the length of σ_1 equal to $\chi_1 - 1$ and the length of σ_2 equal to $r-1$, or vice-versa, then $\chi(\sigma) \neq 0$.

In [5], M. Marcus and M. Newman proved the following result.

Proposition 2.2. *If $S \in \Omega_n$, then*

$$perS \leq 1.$$

Moreover, $perS = 1$ if and only if $S = P(\sigma)$, for some $\sigma \in S_n$.

If $\pi,\ \sigma \in S_n$, we denote by $\pi \circ \sigma$ the composition of these two permutations and we denote by $\sigma(k)$ the image of the value k under the map σ. Furthermore, if $\pi \in S_n$ is a cycle, its length is denoted by $l(\pi)$.

Remark 2.1. Let χ be an irreducible character of S_n. We refer to [1, 6, 7, 8] for a general study in multilinear algebra.

1. $\chi(\sigma) \in \mathbb{Z}$ for all $\sigma \in S_n$, and

$$\sum_{\sigma \in S_n} \chi(\sigma) = \begin{cases} 0 & \text{if } \chi \text{ is not the principal character,} \\ n! & \text{otherwise.} \end{cases}$$

2. $\chi(\sigma^{-1}) = \chi(\sigma)$ for all $\sigma \in S_n$, and $\chi(\pi \circ \sigma \circ \pi^{-1}) = \chi(\sigma)$ for all $\pi,\ \sigma \in S_n$.
3. $\mid \chi(\sigma) \mid \leq \chi(id)$ for all $\sigma \in S_n$.
4. If $n > 4$ and χ is a character of S_n of degree greater than one, then $\mid \chi(\sigma) \mid < \chi(id)$ for all $\sigma \in S_n \backslash \{id\}$, [10].
5. Using direct computation, if χ is a character of S_n of degree greater than one and $\sigma \in S_n \setminus \{id\}$ verify $\mid \chi(\sigma) \mid = \chi(id)$ then $n = 4$, $\chi = [2,2]$ and $\sigma \in \{(12)(34), (13)(24), (14)(23)\}$. Moreover, if $\chi = [2,2]$ and $\sigma \in \{(12)(34), (13)(24), (14)(23)\}$, then $\chi(\pi \circ \sigma) = \chi(\pi)$, $\forall \pi \in S_4$.

From the following proposition, we can conclude that whenever $\chi \neq [2,2]$ and $S \in \Omega_n$, the maximum value of $d_\chi(S)$ is attained when $S = I$, and the minimum value is attained when $S = P(\tau)$, where $\chi(\tau) \leq \chi(\pi)$, for all $\pi \in S_n$.

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Proposition 2.3. *Let χ be an irreducible character of degree greater than 1 of S_n. If $S \in \Omega_n$ then $d_\chi(S) \leq \chi(id)$, and the equality holds if and only if*

$$S = P(\sigma) \quad \text{and} \quad \chi(\sigma) = \chi(id).$$

Moreover, $d_\chi(S) \geq \chi(\tau)$, where $\chi(\tau) \leq \chi(\pi)$ for all $\pi \in S_n$, with equality if and only if

$$S = P(\rho) \quad \text{and} \quad \chi(\rho) = \chi(\tau).$$

Proof. Since

$$\mid d_\chi(S) \mid = \left| \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} S_{j,\sigma(j)} \right| \leq \sum_{\sigma \in S_n} \mid \chi(\sigma) \mid \prod_{j=1}^{n} S_{j,\sigma(j)} \leq \sum_{\sigma \in S_n} \chi(id) \prod_{j=1}^{n} S_{j,\sigma(j)} = \chi(id) perS$$

and since $perS \leq 1$, it follows that $\mid d_\chi(S) \mid \leq \chi(id)$.

If $\chi(id) = \mid d_\chi(S) \mid \leq \chi(id) perS$, then $perS \geq 1$. But as $perS \leq 1$, for all $S \in \Omega_n$, then $perS = 1$. By Proposition 2.2, we have that $S = P(\sigma)$ for some $\sigma \in S_n$. By definition and hypothesis, $\chi(id) = d_\chi(S) = d_\chi(P(\sigma)) = \chi(\sigma)$. Therefore, $\chi(\sigma) = \chi(id)$.

Since $\chi(\tau) < 0$ if $\chi(\tau) = \min\{\chi(\sigma) : \ \sigma \in S_n\}$, we have that

$$d_\chi(S) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{j=1}^{n} S_{j,\sigma(j)} \geq \sum_{\sigma \in S_n} \chi(\tau) \prod_{j=1}^{n} S_{j,\sigma(j)} = \chi(\tau) \sum_{\sigma \in S_n} \prod_{j=1}^{n} S_{j,\sigma(j)} = \chi(\tau) per(S) \geq \chi(\tau).$$

Consequently, $d_\chi(S) \geq \chi(\tau)$. If $d_\chi(S) = \chi(\tau)$, then $per(S) = 1$. By Proposition 2.2, this implies that $S = P(\sigma)$, for some $\sigma \in S_n$. Because $\chi(\tau) = d_\chi(S) = d_\chi(P(\sigma)) = \chi(\sigma)$ then $\chi(\sigma) = \chi(\tau)$. □

Corollary 2.4. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a map from Ω_n into Ω_n that preserves d_χ. If $\chi \neq [2,2]$, then $T(I) = I$. Moreover, when $\chi = [2,2]$, there is $\sigma \in S_4$ such that $T(I) = P(\sigma)$ and $\chi(\sigma) = \chi(id)$.*

Proof. Since $T(I) \in \Omega_n$ and $d_\chi(T(I)) = d_\chi(I) = \chi(id)$, by last proposition, there is $\sigma \in S_n$, such that $T(I) = P(\sigma)$, with $\chi(\sigma) = \chi(id)$. By Remark 2.1, we have that $T(I) = I$ if $\chi \neq [2,2]$. □

Remark 2.2.

1. Using last corollary we conclude that $T(I)$ is invertible.
2. Using the main result of [11] (characterization of the subgroup of M_n, $\mathcal{S}(S_n, \chi) = \{A \in M_n;\ d_\chi(AX) = d_\chi(X),$ for all $X \in M_n\}$) we have that if $\sigma \in S_n$ and $\chi(\sigma) = \chi(id)$, then $d_\chi(P(\sigma)S) = d_\chi(S)$ for all $S \in \Omega_n$.

To prove the following lemmas, we will use the Murnaghan-Nakayama Rule (see the considerations at the beginning of this section and [1]).

Lemma 2.5. *Let $n \geq 4$, and χ be an irreducible character of S_n of degree greater than one. If $i, j, k \in \{1, \ldots, n\}$, are distinct on pairs, then there are $\sigma, \tau \in S_n$ such that*

$$\sigma(i) = j, \quad \sigma(k) = k, \quad \tau = \sigma \circ (ik), \quad \chi(\sigma) \neq 0, \quad \chi(\tau) = 0.$$

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Proof. Suppose that χ is not a single hook, and let p be the number of boundary boxes of $[\chi]$. Then $p \leq n-1$.

If $\sigma \in S_n$ is a cycle of length p such that $\sigma(i)=j, \sigma(k)=k$, since $\tau=\sigma \circ (ik)$ then τ is a cycle of length $p+1$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau)=0$.

Suppose that $\chi=[\chi_1,\ldots,\chi_{v+1}]$ is a single hook, with $\chi_1=u>1$ and $v \geq 1$.

If $u-1 \geq v$, since $n=u+v \geq 4$ then $u-1+v \geq 3$. So, (note that $v \geq 1$ because χ has degree greater than one, $n \geq 4$) $u-1 \geq 2$. Therefore, there exist $\sigma \in S_n$, and disjoint cycles σ_1, σ_2, where $\sigma=\sigma_1 \circ \sigma_2$, $l(\sigma_1)=u-1$ and $l(\sigma_2)=v$, such that $\sigma_1(i)=j$, $\sigma(k)=k$. Consequently, $\tau=\sigma \circ (ik)=\tau_1 \circ \tau_2$ with $\tau_1, \tau_2 \in S_n$ and $l(\tau_1)=u$, $l(\tau_2)=v$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau)=0$.

If $u-1<v$, then, there are $\sigma \in S_n$, disjoint cycles σ_1, σ_2 , where $\sigma=\sigma_1 \circ \sigma_2$, $l(\sigma_1)=u-1$ and $l(\sigma_2)=v$, such that $\sigma_2(i)=j$, and $\sigma(k)=k$. Therefore, $\tau=\sigma \circ (ik)=\tau_1 \circ \tau_2$ with $l(\tau_2)=v+1$. Using the Murnaghan-Nakayama Rule we have that $\chi(\sigma) \neq 0$ and $\chi(\tau)=0$. □

Lemma 2.6. *Let $n \geq 3$, $i,j,k \in \{1,\ldots,n\}$, distinct on pairs and $\sigma, \tau \in S_n$ such that*

$$\sigma(i)=j, \quad \sigma(k)=k, \quad \tau=\sigma \circ (ik).$$

Then for every $\pi \in S_n$, there are $s,\ l \in \{1,\ldots,n\}$ and $l \neq s$ that verify

$$\sigma^{-1}(s) \neq \pi(s), \quad \tau^{-1}(s) \neq \pi(s), \quad \sigma^{-1}(l) \neq \pi(l), \quad \tau^{-1}(l) \neq \pi(l).$$

Proof. Suppose that there is $\pi \in S_n$ with a unique $s \in \{1,\ldots,n\}$ such that

$$\pi(s)=t, \quad \sigma(t) \neq s, \quad \tau(t) \neq s.$$

Consequently,

$$\text{if } \ l \neq s, \ \text{ then } \ \sigma^{-1}(l)=\pi(l) \ \text{ or } \ \tau^{-1}(l)=\pi(l).$$

Let u and v be elements such that $\sigma^{-1}(u)=t=\pi(s),\ \tau^{-1}(v)=t=\pi(s)$, (note that $u \neq s,\ v \neq s$).

