Zero-dilation Index of $S_n$-matrix and Companion Matrix

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ZERO-DILATION INDEX OF $S_n$-MATRIX AND COMPANION MATRIX

HWA-LONG GAU$^\dagger$ AND PEI YUAN WU$^\ddagger$

Abstract. The zero-dilation index $d(A)$ of a square matrix $A$ is the largest $k$ for which $A$ is unitarily similar to a matrix of the form $\begin{bmatrix} 0_k & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix}$, where $0_k$ denotes the $k$-by-$k$ zero matrix. In this paper, it is shown that if $A$ is an $S_n$-matrix or an $n$-by-$n$ companion matrix, then $d(A)$ is at most $\lceil n/2 \rceil$, the smallest integer greater than or equal to $n/2$. Those $A$’s for which the upper bound is attained are also characterized. Among other things, it is shown that, for an odd $n$, the $S_n$-matrix $A$ is such that $d(A) = (n + 1)/2$ if and only if $A$ is unitarily similar to $-A$, and, for an even $n$, every $n$-by-$n$ companion matrix $A$ has $d(A)$ equal to $n/2$.

Key words. Zero-dilation index, $S_n$-Matrix, Companion matrix, Numerical range.

AMS subject classifications. 47A20, 15B99, 15A60, 47A12.

1. Introduction. The zero-dilation index $d(A)$ of an $n$-by-$n$ complex matrix $A$ is defined as the maximum size $k$ of a zero matrix which can be dilated to $A$ or, equivalently, $d(A)$ is the maximum $k$ for which $A$ is unitarily similar to a matrix of the form $\begin{bmatrix} 0_k & \ast \\ \ast & \ast \end{bmatrix}$, where $0_k$ denotes the $k$-by-$k$ zero matrix. The study of $d(A)$ was started in [4], based on the previous work [12] of C.-K. Li and N.-S. Szé on higher-rank numerical ranges. In [4], the matrices $A$ with $d(A) = n - 1$ were completely characterized, and the value of the index for a normal matrix or a weighted permutation matrix with zero diagonals was also determined. The same was done for KMS matrices (cf. [8, Theorem 2.1]). The purpose of this paper is to find the upper bound of $d(A)$ and to characterize those $A$’s which attain this bound among two classes of matrices, namely, the $S_n$-matrices and companion matrices.

Recall that an $n$-by-$n$ matrix $A$ is said to be of class $S_n$ (or simply an $S_n$-matrix) if it is a contraction ($\|A\| \equiv \max_{x \neq 0} \in \mathbb{C}^n \|Ax\|/\|x\| \leq 1$) with all eigenvalues in the open unit disc $\mathbb{D}$ of the complex plane and with $\text{rank}(I_n - A^*A) = 1$, where $I_n$ denotes the $n$-by-$n$ identity matrix. On the other hand, for any monic polynomial

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Zero-dilation Index of $S_n$-Matrix and Companion Matrix

$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, its associated companion matrix $A$ is

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1 & 0 \\
-a_n & -a_{n-1} & \cdots & -a_2 & -a_1
\end{bmatrix}
\]

(1.1)

Note that $p$ is the characteristic polynomial of $A$. Moreover, it is known that both $S_n$-matrices and companion matrices are nonderogatory and form, under similarity, the building block of the Jordan form of (finite-dimensional) $C_0$ contractions and the rational form of general matrices, respectively. A special example of both is the $n$-by-$n$ Jordan block

\[
J_n = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}
\]

For more of their properties, the reader may consult [1, Section 3.1] and [11, Section 3.3].

In Section 2 below, we prove that if $A$ is an $S_n$-matrix, then $d(A)$ is at most $\lceil n/2 \rceil$ (cf. Proposition 2.1), and, moreover, if $n$ is odd, then $d(A) = (n + 1)/2$ if and only if $A$ and $-A$ are unitarily similar or, equivalently, the eigenvalues of $A$ are of the form $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ (cf. Theorem 2.2). An analogous result holds for even $n$ (cf. Theorem 2.3). However, a clear-cut condition on the eigenvalues of $A$ in order that $d(A) = n/2$ is lacking. In fact, the known case of $n = 2$ (an $S_2$-matrix $A$ is such that $d(A) = 1$ if and only if its eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $|\lambda_1 + \lambda_2| + |\lambda_1\lambda_2| \leq 1$; cf. Proposition 2.4) seems to indicate that the conditions should involve one or more inequalities of the eigenvalues.