If $u=v$ then $\pi(u)=\sigma^{-1}(u)=t$ or $\pi(u)=\tau^{-1}(v)=t$. But $\pi(s)=t$, therefore we have a contradiction, $u=s$.

Consequently, $u \neq v$. Since $\tau=\sigma \circ (ik)$ then $(t=i,\ u=j,\ v=k)$ or $(t=k,\ u=k,\ v=j)$. We only prove the case $t=i,\ u=j,\ v=k$, because the proof of the other case is analogous. In the case that we will prove, $\sigma^{-1}(j)=i=\pi(s),\ \tau^{-1}(k)=i=\pi(s)$.

Since $s \neq j,\ s \neq k$ then $\pi(j)=\sigma^{-1}(j)$ or $\pi(j)=\tau^{-1}(j)$. If $\pi(j)=\sigma^{-1}(j)$ then $\pi(j)=\sigma^{-1}(j)=\pi(s)$ and we can conclude that $s=j$ (impossible). So, $\pi(j)=\tau^{-1}(j)=k$. Since $s \neq k$ then $\pi(k)=\sigma^{-1}(k)$ or $\pi(k)=\tau^{-1}(k)$. If $\pi(k)=\tau^{-1}(k)$ then $\pi(k)=\tau^{-1}(k)=\pi(s)$ and we can conclude that $s=k$ (impossible). Therefore, $\pi(k)=\sigma^{-1}(k)=k$. But this implies that $\pi(j)=\pi(k)=k$ which is impossible. □

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3. The injectivity of T. Let χ be an irreducible character of S_n of degree greater than 1 and T be a semilinear map from Ω_n into Ω_n that preserves d_χ. In the main result of this section we will prove that T must be injective.

THEOREM 3.1. *Let χ be an irreducible character of S_n of degree greater than 1 and T be a semilinear map from Ω_n into Ω_n that preserves d_χ. Then T is injective.*

Proof. Let $S, S' \in \Omega_n$ such that $T(S) = T(S')$. Let $B \in \Omega_n$ and $x \in [0,1]$. Since

$$\begin{aligned} d_\chi(xS + (1-x)B) &= d_\chi(T(xS + (1-x)B)) = d_\chi(xT(S) + (1-x)T(B)) \\ &= d_\chi(xT(S') + (1-x)T(B)) = d_\chi(T(xS' + (1-x)B)) \\ &= d_\chi(xS' + (1-x)B), \end{aligned}$$

it follows that $d_\chi(xS + (1-x)B) = d_\chi(xS' + (1-x)B)$.

Case (i) Let $n \geq 4$. If $i, j, k \in \{1, \dots, n\}$ are distinct on pairs, then by Lemma 2.5, there are $\sigma, \tau \in S_n$ such that $\sigma(i) = j$, $\sigma(k) = k$, $\tau = \sigma \circ (ik)$, $\chi(\sigma) \neq 0$, $\chi(\tau) = 0$.

For each $b \in [0,1]$, let us consider the matrix

$$B_b = bP(\sigma) + (1-b)P(\tau).$$

So, for all $p \in \{1, \dots, n\}$,

$$(B_b)_{p\pi(p)} = \begin{cases} 1 & \text{if } \pi(p) = \sigma^{-1}(p) = \tau^{-1}(p), \\ b & \text{if } \pi(p) = \sigma^{-1}(p),\ \pi(p) \neq \tau^{-1}(p), \\ 1-b & \text{if } \pi(p) \neq \sigma^{-1}(p),\ \pi(p) = \tau^{-1}(p), \\ 0 & \text{otherwise.} \end{cases}$$

Now we will compute the coefficient of the term associated with x of the polynomial

$$d_\chi(xS + (1-x)B_b) = \sum_{\pi \in S_n} \chi(\pi) \prod_{l=1}^{n} (xS + (1-x)B_b)_{l\pi(l)}.$$

If there is $s \in \{1, \dots, n\}$ such that for some $\pi \in S_n$, $\pi(s) \neq \sigma^{-1}(s)$ and $\pi(s) \neq \tau^{-1}(s)$ then

$$(xS + (1-x)B_b)_{s\pi(s)} = xS_{s\pi(s)}.$$

To obtain the coefficient of the term associated with x of the polynomial $\chi(\pi) \prod_{l=1}^{n} (xS + (1-x)B_b)_{l\pi(l)}$ the other terms of $\prod_{l=1, l\neq s}^{n} (xS + (1-x)B_b)_{l\pi(l)}$ must verify $(B_b)_{l\pi(l)} \neq 0$. Consequently, if $l \neq s$ then $\pi(l) = \sigma^{-1}(l)$ or $\pi(l) = \tau^{-1}(l)$. But this is impossible by Lemma 2.6. Therefore, if $s \in \{1, \dots, n\}$ and $\pi \in S_n$, then $\pi(s) = \sigma^{-1}(s)$ or $\pi(s) = \tau^{-1}(s)$. Since $\tau = \sigma \circ (ik)$ then $\pi(s) = \sigma^{-1}(s) = \tau^{-1}(s)$, when $s \in \{1, \dots, n\} \setminus \{j, k\}$. Because $\pi(j) = \sigma^{-1}(j)$ or $\pi(j) = \tau^{-1}(j)$, and $\pi(k) = \sigma^{-1}(k)$ or $\pi(k) = \tau^{-1}(k)$ then $\pi(j) = i$ or $\pi(j) = k$, and $\pi(k) = k$ or $\pi(k) = i$. But π is a bijection, so we have two cases:

- If $\pi(j) = i$, then $\pi(k) = k$ and $\pi = \sigma^{-1}$.
- If $\pi(j) = k$, then $\pi(k) = i$ and $\pi = \tau^{-1}$.

Therefore, the coefficient of the term associated with x of the polynomial $d_\chi(xS + (1-x)B_b)$ appears when $\pi = \sigma^{-1}$ or $\pi = \tau^{-1}$.

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As $\chi(\tau^{-1}) = 0$, it is enough to compute $\chi(\sigma^{-1}) \prod_{l=1}^{n}(xS + (1-x)B_b)_{l\sigma^{-1}(l)}$. Since $\sigma(\sigma^{-1}(l)) = l$, for all $l \in \{1, \ldots, n\}$ and $\tau(\sigma^{-1}(l)) \neq l$ when $l \in \{j, k\}$, then

$$\chi(\sigma^{-1}) \prod_{l=1}^{n}(xS+(1-x)B_b)_{l\sigma^{-1}(l)} = \chi(\sigma^{-1})(xS_{j\sigma^{-1}(j)}+(1-x)b)(xS_{k\sigma^{-1}(k)}+(1-x)b) \prod_{l\neq j,k}(xS_{l\sigma^{-1}(l)}+(1-x)).$$

Consequently, the coefficient of the term associated with x in the polynomial $d_\chi(xS + (1-x)B_b)$ is

$$\chi(\sigma^{-1})((S_{j\sigma^{-1}(j)} - b)b + (S_{k\sigma^{-1}(k)} - b)b + b^2 \sum_{l\neq j,k}(S_{l\sigma^{-1}(l)} - 1)).$$

Since $\sigma^{-1}(j) = i$ and $\sigma^{-1}(k) = k$ then the coefficient of the term associated with x in the polynomial $d_\chi(xS + (1-x)B_b)$ is $\chi(\sigma^{-1})((S_{ji} - b)b + (S_{kk} - b)b + b^2 \sum_{l\neq j,k}(S_{l\sigma^{-1}(l)} - 1))$.

Using the fact that

$$d_\chi(xS + (1-x)B_b) = d_\chi(xS' + (1-x)B_b)$$

for all $b \in [0, 1]$, we have that

$$\chi(\sigma^{-1})((S_{ji}-b)b+(S_{kk}-b)b+b^2 \sum_{l\neq j,k}(S_{l\sigma^{-1}(l)}-1)) = \chi(\sigma^{-1})((S'_{ji}-b)b+(S'_{kk}-b)b+b^2 \sum_{l\neq j,k}(S'_{l\sigma^{-1}(l)}-1)),$$

for all $b \in [0, 1]$. Consequently,

$$(S_{ji} + S_{kk})b + b^2 \left(\sum_{l\neq j,k}(S_{l\sigma^{-1}(l)} - 1) - 2\right) = (S'_{ji} + S'_{kk})b + b^2 \left(\sum_{l\neq j,k}(S'_{l\sigma^{-1}(l)} - 1) - 2\right)$$

for all $b \in [0, 1]$.

Then the coefficient of the term associated with b of the last polynomials are equal, i.e.,

$$S_{ji} + S_{kk} = S'_{ji} + S'_{kk} \tag{3.1}$$

for all $i, j, k \in \{1, \ldots, n\}$, distinct on pairs. Since $n \geq 4$, there is $p \notin \{i, j, k\}$ such that

$$S_{ji} + S_{pp} = S'_{ji} + S'_{pp}, \tag{3.2}$$

and subtracting the equalities (3.1) and (3.2), we obtain that

$$S_{kk} - S'_{kk} = S_{pp} - S'_{pp}$$

for all $k, p \in \{1, \ldots, n\}$.

If c is the constant defined by $c = S_{kk} - S'_{kk}$, then $S_{kk} = S'_{kk} + c$, and by (3.1), we obtain $S_{ji} = S'_{ji} - c$, for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

As $S, S' \in \Omega_n$, we have $S_{jj} + \sum_{j=1, j\neq i}^{n} S_{ji} = 1$ and $S'_{jj} + c + \sum_{j=1, j\neq i}^{n}(S'_{ji} - c) = 1$, which implies that $\sum_{j=1}^{n} S'_{ji} + (2-n)c = 1$. Since $n \neq 2$ then $c = 0$, which means that $S_{kk} = S'_{kk}$ and $S_{ji} = S'_{ji}$, for all $k, i, j \in \{1, \ldots, n\}$. Therefore, $S = S'$, and T is injective.

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Case (ii) Let $n = 3$ and $\chi = [2,1]$. Let us consider $\sigma = (ij)$ and $\tau = (ijk)$ for $\{i,\ j,\ k\} = \{1,\ 2,\ 3\}$. Then $\chi(\sigma) = 0$ and $\chi(\tau) \neq 0$. For each $b \in [0,1]$, consider the matrix $B_b = bP(\sigma) + (1-b)P(\tau)$. So, for all $p \in \{1,2,3\}$ and $\pi \in S_3$,

$$(B_b)_{p\pi(p)} = \begin{cases} 1 & \text{if } \pi(p) = \sigma^{-1}(p) = \tau^{-1}(p), \\ b & \text{if } \pi(p) = \sigma^{-1}(p),\ \pi(p) \neq \tau^{-1}(p), \\ 1-b & \text{if } \pi(p) \neq \sigma^{-1}(p),\ \pi(p) = \tau^{-1}(p), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} d_\chi(xS + (1-x)B_b) &= \sum_{\pi \in S_3} \chi(\pi) \prod_{l=1}^{3} (xS + (1-x)B_b)_{l\pi(l)} \\ &= \chi(\tau)(xS_{i\tau(i)} + (1-x)b)(xS_{j\tau(j)})(xS_{k\tau(k)}) + \chi(\tau^{-1})(xS_{i\tau^{-1}(i)} + (1-x)(1-b)) \\ &\quad \cdot(xS_{j\tau^{-1}(j)} + (1-x))(xS_{k\tau^{-1}(k)} + (1-x)(1-b)) + \chi(id)(xS_{ii})(xS_{jj})(xS_{kk} + (1-x)b). \end{aligned}$$

Since $\tau^{-1}(i) = k$, $\tau^{-1}(j) = i$ and $\tau^{-1}(k) = j$, the coefficient of the term associated with x of the polynomial $d_\chi(xS + (1-x)B_b)$ is $\chi(\tau^{-1})((1-b)((S_{ik} + S_{ji} + S_{kj} - 3) + b(-S_{ji} + 3)))$ for all $b \in [0,1]$.

Using the fact that

$$d_\chi(xS + (1-x)B_b) = d_\chi(xS' + (1-x)B_b),$$

we have that

$$\chi(\tau^{-1})((1-b)((S_{ij} + S_{ji} + S_{kj} - 3) + b(-S_{ji} + 3))) = \chi(\tau^{-1})((1-b)((S'_{ij} + S'_{ji} + S'_{kj} - 3) + b(-S'_{ji} + 3)))$$

for all $b \in [0,1]$. So, the coefficient of the term associated with b^2 of last polynomials are equal and this implies that

$$S_{ji} = S'_{ji}$$

for all $i \neq j$. Since $S_{ii} + S_{ji} + S_{ki} = 1 = S'_{ii} + S'_{ji} + S'_{ki}$, then $S_{ii} = S'_{ii}$, for all $i \in \{1,2,3\}$. Consequently, $S = S'$. So, T is injective. ◻

4. The image of a permutation matrix by T. Let $C \subseteq \Omega_n$ be a convex polyhedron. An element $S \in C$ is a vertex of C, if S satisfies:

$$\forall\, S_1, S_2 \in C: \ S = \alpha S_1 + (1-\alpha)S_2, \text{ with } \alpha \in]0,1[, \text{ it follows } S_1 = S_2 = S.$$

Let T be a semilinear map from Ω_n into Ω_n that preserves d_χ. Since Ω_n and $T(\Omega_n)$ are convex polyhedrons, and the permutation matrices are the vertices of Ω_n (see [2]), in the next step we will see that if $\sigma \in S_n$ then $T(P(\sigma))$ is a vertex of $T(\Omega_n)$.

Proposition 4.1. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear map from Ω_n into Ω_n that preserves d_χ. If $\sigma \in S_n$ then $T(P(\sigma))$ is a vertex of the convex polyhedron $T(\Omega_n)$.*

Proof. Let $S_1, S_2 \in \Omega_n$ and $\sigma \in S_n$ such that $T(P(\sigma)) = \alpha T(S_1) + (1-\alpha)T(S_2)$, for some $\alpha \in]0,1[$. Then by semilinearity of T we have $T(P(\sigma)) = T(\alpha S_1 + (1-\alpha)S_2)$. Using Theorem 3.1, $P(\sigma) = \alpha S_1 + (1-\alpha)S_2$,

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with $\alpha \in]0,1[$. As $P(\sigma)$ is a vertex of Ω_n, then $S_1 = S_2 = P(\sigma)$, which means that $T(S_1) = T(S_2) = T(P(\sigma))$, and $T(P(\sigma))$ is a vertex of $T(\Omega_n)$. □

In what follows, we consider that the semilinear map T from Ω_n into Ω_n is surjective. Since T preserves d_χ, we have that T is bijective and $T(\Omega_n) = \Omega_n$.

Corollary 4.2. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear surjective map from Ω_n into Ω_n that preserves d_χ. Then for each $\sigma \in S_n$ there is a $\pi \in S_n$, such that*

$$T(P(\sigma)) = P(\pi), \text{ where } \chi(\sigma) = \chi(\pi).$$

Definition 4.3. We say that two matrices S_1 and S_2 are equal to one in the position (i,j), if $(S_1)_{ij} = (S_2)_{ij} = 1$.

We denote by $c[S_1, S_2]$ the number of positions where S_1 and S_2 are equal to one. Consequently, if P is a permutation matrix and $S \in \Omega_n$, then $c[P, S]$ is equal to the number of ones of the matrix $xP + (1-x)S$, for all $x \in]0,1[$. In particular $c[I, S]$ is equal to the number of ones in the main diagonal of S.

Proposition 4.4. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear surjective map from Ω_n into Ω_n that preserves d_χ. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $S \in \Omega_n$. If $T(P(\sigma)) = P(\pi)$ and $T(S) = S'$, then*

$$\sum_{j=1}^{n} S_{j\sigma^{-1}(j)} = \sum_{j=1}^{n} S'_{j\pi^{-1}(j)}.$$

Proof. Let $x \in [0,1]$. First we will compute the coefficient of the term associated with x of the polynomial $d_\chi(xS + (1-x)P(\sigma)) = \sum_{\tau \in S_n} \chi(\tau) \prod_{j=1}^{n} (xS + (1-x)P(\sigma))_{j\tau(j)}$. If $\tau \neq \sigma^{-1}$, then there is $s \in \{1,\ldots,n\}$ such that $(xS + (1-x)P(\sigma))_{s\tau(s)} = xS_{s\tau(s)}$. Since τ and σ are bijections, there are, at least two integers $s, h \in \{1,\ldots,n\}$ with $s \neq h$ and $(xS + (1-x)P(\sigma))_{s\tau(s)} = xS_{s\tau(s)}$, $(xS + (1-x)P(\sigma))_{h\tau(h)} = xS_{h\tau(h)}$. Consequently, $\prod_{j=1}^{n} (xS + (1-x)P(\sigma))_{j\tau(j)}$ is a polynomial with the coefficient associated with x equal to zero. So, the coefficient of the term associated with x of the polynomial $d_\chi(xS + (1-x)P(\sigma))$ is obtained when $\tau = \sigma^{-1}$ and is equal to

$$\chi(\sigma^{-1}) \sum_{j=1}^{n} (S_{j\sigma^{-1}(j)} - 1).$$

As $d_\chi(xS + (1-x)P(\sigma)) = d_\chi(xS' + (1-x)P(\pi))$ we have that

$$\chi(\sigma^{-1}) \sum_{j=1}^{n} (S'_{j\sigma^{-1}(j)} - 1) = \chi(\pi^{-1}) \sum_{j=1}^{n} (S_{j\pi^{-1}(j)} - 1).$$

Consequently, we get the desired conclusion using Corollary 4.2 and the fact that $\chi(\sigma) \neq 0$. □

Corollary 4.5. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear surjective map from Ω_n into Ω_n that preserves d_χ. Let $\sigma \in S_n$ such that $\chi(\sigma) \neq 0$ and $\rho \in S_n$. If $T(P(\sigma)) = P(\pi)$, then*

$$c[P(\sigma), P(\rho)] = c[P(\pi), T(P(\rho))].$$

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Proof. By Proposition 4.4,

$$\sum_{j=1}^{n} P(\rho)_{j\sigma^{-1}(j)} = \sum_{j=1}^{n} T(P(\rho))_{j\pi^{-1}(j)}.$$

So we get the desired conclusion. □

Lemma 4.6. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear surjective map from Ω_n into Ω_n that preserves d_χ. Let $\rho, \theta \in S_n$ such that $T(P(\rho)) = P(\theta)$. If ρ is a transposition then θ is a transposition, and if ρ is a cycle of length three then θ is a cycle of length three.*

Proof. Let ρ be a cycle of length $2 \leq l \leq 3$, such that $T(P(\rho)) = P(\theta)$, then by Corollary 4.5,

$$c[I, P(\rho)] = n - l = c[I, P(\theta)].$$

If $l = 2$, then there are $i, j \in \{1, \dots, n\}$ such that $i \neq j$, $P(\theta)_{ii} = P(\theta)_{jj} = 0$, $P(\theta)_{kk} = 1$, for all $k \neq i, j$, and consequently, $P(\theta)_{ij} = P(\theta)_{ji} = 1$.