The study of the zero-dilation index for companion matrices is taken up in Section 3. Here the straightforward case is for the even $n$. We show that if $A$ is an $n$-by-$n$ companion matrix, then $d(A) \leq \lfloor n/2 \rfloor$, and if, moreover, $n$ is even, then $d(A) = n/2$ (cf. Theorem 3.2). For an odd $n$, characterizations of those $A$’s with $d(A) = (n + 1)/2$ are similar to the ones for $S_n$-matrices; this is the case if and only if $A$ and $-A$ are unitarily similar (cf. Theorem 3.3). What is lacking is a condition in terms of the numerical range of $A$. Recall that the numerical range $W(A)$ of an $n$-by-$n$ matrix $A$ is the subset $\{ \langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1 \}$ of the plane, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard inner product and norm of vectors in $\mathbb{C}^n$. [11, Chapter 1] is our main
reference for properties of the numerical range. An $S_n$-matrix is determined, up to unitary similarity, by its numerical range (cf. [5, Theorem 3.2]). This is not the case for companion matrices; there are two 3-by-3 (invertible) companion matrices $A_1$ and $A_2$ with $W(A_1) = W(A_2)$ which are not unitarily similar (cf. [7, Example 2.1]). Back to our problem, it is unknown whether, for a noninvertible companion matrix $A$ with odd size, the equality $W(A) = -W(A)$ would guarantee the unitary similarity of $A$ and $-A$.

There is another expression for the zero-dilation index, which is in terms of the higher-rank numerical ranges. Recall that the rank-$k$ numerical range $\Lambda_k(A)$ ($1 \leq k \leq n$) of an $n$-by-$n$ matrix $A$ is the subset $\{ \lambda \in \mathbb{C} : \lambda I_k \text{ dilates to } A \}$ of the plane. In particular, $\Lambda_1(A)$ is simply the classical numerical range $W(A)$. Obviously, $d(A)$ equals the maximum $k$ for which $\Lambda_k(A)$ contains 0. A more useful description of $\Lambda_k(A)$ was given by Li and Sze in [12, Theorem 2.2], namely,

$$\Lambda_k(A) = \bigcap_{\theta \in \mathbb{R}} \{ \lambda \in \mathbb{C} : \text{Re}(e^{i\theta}A) \leq \lambda_k(\text{Re}(e^{i\theta}A)) \},$$

where $\text{Re}(z) = (z + \overline{z})/2$ (resp., $\text{Re} B = (B + B^*)/2$) denotes the real part of a complex number (resp., a matrix $B$), and, for an $n$-by-$n$ Hermitian matrix $C$, $\lambda_1(C) \geq \cdots \geq \lambda_n(C)$ denote its ordered eigenvalues. In terms of this description, $d(A)$ can be expressed as

$$d(A) = \min\{ k \theta : \lambda_k(\text{Re}(e^{i\theta}A)) \geq 0 > \lambda_{k+1}(\text{Re}(e^{i\theta}A)), \theta \in \mathbb{R} \}$$

(cf. [12, Theorem 3.1]). For a Hermitian $C$, let $i_{\geq 0}(C)$ denote the number of nonnegative eigenvalues of $C$ (counting multiplicity). From above, it follows that

$$(1.2) \quad d(A) = \min\{ i_{\geq 0}(\text{Re}(e^{i\theta}A)) : \theta \in \mathbb{R} \}.$$  

This is the expression we use most often in the subsequent discussions. Indeed, the proofs of the upper bounds for $d(A)$ and the attainment of these bounds make use of [11, Corollaries 2.5 and 2.6], which were derived before from [12].