The case $l = 3$ can be proved using the same arguments. □

A semilinear map T is called unital if $T(I) = I$. When T is a semilinear map from Ω_n into Ω_n the case of a nonunital map can be reduced to the unital case by considering the semilinear map Φ defined by $\Phi(S) = T(I)^{-1}T(S)$, since $T(I)$ is invertible. Recall that by Corollary 2.4, if the irreducible character of degree greater than one, χ, verifies $\chi \neq [2, 2]$ and T preserves d_χ then $T(I) = I$.

Proposition 4.7. *Let χ be an irreducible character of degree greater than 1 of S_n. Let T be a semilinear unital surjective map from Ω_n into Ω_n that preserves d_χ. Then there is $\alpha \in S_n$ such that for all $i, j \in \{1, \dots, n\}$, $i \neq j$,*

$$T(P(ij)) = P(\alpha(i)\alpha(j)).$$

Proof. First we will prove two claims.

If X is a set, we denote by $|X|$ the cardinality of X.

Claim 1. Let $i, j, l, a, e, c, d \in \{1, \dots, n\}$ with i, j, l distinct on pairs. If $T(P(ij)) = P(ae)$ and $T(P(il)) = P(cd)$ then $|\{a, e, c, d\}| = 3$.

Proof of Claim 1. Using Lemma 4.6, since T is injective, $|\{a, e, c, d\}| \neq 2$.

Suppose that $|\{a, e, c, d\}| = 4$, which does not happen if $n = 3$. Let $S = bP(ij) + (1 - b)P(il)$, with $b \in [0, 1]$. Since, $T(S) = bP(ae) + (1 - b)P(cd)$, where $b \in [0, 1]$, and $d_\chi(xS + (1 - x)I) = d_\chi(xT(S) + (1 - x)I)$, then the coefficient of the term associated with x^2b^2 of both polynomials must be equal.

First we will compute the term associated with x^2b^2 of the polynomial

$$d_\chi(xS + (1 - x)I) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^{n} (xS + (1 - x)I)_{s\pi(s)}.$$

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When $\pi \in S_n$,

$$(S)_{s\pi(s)} = (bP(ij) + (1-b)P(il))_{s\pi(s)} = \begin{cases} 1 & \text{if } s \in \{1,\ldots,n\} \setminus \{i,j,l\}, \quad \pi(s) = s, \\ b & \text{if } (s,\pi(s)) \in \{(l,l),(i,j),(j,i)\}, \\ 1-b & \text{if } (s,\pi(s)) \in \{(l,i),(i,l),(j,j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, when $\pi \in S_n$,

$$(xS + (1-x)I)_{s\pi(s)} = \begin{cases} 1 & \text{if } s \in \{1,\ldots,n\} \setminus \{i,j,l\}, \quad \pi(s) = s, \\ 1-x & \text{if } (s,\pi(s)) = (i,i), \\ 1-xb & \text{if } (s,\pi(s)) = (j,j), \\ 1-x(1-b) & \text{if } (s,\pi(s)) = (l,l), \\ xb & \text{if } (s,\pi(s)) \in \{(i,j),(j,i)\}, \\ x(1-b) & \text{if } (s,\pi(s)) \in \{(l,i),(i,l)\}, \\ 0 & \text{otherwise.} \end{cases}$$

So, if $\pi \notin \{id,(ij),(il)\}$ and there is $h \in \{1,\ldots,n\} \setminus \{i,j,l\}$ with $\pi(h) \neq h$ then $(xS + (1-x)I)_{h\pi(h)} = 0$ and $\chi(\pi)\prod_{s=1}^{n}(xS + (1-x)I)_{s\pi(s)} = 0$. Consequently, if $\chi(\pi)\prod_{s=1}^{n}(xS + (1-x)I)_{s\pi(s)} \neq 0$ then $\pi(h) = h$, for all $h \in \{1,\ldots,n\} \setminus \{i,j,l\}$ and $\pi \in \{id,(ij),(il),(jl),(ijl),(ilj)\}$.

If $\pi = (jl)$ or $\pi = (ijl)$ then $(xS + (1-x)I)_{j\pi(j)} = 0$ and $\chi(\pi)\prod_{s=1}^{n}(xS + (1-x)I)_{s\pi(s)} = 0$.

If $\pi = (ilj)$ then $(xS + (1-x)I)_{l\pi(l)} = 0$ and $\chi(\pi)\prod_{s=1}^{n}(xS + (1-x)I)_{s\pi(s)} = 0$.

So, $d_\chi(xS+(1-x)I) = \chi(ij)(1-x(1-b))(xb)(xb)+\chi(id)(1-x(1-b))(1-xb)(1-x)+\chi(il)(1-xb)(x(1-b))^2$. Therefore, the coefficient of the term associated with x^2b^2 of the polynomial $d_\chi(xS + (1-x)I)$ is

$$-\chi(id) + \chi(ij) + \chi(il).$$

Now we will compute the term associated with x^2b^2 of the polynomial

$$d_\chi(xT(S) + (1-x)I) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^{n}(xT(S) + (1-x)I)_{s\pi(s)}.$$

When $\pi \in S_n$,

$$(T(S))_{s\pi(s)} = (bP(ae) + (1-b)P(cd))_{s\pi(s)} = \begin{cases} 1 & \text{if } s \in \{1,\ldots,n\} \setminus \{a,e,c,d\}, \quad \pi(s) = s, \\ b & \text{if } (s,\pi(s)) \in \{(a,e),(e,a),(c,c),(d,d)\}, \\ 1-b & \text{if } (s,\pi(s)) \in \{(c,d),(d,c),(a,a),(e,e)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, when $\pi \in S_n$,

$$(xT(S) + (1-x)I)_{s\pi(s)} = \begin{cases} 1 & \text{if } s \in \{1,\ldots,n\} \setminus \{a,e,c,d\}, \quad \pi(s) = s \\ 1-xb & \text{if } (s,\pi(s)) \in \{(a,a),(e,e)\}, \\ 1-x(1-b) & \text{if } (s,\pi(s)) \in \{(d,d),(c,c)\}, \\ xb & \text{if } (s,\pi(s)) \in \{(a,e),(e,a)\}, \\ x(1-b) & \text{if } (s,\pi(s)) \in \{(c,d),(d,c)\}, \\ 0 & \text{otherwise.} \end{cases}$$

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So, if $\pi \notin \{id, (ae), (cd)\}$ and there is $h \in \{1, \ldots, n\} \backslash \{a, e, c, d\}$ with $\pi(h) \neq h$ then $(xT(S)+(1-x)I)_{h\pi(h)} = 0$ and $\chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} = 0$. Consequently, if $\chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} \neq 0$ then $\pi(h) = h$, for all $h \in \{1, \ldots, n\} \setminus \{a, e, c, d\}$.

If $\pi(r) \in \{\pi(a), \pi(e)\} \subseteq \{c, d\}$ or $\pi(r) \in \{\pi(c), \pi(d)\} \subseteq \{a, e\}$, then $(xT(S) + (1-x)I)_{r\pi(r)} = 0$ and $\chi(\pi) \prod_{s=1}^{n} (xT(S) + (1-x)I)_{s\pi(s)} = 0$.

So, $d_\chi(xT(S)+(1-x)I) = \chi(ae)(1-x(1-b))^2(xb)^2+\chi(id)(1-x(1-b))^2(1-xb)^2+\chi(cd)(1-xb)^2(x(1-b))^2$. Therefore, the coefficient of the term associated with x^2b^2 of the polynomial $d_\chi(xT(S) + (1-x)I)$ is

$$-2\chi(id) + \chi(ae) + \chi(cd).$$

Since the polynomials $d_\chi(xS + (1-x)I)$ and $d_\chi(xT(S) + (1-x)I)$ are equal then the coefficients of the term associated with x^2b^2 of each polynomial are equal, i.e.,

$$-\chi(id) + \chi(ij) + \chi(il) = -2\chi(id) + \chi(ae) + \chi(cd).$$

Because $\chi(id) \neq 0$, we obtain a contradiction. Consequently, $|\{a, e, c, d\}| = 3$. ∎

Claim 2. Let $i, j, l, a, e, d \in \{1, \ldots, n\}$ with i, j, l distinct on pairs and a, e, d distinct on pairs. If $T(P(ij)) = P(ae)$ and $T(P(il)) = P(ad)$, then

$$T(P(jl)) = P(ed).$$

Proof of Claim 2. If $T(P(jl)) = P(gf)$, using Claim 1, we conclude that $|\{a, e, g, f\}| = 3$ and $|\{a, d, g, f\}| = 3$.

Let us assume that $g = a$. Then $f \neq a$, $f \neq e$ and $f \neq d$, and consequently $|\{a, e, d, f\}| = 4$.

Let $S = b_1P(ij) + b_2P(il) + (1-(b_1+b_2))P(jl)$, with $b_1, b_2 \in [0, 1]$ and $b_1 + b_2 \leq 1$. Since, $d_\chi(xS + (1-x)I) = d_\chi(xT(S) + (1-x)I)$, then the coefficient of the term associated with $x^4b_1b_2$ of both polynomials must be equal.

First we will compute the term associated with $x^4b_1b_2$ of the polynomial

$$d_\chi(xS + (1-x)I) = \sum_{\pi \in S_n} \chi(\pi) \prod_{s=1}^{n} (xS + (1-x)I)_{s\pi(s)}.$$

When $\pi \in S_n$,

$$\begin{aligned} (S)_{s\pi(s)} &= (b_1P(ij) + b_2P(il) + (1-(b_1+b_2))P(jl))_{s\pi(s)} \\ &= \begin{cases} 1 & \text{if } s \in \{1, \ldots, n\} \setminus \{i, j, l\}, \quad \pi(s) = s, \\ b_1 & \text{if } (s, \pi(s)) \in \{(l, l), (i, j), (j, i)\}, \\ b_2 & \text{if } (s, \pi(s)) \in \{(j, j), (i, l), (l, i)\}, \\ 1-(b_1+b_2) & \text{if } (s, \pi(s)) \in \{(i, i), (j, l), (l, j)\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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Consequently, when $\pi \in S_n$,

$$(xS+(1-x)I)_{s\pi(s)} = \begin{cases} 1 & \text{if } s \in \{1,\dots,n\}\setminus\{i,j,l\}, \quad \pi(s)=s, \\ 1-x(b_1+b_2) & \text{if } (s,\pi(s))=(i,i), \\ 1-x(1-b_2) & \text{if } (s,\pi(s))=(j,j), \\ 1-x(1-b_1) & \text{if } (s,\pi(s))=(l,l), \\ xb_1 & \text{if } (s,\pi(s))\in\{(i,j),(j,i)\}, \\ xb_2 & \text{if } (s,\pi(s))\in\{(i,l),(l,i)\}, \\ x(1-(b_1+b_2)) & \text{if } (s,\pi(s))\in\{(l,j),(j,l)\}, \\ 0 & \text{otherwise.} \end{cases}$$

So, if $\pi \in S_n$ and there is $h \in \{1,\dots,n\}\setminus\{i,j,l\}$ with $\pi(h)\neq h$ then $(xS+(1-x)I)_{h\pi(h)}=0$ and $\chi(\pi)\prod_{s=1}^n(xS+(1-x)I)_{s\pi(s)}=0$. If $\pi\in S_n$ and for all $h\in\{1,\dots,n\}\setminus\{i,j,l\}$, $\pi(h)=h$ then $(xS+(1-x)I)_{h\pi(h)}=1$. Consequently, the degree of the polynomial $d_\chi(xS+(1-x)I)$ is less than or equal to three. Therefore, the coefficient of the term associated with $x^4b_1b_2$ of the polynomial $d_\chi(xS+(1-x)I)$ is zero.

Now we will compute the term associated with $x^4b_1b_2$ of the polynomial $d_\chi(xT(S)+(1-x)I)=\sum_{\pi\in S_n}\chi(\pi)\prod_{s=1}^n(xT(S)+(1-x)I)_{s\pi(s)}$. When $\pi\in S_n$,

$$\begin{aligned}(T(S))_{s\pi(s)} &= (b_1P(ae)+b_2P(ad)+(1-(b_1+b_2))P(af))_{s\pi(s)} \\ &= \begin{cases} 1 & \text{if } s\in\{1,\dots,n\}\setminus\{a,e,c,d\}, \quad \pi(s)=s, \\ b_1 & \text{if } (s,\pi(s))\in\{(a,e),(e,a)\}, \\ b_2 & \text{if } (s,\pi(s))\in\{(a,d),(d,a)\}, \\ 1-(b_1+b_2) & \text{if } (s,\pi(s))\in\{(a,f),(f,a)\}, \\ 1-b_1 & \text{if } (s,\pi(s))=(e,e), \\ 1-b_2 & \text{if } (s,\pi(s))=(d,d), \\ b_1+b_2 & \text{if } (s,\pi(s))=(f,f), \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Consequently, when $\pi \in S_n$,

$$(xT(S)+(1-x)I)_{s\pi(s)} = \begin{cases} 1 & \text{if } s\in\{1,\dots,n\}\setminus\{a,e,c,d\}, \quad \pi(s)=s, \\ xb_1 & \text{if } (s,\pi(s))\in\{(a,e),(e,a)\}, \\ xb_2 & \text{if } (s,\pi(s))\in\{(a,d),(d,a)\}, \\ x(1-(b_1+b_2)) & \text{if } (s,\pi(s))\in\{(a,f),(f,a)\}, \\ 1-xb_1 & \text{if } (s,\pi(s))=(e,e), \\ 1-xb_2 & \text{if } (s,\pi(s))=(d,d), \\ 1-x(1-(b_1+b_2)) & \text{if } (s,\pi(s))=(f,f), \\ 1-x & \text{if } (s,\pi(s))=(a,a), \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi\notin\{id,(ae),(af),(ad)\}$ then there is $h\in\{1,\dots,n\}$ with $(h,\pi(h))\notin\{(h,h),(a,e),(e,a),(a,f),(f,a),(a,d),(d,a)\}$. Consequently, $(xT(S)+(1-x)I)_{h\pi(h)}=0$. Then $\chi(\pi)\prod_{h=1}^n(xT(S)+(1-x)I)_{h\pi(h)}=0$. So, $d_\chi(xT(S)+(1-x)I)=\chi(id)(1-xb_1)(1-xb_2)(1-x)(1-x(1-(b_1+b_2)))+\chi(ae)(xb_1)^2(1-xb_2)(1-x(1-(b_1+b_2)))+\chi(ad)(xb_2)^2(1-xb_1)(1-x(1-(b_1+b_2)))+\chi(af)x^2(1-(b_1+b_2))^2(1-xb_1)(1-xb_2)=\chi(id)(1-x(b_1+b_2+1)+x^2(b_1+b_2+b_1b_2)-x^3b_1b_2)(1-x(1-(b_1+b_2)))+\cdots+\chi(af)x^2(1-2b_1-2b_2+b_1^2+b_2^2+2b_1b_2)(1-x(b_1+b_2)+x^2b_1b_2)$.

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Then the coefficient of the polynomial $d_\chi(xT(S)+(1-x)I)$ associated with $x^4b_1b_2$ is $\chi(id)+\chi(af)$.

Since $d_\chi(xS+(1-x)I)=d_\chi(xT(S)+(1-x)I)$, then the coefficient of the term associated with $x^4b_1b_2$ of both polynomials must be equal, i.e.,

$$0=\chi(id)+\chi(af).$$

But this is impossible by Remark 2.1 when $\chi\neq[2,2]$, and because $\chi(id)=2$, $\chi(af)=0$, when $\chi=[2,2]$. Thus, $g\neq a$, and using the same argument, we have that $f\neq a$. Therefore, $g=e$, so $f=d$ since $|\{a,d,g,f\}|=3$, and we conclude that $T(P(jl))=P(ed)$. ∎

For all $i,j\in\{1,\dots,n\}$ with $i\neq j$ let us consider $k\in\{1,\dots,n\}$, such that $k\neq i,\ k\neq j$. Let us assume that $\{i,j,k\}=\{1,2,3\}$.

Using Lemma 4.6, there are $j_1,j_2,j_3,j_4\in\{1,\dots,n\}$ such that $T(P(12))=P(j_1j_2)$, $T(P(13))=P(j_3j_4)$. By Claim 1 $|\{j_1,j_2,j_3,j_4\}|=3$. Let $\alpha(1)=i_1$ where $i_1\in\{j_1,j_2\}\cap\{j_3,j_4\}$. Let $\alpha(2)=i_2$, where $i_2\in\{j_1,j_2\}\setminus\{i_1\}$, and $\alpha(3)=i_3$, where $i_3\in\{j_3,j_4\}\setminus\{i_1\}$.

Using this construction, we can define a function

$$\alpha:\{1,\dots,n\}\longrightarrow\{1,\dots,n\},$$

where $\alpha(r)=i_r$.

Using Claim 2 and the injectivity of T, we conclude that $\alpha\in S_n$. □

5. Proof of the main result. Let χ be an irreducible character of degree greater than 1 of S_n. In this section, we characterize the semilinear surjective maps T from Ω_n into Ω that preserve d_χ (Theorem 1.2).

By the Murnaghan-Nakayama Rule (mentioned in Section 2), if χ is an irreducible character of S_n and p is the number of boundary boxes of the Young Diagram associated with χ, then $\chi(\xi)\neq 0$ whenever ξ is a cycle of length p. On what follows we consider $\alpha\in S_n$ obtained using Proposition 4.7.

PROPOSITION 5.1. *Let χ be an irreducible character of S_n of degree greater than one and p be the number of boundary boxes of the Young Diagram associated with χ. Let T be a semilinear unital surjective map from Ω_n into Ω_n that preserves d_χ. Let $\xi\in S_n$ be a cycle of length p and $T(P(\xi))=P(\rho)$. Then*

$$\rho=\alpha\circ\xi\circ\alpha^{-1}$$

or

$$\rho=\alpha\circ\xi^{-1}\circ\alpha^{-1}.$$

Proof. Let $\xi=(i_1i_2\cdots i_p)$. Then, by Corollary 4.5, $c[I,P(\xi)]=c[I,P(\rho)]=n-p$. Let $S=P(i_1i_2)$. Then

$$S'=T(S)=T(P(i_1i_2))=P(\alpha(i_1)\alpha(i_2))$$

and, by Corollary 4.5,

$$c[P(\xi),P(i_1i_2)]=n-p+1=c[P(\rho),P(\alpha(i_1)\alpha(i_2))],$$

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i.e., $\rho^{-1}(\alpha(i_1)) = \alpha(i_2)$, or $\rho^{-1}(\alpha(i_2)) = \alpha(i_1)$, and both cases cannot happen at the same time because ρ is not a transposition.