For any nonzero complex number $z$, $\text{arg} z$ is the unique number in $[0, 2\pi)$ satisfying $z = |z|e^{i(\text{arg} z)}$. The diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ is denoted by $\text{diag}(\lambda_1, \ldots, \lambda_n)$.

The study undertaken in this paper reveals more common properties of the $S_n$-matrices and companion matrices. Hopefully, results of this nature may lead to the further unlocking of the full potential of higher-rank numerical ranges of these two classes of matrices.

2. $S_n$-matrix. We start with the following upper bound of $d(A)$ for $A$ an $S_n$-matrix.

**Proposition 2.1.** If $A$ is an $S_n$-matrix, then $d(A) \leq \lfloor n/2 \rfloor$. 

Proof. Since $e^{i\theta}A$ is also an $S_n$-matrix for any real $\theta$, its real part $\text{Re}(e^{i\theta}A)$ has only simple eigenvalues (cf. [5, Corollary 2.7]). In particular, this implies that $\dim \ker \left(\text{Re}(e^{i\theta}A)\right) \leq 1$ for all $\theta$. Thus, $d(A) \leq \lfloor n/2 \rfloor$ by [4, Corollary 2.5]. □

For an odd $n$, the next theorem gives equivalent conditions for the extremum case $d(A) = \lfloor n/2 \rfloor$.

**Theorem 2.2.** For an $S_n$-matrix ($n$ odd), the following conditions are equivalent:

(a) $d(A) = (n + 1)/2$,
(b) $\lambda_{(n+1)/2}(\text{Re}(e^{i\theta}A)) = 0$ for all real $\theta$,
(c) $A$ is unitarily similar to a matrix of the form

\[
\begin{bmatrix}
0_{(n+1)/2} & A' \\
1 & 0 \\
\vdots & \vdots \\
1 & 0
\end{bmatrix},
\]

where $A'$ is some $(n + 1)/2$-by-$(n - 1)/2$ matrix,
(d) $A$ and $-A$ are unitarily similar,
(e) the eigenvalues of $A$ are of the form $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ with $b_1, \ldots, b_{(n-1)/2}$ in $\mathbb{C}$,
(f) $W(A) = -W(A)$.

Proof. (a) $\Rightarrow$ (b). This holds for any $n$-by-$n$ matrix $A$. If $\lambda_{(n+1)/2}(\text{Re}(e^{i\theta}A)) < 0$ for some real $\theta$, then $i_{\geq 0}(\text{Re}(e^{i\theta}A)) < (n + 1)/2$, which implies, by (1.2), that $d(A) < (n + 1)/2$, contradicting (a). Similarly, if $\lambda_{(n+1)/2}(\text{Re}(e^{i\theta}A)) > 0$, then $i_{\geq 0}(\text{Re}(e^{i(\theta+\pi)}A)) < (n + 1)/2$ and hence $d(A) < (n + 1)/2$, again a contradiction. Thus, $\lambda_{(n+1)/2}(\text{Re}(e^{i\theta}A)) = 0$ for all real $\theta$, that is, (b) holds.

(b) $\Rightarrow$ (a). Under (b), we have $i_{\geq 0}(\text{Re}(e^{i\theta}A)) \geq (n + 1)/2$ for all real $\theta$, and thus, $d(A) \geq (n + 1)/2$. Then (a) follows from Proposition 2.1.

(a) $\Rightarrow$ (c). We may assume that $A = \begin{bmatrix} 0_{(n+1)/2} & B \\ C & D \end{bmatrix}$, where $B$, $C$ and $D$ are $(n + 1)/2$-by-$(n + 1)/2$, $(n - 1)/2$-by-$(n + 1)/2$ and $(n - 1)/2$-by-$(n - 1)/2$ matrices, respectively. Since $I_n - A^*A = \begin{bmatrix} I_{(n+1)/2} - C^*C & * \\ * & * \end{bmatrix}$ has rank one, we have $\text{rank}(I_{(n+1)/2} - C^*C) \leq 1$. Note that $\text{rank} C^*C = \text{rank} C \leq (n - 1)/2$. Thus, $C^*C$ is unitarily similar to $\text{diag}(c_1, \ldots, c_{(n-1)/2}, 0)$ for some $c_j$ satisfying $0 \leq c_j \leq 1$ for all $j$. Hence, $I_{(n+1)/2} - C^*C$ is unitarily similar to $\text{diag}(1 - c_1, \ldots, 1 - c_{(n-1)/2}, 1)$. From $\text{rank}(I_{(n+1)/2} - C^*C) \leq 1$, we derive that $c_j = 1$ for all $j$, $1 \leq j \leq (n - 1)/2$. It follows that $C^*C$ is unitarily similar to $\text{diag}(1, \ldots, 1, 0)$. Note that the singular value decomposition of $C$ yields the existence of unitary matrices $U$ and $V$ of sizes $(n - 1)/2$.
and \((n + 1)/2\), respectively, such that

\[
C = U \begin{bmatrix}
1 & 0 \\
\ddots & \ddots \\
& 1 & 0
\end{bmatrix} V
\]

(cf. [11, Theorem 2.6.3]). If \(W\) denotes the \(n\)-by-\(n\) unitary matrix \(V^* \oplus U\), then

\[
W^*AW = \begin{bmatrix}
0_{(n+1)/2} & * \\
U^*CV^* & *
\end{bmatrix} = \begin{bmatrix}
0_{(n+1)/2} & * \\
1 & 0 \\
& \ddots & \ddots \\
& & 1 & 0
\end{bmatrix}.
\]

Since \(\|A\| = 1\), the matrix on the right-hand side of the above expression is of the asserted form in (c).

(c) \(\Rightarrow\) (d). Let \(A''\) denote the matrix in (c) and let \(U = I_{(n+1)/2} \oplus (-I_{(n-1)/2})\). Then \(U^*A''U = -A''\). It follows that \(A\) is unitarily similar to \(-A\).

(d) \(\Rightarrow\) (a). The unitary similarity of \(A\) and \(-A\) implies, by [4, Corollary 2.6], that \(d(A) \geq (n + 1)/2\), which together with \(\dim \ker (\text{Re}(e^{i\theta}A)) \leq 1\) for all real \(\theta\) [5, Corollary 2.7] yields \(d(A) = (n + 1)/2\).

(d) \(\Rightarrow\) (e). Let the eigenvalues of \(A\) be \(\lambda_1, \ldots, \lambda_n\). Then (d) implies the coincidence of \(\lambda_1, \ldots, \lambda_n\) and \(-\lambda_1, \ldots, -\lambda_n\). In particular, we have

\[
\det A = \prod_j \lambda_j = \prod_j (-\lambda_j) = -\prod_j \lambda_j = -\det A.
\]

Hence, \(\det A = 0\) and, therefore, \(\lambda_j = 0\) for some \(j\). We may assume that \(\lambda_1 = 0\). The coincidence of \(\lambda_2, \ldots, \lambda_n\) and \(-\lambda_2, \ldots, -\lambda_n\) implies that either \(-\lambda_2 = \lambda_2\) or \(-\lambda_2 = \lambda_j\) for some \(j\), \(3 \leq j \leq n\). The former yields \(\lambda_2 = 0\) and the latter \(\{\lambda_2, \lambda_j\} = \{\pm \lambda_2\}\). Hence, either \(\lambda_3, \ldots, \lambda_n\) coincide with \(-\lambda_3, \ldots, -\lambda_n\) or \(\lambda_3, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n\) coincide with \(-\lambda_3, \ldots, -\lambda_{j-1}, -\lambda_{j+1}, \ldots, -\lambda_n\). Continuing in this fashion, we obtain the assertion in (e).

(e) \(\Rightarrow\) (d). Obviously, (e) implies that the eigenvalues of \(A\) and \(-A\) coincide. Thus, \(A\) and \(-A\) are unitarily similar by, say, [6, Corollary 1.3].