Repeating the same argument with $S = P(i_t i_{t+1})$, where $t \in \{2, \dots, p-1\}$, and using the bijectivity of ρ and α, we must have

(1) $\rho^{-1}(\alpha(i_1)) = \alpha(i_2)$, $\rho^{-1}(\alpha(i_2)) = \alpha(i_3), \dots$ $\rho^{-1}(\alpha(i_p)) = \alpha(i_1)$, or
(2) $\rho^{-1}(\alpha(i_1)) = \alpha(i_p)$, $\rho^{-1}(\alpha(i_p)) = \alpha(i_{p-1}), \dots$ $\rho^{-1}(\alpha(i_2)) = \alpha(i_1)$.

So by definition of ξ we have that

(1) $\alpha^{-1} \circ \rho^{-1} \circ \alpha = \xi^{-1}$, or
(2) $\alpha^{-1} \circ \rho^{-1} \circ \alpha = \xi$.

Then $\rho = \alpha \circ \xi \circ \alpha^{-1}$ or $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$. □

In Proposition 5.2, we will prove that if $\beta,\ \gamma \in S_n$ are cycles of length p, then we cannot have $T(P(\beta)) = P(\alpha \circ \beta \circ \alpha^{-1})$, and $T(P(\gamma)) = P(\alpha \circ \gamma^{-1} \circ \alpha^{-1})$.

Proposition 5.2. *Let χ be an irreducible character of S_n of degree greater than one and p be the number of boundary boxes of the Young Diagram associated with χ. Let T be a semilinear unital surjective map from Ω_n into Ω_n that preserves d_χ. Suppose that if $p = 4$ then $n \neq p$. Let $\xi \in S_n$ be a cycle of length p and $T(P(\xi)) = P(\rho)$.*

1) *If $\rho = \alpha \circ \xi \circ \alpha^{-1}$ then $T(P(\theta)) = P(\alpha \circ \theta \circ \alpha^{-1})$ whenever $\theta \in S_n$ is a cycle of length p.*
2) *If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$ then $T(P(\theta)) = P(\alpha \circ \theta^{-1} \circ \alpha^{-1})$ whenever $\theta \in S_n$ is a cycle of length p.*

Proof. We will prove part 1). The proof will be divided into two cases:

Case 1. Let $p \neq n$.

Claim 1. If $i, j \in \{1, \dots, n\}$ with $i \neq j$, verify $\xi(i) = i$ and $\xi(j) \neq j$ then $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$.

Proof of Claim 1. Since $(ij) \circ \xi \circ (ij)$ is a cycle of length p, by Proposition 5.1, $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$ or $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$. Suppose that $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$.

Let $S = P((ij) \circ \xi \circ (ij))$. By Proposition 4.4, $S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)}$, where $S' = T(S)$. Since

$$S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = |\{a : (ij) \circ \xi^{-1}(a) = \xi^{-1} \circ (ij)(a)\}| = n - 3$$

and

$$S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)} = |\{a : (ij) \circ \xi^{-1}(a) = \xi \circ (ij)(a)\}| = n - p - 1,$$

then $p = 2$ (impossible). So $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$. ∎

Let $\theta = (a\ \theta(a)\ \cdots\ \theta^{p-1}(a))$ be a cycle of length p, $\theta \neq \xi$, where $a \in \{1, \dots, n\}$ and

$$\theta^l(a) = \begin{cases} a & \text{if } l = 0, \\ \theta(\theta^{l-1}(a)) & \text{if } l > 0. \end{cases}$$

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If $\xi(a) = a$, let t be an integer such that $\xi(t) \neq t$. Using Claim 1,

$$TP((at) \circ \xi \circ (at)) = P(\alpha \circ (at) \circ \xi \circ (at) \circ \alpha^{-1})$$

and if $\beta = (at) \circ \xi \circ (at)$ then β is a cycle of length p verifying $\beta(a) \neq a$. So, we can assume that $\xi(a) \neq a$.

Let s be the smallest positive integer that $\theta^s(a) \neq \xi(\theta^{s-1}(a))$. Consequently, $s < p$ and $\theta^u(a) = \xi^u(a)$, for $u = 0, \dots, s-1$.

- If $\xi(\theta^s(a)) = \theta^s(a)$, let $\xi(\theta^{s-1}(a)) = r$ (note that we have $\xi(\theta^{s-1}(a)) \neq \theta^{s-1}(a)$ because $\xi(\theta^{s-1}(a)) = \theta^{s-1}(a)$ implies that $\xi(a) = a$). Using Claim 1,

$$TP((\theta^s(a)r) \circ \xi \circ (\theta^s(a)r)) = P(\alpha \circ (\theta^s(a)r) \circ \xi \circ (\theta^s(a)r) \circ \alpha^{-1})$$

  and if $\beta_1 = (\theta^s(a)r) \circ \xi \circ (\theta^s(a)r)$ then β_1 is a cycle of length p verifying $\beta_1^u(a) = \theta^u(a)$, for $u = 0, \dots, s$.
- If $\xi(\theta^s(a)) \neq \theta^s(a)$, let $\xi(\theta^{s-1}(a)) = r$. Since $n \neq p$, let k be an integer such that $\xi(k) = k$. Using Claim 1,

$$TP((\theta^s(a)k) \circ \xi \circ (\theta^s(a)k)) = P(\alpha \circ (\theta^s(a)k) \circ \xi \circ (\theta^s(a)k) \circ \alpha^{-1})$$

  and if $\beta_2 = (\theta^s(a)k) \circ \xi \circ (\theta^s(a)k)$ then β_2 is a cycle of length p verifying $\beta_2^u(a) = \theta^u(a)$, for $u = 0, \dots, s-1$, $\beta_2(\theta^s(a)) = \theta^s(a)$ and $\beta_2(\theta^{s-1}(a)) = r$. Using what we proved above, we conclude that there is a cycle of length p, β_3, such that $\beta_3^u(a) = \theta^u(a)$, for $u = 0, \dots, s$ and $TP(\beta_3) = P(\alpha\beta_3\alpha^{-1})$.

Repeating this argument, we prove the result.

Case 2. Let $n = p \neq 4$.

Claim 2. If $i, j \in \{1, \dots, n\}$, with $i \neq j$, verify $\xi(i) = j$, then $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$.

Proof of Claim 2. Using a similar argument as in Claim 1, suppose that $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi^{-1} \circ (ij) \circ \alpha^{-1})$.

Let $S = P((ij) \circ \xi \circ (ij))$. By Proposition 4.4, $S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)}$, where $S' = T(S)$. Since

$$S_{1\xi^{-1}(1)} + \cdots + S_{n\xi^{-1}(n)} = |\{a : (ij) \circ \xi^{-1}(a) = \xi^{-1} \circ (ij)(a)\}| = n - 3$$

and

$$S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)} = |\{a : (ij) \circ \xi^{-1}(a) = \xi \circ (ij)(a)\}| = n, \text{ if } p = 3$$

or

$$S'_{1\rho^{-1}(1)} + \cdots + S'_{n\rho^{-1}(n)} = |\{a : (ij) \circ \xi^{-1}(a) = \xi \circ (ij)(a)\}| = n - p + 1, \text{ if } p > 3$$

then $p = 4$ (impossible). So $T(P((ij) \circ \xi \circ (ij))) = P(\alpha \circ (ij) \circ \xi \circ (ij) \circ \alpha^{-1})$. ∎

Let $\theta = (a\ \theta(a)\ \cdots\ \theta^{p-1}(a))$ be a cycle of length p, $\theta \neq \xi$, where $a \in \{1, \dots, n\}$ and

$$\theta^l(a) = \begin{cases} a & \text{if } l = 0, \\ \theta(\theta^{l-1}(a)) & \text{if } l > 0. \end{cases}$$

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Since $n = p$ then $\xi(a) \neq a$. Let s be the smallest positive integer that $\theta^s(a) \neq \xi(\theta^{s-1}(a))$. Consequently, $s < p-1$ and $\theta^u(a) = \xi^u(a)$, for $u = 0, \ldots, s-1$. Since $n = p$, there is an integer k such that $p-1 \geq k > s$ and $\xi^k(a) = \theta^s(a)$. Using Claim 2,

$$TP((\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a)) = P(\alpha \circ (\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a)) \circ \alpha^{-1}),$$

and if $\beta_4 = (\xi^k(a)\xi^{k-1}(a)) \circ \xi \circ (\xi^k(a)\xi^{k-1}(a))$, then β_4 is a cycle of length p verifying $\beta_4^u(a) = \theta^u(a)$, for $u = 0, \ldots, s-1$ and $\beta_4^{k-1}(a) = \xi^k(a) = \theta^s(a)$. Using this argument we obtain a cycle of length p, β_5, such that $\beta_5^u(a) = \theta^u(a)$, for $u = 0, \ldots, s$ and $TP(\beta_5) = P(\alpha\beta_5\alpha^{-1})$.

Repeating this argument, we prove the result.

The proof of part 2) is analogous. □

For each $i, j \in \{1, \ldots, n\}$ let $U_{i,j}$ be the subset of Ω_n such that

$$U_{i,j} = \{P \in \Omega_n : P \text{ is a permutation matrix and } P_{ij} = 1\}.$$

These sets are very important for our study.

Proposition 5.3. *Let χ be an irreducible character of S_n of degree greater than one, χ, and p be the number of boundary boxes of the Young Diagram associated with χ. Let T be a unital semilinear surjective map from Ω_n into Ω_n that preserves d_χ. Let $i, j \in \{1, \ldots, n\}$ where $i \neq j$, and P be a permutation matrix, such that $P \in U_{i,j}$. Assume that ξ is a cycle of length p, and $T(P(\xi)) = P(\rho)$. Then one of the following conditions must hold:*

(1) *If $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(P) \in U_{\alpha(i),\alpha(j)}$.*
(2) *If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(P) \in U_{\alpha(j),\alpha(i)}$.*

Proof. We will prove (1). Let $\pi \in S_n$ such that $\pi(j) = i$. Therefore, $P(\pi) \neq I$ and $P(\pi) \in U_{i,j}$. By hypothesis, $\rho = \alpha \circ \xi \circ \alpha^{-1}$. We will see that $T(P(\pi)) \in U_{\alpha(i),\alpha(j)}$. Let $P(\theta) = T(P(\pi))$. We shall consider several cases:

Case 1. Let $n \geq 5$. If $n \geq 5$, and the number of boundary boxes of the Young diagram associated with χ is p, then $p \geq 4$. Suppose that $T(P(\pi)) = P(\theta) \notin U_{\alpha(i),\alpha(j)}$, i.e.,

$$\alpha^{-1} \circ \theta \circ \alpha(j) \neq i$$

. Let $\theta' = \alpha^{-1} \circ \theta \circ \alpha$, then by Corollary 4.5,

$$c[P(\varsigma), P(\pi)] = c[T(P(\varsigma)), P(\theta)],$$

whenever ς is a cycle of length p.

Since $n \geq 5$, we can choose $a \in \{1, \ldots, n\}$ such that

$$a \neq i, \quad a \neq j, \quad \pi(a) \neq j,$$

and we can choose $b \in \{1, \ldots, n\}$ such that

$$b \neq i, \quad b \neq j, \quad b \neq a, \quad \theta'(a) \neq b, \quad \text{and} \quad \theta'(b) \neq j.$$

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Let us consider the cycles ξ_1 and η of length p, defined by

$$\xi_1(a) = b, \quad \xi_1(b) = j, \quad \xi_1(j) = i, \quad \eta(a) = j, \quad \eta(j) = b, \quad \eta(b) = i,$$

and $\xi_1(q) = \eta(q)$ for all $q \notin \{a, b, j\}$.

Since $\xi_1(j) = \pi(j)$ and $\eta(q) \neq \pi(q)$ for all $q \in \{a, b, j\}$, then

$$c[P(\xi_1), P(\pi)] > c[P(\eta), P(\pi)],$$

which implies that

$$c[T(P(\xi_1)), P(\theta)] > c[T(P(\eta)), P(\theta)].$$

By Proposition 5.2, we have

$$c[P(\alpha \circ \xi_1 \circ \alpha^{-1}), P(\theta)] > c[P(\alpha \circ \eta \circ \alpha^{-1}), P(\theta)].$$

Since $\xi_1(q) \neq \theta'(q)$ for all $q \in \{a, b, j\}$, then

$$c[P(\alpha \circ \xi_1 \circ \alpha^{-1}), P(\theta)] \leq c[P(\alpha \circ \eta \circ \alpha^{-1}), P(\theta)],$$

which is a contradiction. So $T(P(\pi)) \in U_{\alpha(i),\alpha(j)}$.

Case 2. Let $n = 3$ and $\chi = [2, 1]$. Since $p = 3$, if π is a cycle of length 3, then the result is obtained using Proposition 5.2. If π is a cycle of length 2, then the result is obtained using Proposition 4.7.

Case 3. Let $n = 4$ and $\chi = [3, 1]$ or $\chi = [2, 1, 1]$. In this case, we can not use Proposition 5.2 since the number of boundary boxes of the Young Diagram associated with χ is $p = 4$. If π is a cycle of length 2, then the result is obtained using Proposition 4.7.

Let $\pi = (ij) \circ (kl)$ with i, j, k, l distinct on pairs, then by Corollary 4.5 (in this case, if σ is a transposition then $\chi(\sigma) = 1$ or -1),

$$c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))].$$

Since $c[I, P(\pi)] = 0$ then $c[I, T(P(\pi))] = 0$. So, $\theta(\alpha(i)) = \alpha(j)$ and $\theta(\alpha(j)) = \alpha(i)$. Therefore, $P(\theta) \in U_{\alpha(i),\alpha(j)}$.

Let i, j, k distinct on pairs. If $\pi = (jik)$, using Lemma 4.6, $T(P(jik)) = P(abc)$, where a, b, c are distinct on pairs. Since $\chi(ij) \neq 0$ (in this case, $\chi(ij) = 1$ or $\chi(ij) = -1$), by Corollary 4.5 we have $c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))]$. Since $c[I, P(\pi)] = 1$ then $c[I, T(P(\pi))] = 1$. So,

$$(abc)(\alpha(i)) = \alpha(j) \quad \text{or} \quad (abc)(\alpha(j)) = \alpha(i),$$

(only one of these conditions because (abc) is not a transposition).

In the same way, using the transposition (ik),

$$(abc)(\alpha(i)) = \alpha(k) \quad \text{or} \quad (abc)(\alpha(k)) = \alpha(i)$$

and using the transposition (kj),

$$(abc)(\alpha(k)) = \alpha(j) \quad \text{or} \quad (abc)(\alpha(j)) = \alpha(k).$$

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Consequently,

$$(abc) = (\alpha(i)\alpha(j)\alpha(k)) \quad \text{or} \quad (abc) = (\alpha(j)\alpha(i)\alpha(k)).$$

Since ξ is a cycle of length 4, then ξ is one of the following permutations

$$(jikl) \quad \text{or} \quad (jilk) \quad \text{or} \quad (jlik) \tag{5.1}$$

or

$$(jlki) \quad \text{or} \quad (jkil) \quad \text{or} \quad (jkli), \tag{5.2}$$

with $l \in \{1, 2, 3, 4\} \setminus \{j, i, k\}$.

If ξ is equal to a permutation of (5.2), then $c[P(\xi), P(\pi)] = 0$. Using Corollary 4.5 (recall that $\chi(\xi) \neq 0$), $c[P(\alpha \circ \xi \circ \alpha^{-1}), P(abc)] = 0$. Since $\alpha \circ \xi \circ \alpha^{-1}(\alpha(i)) = \alpha(j)$ or $\alpha \circ \xi \circ \alpha^{-1}(\alpha(j)) = \alpha(k)$, we conclude that $(abc) = (\alpha(j)\alpha(i)\alpha(k))$.

If ξ is equal to a permutation of (5.1), then $c[P(\xi), P(\pi)] = 2$. Using Corollary 4.5, $c[P(\alpha\circ\xi\circ\alpha^{-1}), P(abc)] = 2$. Since $(abc)(\alpha(l)) = \alpha(l)$, we conclude that $(abc) = (\alpha(j)\alpha(i)\alpha(k))$. Therefore, $P(\theta) = T(P(jik)) = P(\alpha(j)\alpha(i)\alpha(k)) \in U_{\alpha(i),\alpha(j)}$.

If $\pi = (jikl)$ is a cycle of length 4, with i, j, k, l distinct on pairs, then $c[I, P(\pi)] = 0 = c[I, P(\theta)]$. Considering the transposition (ij) and using Corollary 4.5 we get $c[P(ij), P(\pi)] = 1 = c[P(\alpha \circ (ij) \circ \alpha^{-1}), P(\theta)]$. Then

$$\theta(\alpha(j)) = \alpha(i) \quad \text{or} \quad \theta(\alpha(i)) = \alpha(j).$$

Suppose that $\theta(\alpha(i)) = \alpha(j)$. Considering the permutation (jik) and using Corollary 4.5, we get $c[P(jik), P(\pi)] = 2 = c[P(\alpha \circ (jik) \circ \alpha^{-1}), P(\theta)]$. Then

$$\theta(\alpha(i)) = \alpha(k) \quad \text{and} \quad \theta(\alpha(k)) = \alpha(j).$$

So, $\alpha(k) = \theta(\alpha(i)) = \alpha(j)$. Impossible because θ is a permutation. Consequently, $\theta(\alpha(j)) = \alpha(i)$ and $P(\theta) = T(P(jikl)) \in U_{\alpha(i),\alpha(j)}$.

Case 4. Let $n = 4$ and $\chi = [2, 2]$. Since $p = 3$, if π is a cycle of length 3, then the result is obtained using Proposition 5.2. If π is a cycle of length 2, then the result is obtained using Proposition 4.7.

Let i, j, k, l distinct on pairs. Let $\pi = (ij) \circ (kl)$ then

$$c[P(ij), P(\pi)] = 2 = c[P(\alpha(i)\alpha(j)), T(P(\pi))]$$

(in this case, $\chi((ij) \circ (kl)) = 2 \neq 0$). Since $c[I, P(\pi)] = 0$ then $c[I, T(P(\pi))] = 0$. So, $\theta(\alpha(i)) = \alpha(j)$ and $\theta(\alpha(j)) = \alpha(i)$. Therefore, $P(\theta) \in U_{\alpha(i),\alpha(j)}$.

Let $\pi = (jikl)$ with i, j, k, l distinct on pairs, then

$$c[P(jik), P(\pi)] = 2 = c[P(\alpha(j)\alpha(i)\alpha(k)), T(P(\pi))]$$

(in this case, $\chi(jik) = -1 \neq 0$). Since $c[I, P(\pi)] = 0$ then $c[I, T(P(\pi))] = 0$. So, we must have two of these cases, $\theta(\alpha(j)) = \alpha(i)$ or $\theta(\alpha(i)) = \alpha(k)$ or $\theta(\alpha(k)) = \alpha(j)$, (recall that $P(\theta) = T(P(\pi))$). In the

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same way, using (jil) we must have two of these cases, $\theta(\alpha(i)) = \alpha(l)$ or $\theta(\alpha(l)) = \alpha(j)$ or $\theta(\alpha(j)) = \alpha(i)$. If $\theta(\alpha(j)) \neq \alpha(i)$ then $\theta(\alpha(i)) = \alpha(k)$, $\theta(\alpha(k)) = \alpha(j)$ and $\theta(\alpha(i)) = \alpha(l)$. Impossible because θ is a permutation. Consequently, $\theta(\alpha(j)) = \alpha(i)$.

Therefore, $P(\theta) = T(P(ijkl)) \in U_{\alpha(i),\alpha(j)}$.

The proof of part 2) is analogous. □

Now we are in conditions to prove the main result of this paper.

Proof of Theorem 1.2. If there are $\sigma, \alpha \in S_n$, with $\chi(\sigma) = \chi(id)$, such that

$$T(S) = P(\sigma)P(\alpha)SP(\alpha^{-1}),$$

for all $S \in \Omega_n$, we have that

$$d_\chi(T(S)) = \sum_{\pi \in S_n} \chi(\pi) \prod_{j=1}^{n} T(S)_{j\pi(j)} = \sum_{\rho \in S_n} \chi(\alpha \circ \rho \circ \alpha^{-1} \circ \sigma^{-1}) \prod_{j=1}^{n} S_{j\rho(j)}.$$

Since $\chi(\sigma) = \chi(id)$ then $\chi(\alpha \circ \rho \circ \alpha^{-1} \circ \sigma^{-1}) = \chi(\alpha \circ \rho \circ \alpha^{-1}) = \chi(\rho)$ (see Remark 2.1). Consequently, $d_\chi(T(S)) = \sum_{\rho \in S_n} \chi(\rho) \prod_{j=1}^{n} S_{j\rho(j)} = d_\chi(S)$. Therefore, the map T preserves d_χ.

The proof of the case when $T(S) = P(\sigma)P(\alpha)S^T P(\alpha^{-1})$ is similar.

Conversely, suppose that the map T preserves d_χ and is unital.

Let p be the number of boundary boxes of the Young Diagram associated with χ and let $\alpha \in S_n$ obtained using Proposition 4.7.

Claim 1. Let P be a permutation matrix, such that $P \in U_{ii}$. Then $T(P) \in U_{\alpha(i)\alpha(i)}$.

Proof of Claim 1. Suppose that $P = P(\pi)$ with $\pi \in S_n$. Let $k = c[P, I]$. By Corollary 4.5, $k = c[T(P), I]$. Let $i_1, \ldots, i_{n-k}$ be distinct on pairs, such that $\pi(i_j) \neq i_j$, for all $j \in \{1, \ldots, n-k\}$.

Assume that ξ is a cycle of length p, $T(P(\xi)) = P(\rho)$, with $\rho = \alpha \circ \xi \circ \alpha^{-1}$ (condition 1) of Proposition 5.3). Since $P \in U_{\pi(i_j)i_j}$, then $T(P) \in U_{\alpha(\pi(i_j))\alpha(i_j)}$, for all $j \in \{1, \ldots, n-k\}$. As $k = c[T(P), I]$, then $T(P) \in U_{r_t r_t}$, where $r_t \in \{1, \ldots, n\} \setminus \{\alpha(i_1), \ldots, \alpha(i_{n-k})\}$.

Let us consider p_t, for all $t \in \{1, \ldots, k\}$, such that $\alpha(p_t) = r_t$, then $p_1, \ldots, p_k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{n-k}\}$. Since $P \in U_{i,i}$ then $\pi(i) = i$ and there exists $p_j \in \{p_1, \ldots, p_k\}$ such that $p_j = i$. Since $\alpha(i) = \alpha(p_j) = r_j$ then $T(P) \in U_{\alpha(i)\alpha(i)}$.

If we are in the condition 2) of Proposition 5.3, the proof is analogous. ∎

Claim 2. Assume that ξ is a cycle of length p, and $T(P(\xi)) = P(\rho)$. Then one of the following conditions must hold:

(1) If $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(U_{i,j}) = U_{\alpha(i),\alpha(j)}$, $\forall i, j$.
(2) If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(U_{i,j}) = U_{\alpha(j),\alpha(i)}$, $\forall i, j$.

Proof of Claim 2. By Propositions 5.3 and Claim 1, we know that

(1) if $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(U_{i,j}) \subseteq U_{\alpha(i),\alpha(j)}$, $\forall i, j$;
(2) if $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(U_{i,j}) \subseteq U_{\alpha(j),\alpha(i)}$, $\forall i, j$.

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Since

$$\varphi : U_{i,j} \longrightarrow U_{k,l}$$

$$P \longmapsto P(ik)PP(jl)$$

is a bijective map, then

$$|U_{i,j}| = |U_{k,l}|, \quad \forall i, j, k, l.$$

So,

(1) if $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(U_{i,j}) = U_{\alpha(i),\alpha(j)}$, $\forall i, j$;
(2) if $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(U_{i,j}) = U_{\alpha(j),\alpha(i)}$, $\forall i, j$. ∎

Claim 3. Assume that ξ is a cycle of length p, and $T(P(\xi)) = P(\rho)$. Then one of the following conditions must hold:

(1) If $\rho = \alpha \circ \xi \circ \alpha^{-1}$, then $T(A) = P(\alpha)AP(\alpha^{-1})$, for all $A \in \Omega_n$.
(2) If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, then $T(A) = P(\alpha)A^T P(\alpha^{-1})$, for all $A \in \Omega_n$.

Proof of Claim 3. Since there exist $\sigma_1, \ldots, \sigma_t \in S_n$ and $\lambda_1, \ldots, \lambda_t \in [0, 1]$ with $\lambda_1 + \cdots + \lambda_t = 1$ such that $A = \lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)$ then

(1) if $\rho = \alpha \circ \xi \circ \alpha^{-1}$, by Claim 2,

$$\begin{aligned} T(A) &= T(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)) = \lambda_1 T(P(\sigma_1)) + \cdots + \lambda_t T(P(\sigma_t)) \\ &= \lambda_1 P(\alpha \circ \sigma_1 \circ \alpha^{-1}) + \cdots + \lambda_t P(\alpha \circ \sigma_t \circ \alpha^{-1}) \\ &= P(\alpha)(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t))P(\alpha^{-1}) \\ &= P(\alpha)AP(\alpha^{-1}). \end{aligned}$$

(2) If $\rho = \alpha \circ \xi^{-1} \circ \alpha^{-1}$, by Claim 2,

$$\begin{aligned} T(A) &= T(\lambda_1 P(\sigma_1) + \cdots + \lambda_t P(\sigma_t)) = \lambda_1 T(P(\sigma_1)) + \cdots + \lambda_t T(P(\sigma_t)) \\ &= \lambda_1 P(\alpha \circ \sigma_1^{-1} \circ \alpha^{-1}) + \cdots + \lambda_t P(\alpha \circ \sigma_t^{-1} \circ \alpha^{-1}) \\ &= P(\alpha)(\lambda_1 P(\sigma_1^{-1}) + \cdots + \lambda_t P(\sigma_t^{-1}))P(\alpha^{-1}) \\ &= P(\alpha)(\lambda_1 P(\sigma_1 + \cdots + \lambda_t P(\sigma_t))^T P(\alpha^{-1}) \\ &= P(\alpha)A^T P(\alpha^{-1}). \ \blacksquare \end{aligned}$$

Using Corollary 2.4, we have that if $\chi \neq [2, 2]$, then $T(I) = I$. By Claim 3 and Corollary 4.2, the map T must have one of the forms (1) or (2).

If the map T is nonunital, then $T(I) \neq I$, and in this case, by Corollary 2.4, we must have $\chi = [2, 2]$. Since $T(I) = P(\sigma)$ with $\chi(\sigma) = \chi(id)$, we can consider the semilinear map Φ defined by $\Phi(S) = T(I)^{-1}T(S)$, since $T(I)$ is invertible. The map Φ is unital, and

$$d_\chi(\Phi(S)) = d_\chi(T(I)^{-1}T(S)) = d_\chi(P(\sigma^{-1})T(S)).$$

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Using Remark 2.1 and

$$\begin{aligned} d_\chi(P(\sigma^{-1})T(S)) &= \sum_{\rho\in S_4}\chi(\rho)\prod_{j=1}^{4}(P(\sigma^{-1})T(S))_{j\rho(j)} = \sum_{\pi\in S_4}\chi(\pi\circ\sigma)\prod_{j=1}^{4}(T(S))_{j\pi(j)} \\ &= \sum_{\pi\in S_4}\chi(\pi)\prod_{j=1}^{4}(T(S))_{j\pi(j)} = d_\chi(T(S)) = d_\chi(S), \end{aligned}$$

we conclude that Φ preserves d_χ.

By Claim 3 and Corollary 4.2, the result follows. □

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