(d) \(\Leftrightarrow\) (f). This follows by [5, Theorem 3.2]. \(\Box\)

We now turn to the case of even \(n\).
Zero-dilation Index of $S_n$-Matrix and Companion Matrix

**Theorem 2.3.** For an $S_n$-matrix $A$ with $n$ even, the following conditions are equivalent:

1. $d(A) = n/2$,
2. $\lambda_{(n/2)+1}(\Re(e^{i\theta}A)) \leq 0 \leq \lambda_{n/2}(\Re(e^{i\theta}A))$ for all real $\theta$,
3. $A$ is unitarily similar to a matrix of the form
   \[
   \begin{pmatrix}
   0_{n/2} & * & * \\
   1 & 0 & 0 \\
   \vdots & \vdots & \vdots \\
   0 & \cdots & 0 & * & * \\
   \end{pmatrix},
   \]
4. there is an $(n-1)$-by-$(n-1)$ compression $B$ of $A$, that is, $A$ is unitarily similar to a matrix of the form
   \[
   \begin{pmatrix}
   B & * \\
   * & * \\
   \end{pmatrix}
   \]
   such that $B$ and $-B$ are unitarily similar,
5. for any $(n+1)$-by-$(n+1)$ unitary dilation $U$ of $A$ with eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$ arranged so that $\arg \lambda_1 < \cdots < \arg \lambda_{n+1}$, $-\lambda_j$ lies in the circular arc of $\partial \mathbb{D}$ between $\lambda_{(n+2)+j}$ and $\lambda_{(n+2)+j+1}$ for all $j$ (here $\lambda_k$ is interpreted as $\lambda_k - (n+1)$ if $k > n+1$).

**Proof.** The proof of (a) $\iff$ (b) is analogous to the one for Theorem 2.2 (a) $\iff$ (b), which we omit.

(a) $\Rightarrow$ (c). As in the proof of the corresponding implication in Theorem 2.2 we assume that $A = \begin{pmatrix} 0_{n/2} & B \\ C & D \end{pmatrix}$, where $B$, $C$ and $D$ are all of size $n/2$. As before, we have $\text{rank}(I_{n/2} - C^*C) \leq 1$. Let $C^*C$ be unitarily similar to $\text{diag}(\lambda_1, \ldots, \lambda_{n/2})$, where the $\lambda_j$’s satisfy $0 \leq \lambda_{n/2} \leq \cdots \leq \lambda_1 \leq 1$. Thus, $I_{n/2} - C^*C$ is unitarily similar to $\text{diag}(1 - \lambda_1, 1 - \lambda_{n/2})$. The rank condition of $I_{n/2}^2 - C^*C$ yields that $\lambda_j = 1$ for all $j$, $1 \leq j \leq (n/2) - 1$. Thus, $C = U$ $\text{diag}(1, \ldots, 1, \sqrt{\lambda_{n/2}})V$ for some $(n/2)$-by-$(n/2)$ unitary matrices $U$ and $V$. It follows that $A$ is unitarily similar to a matrix of the form in (c).

(c) $\Rightarrow$ (d). If $B$ is the $(n-1)$-by-$(n-1)$ leading principal submatrix of the matrix in (c), then $B$ is unitarily similar to $-B$ as in the proof of Theorem 2.2 (c) $\Rightarrow$ (d). This proves (d).

(d) $\Rightarrow$ (a). The unitary similarity of $B$ and $-B$ implies that $d(B) \geq n/2$ by [4, Corollary 2.6]. Thus, $d(A) \geq n/2$. But $d(A) \leq n/2$ also holds by [4, Corollary 2.5] since $\dim \ker(\Re(e^{i\theta}A)) \leq 1$ for all real $\theta$. Therefore, $d(A) = n/2$.

(a) $\iff$ (e). This is a consequence of [3, Theorem 1.2] and [4, Theorem 4.1 (b)]. Indeed, the condition in (e) and [4, Theorem 4.1 (b)] imply that every $(n+1)$-by-$(n+1)$ unitary dilation $U$ of $A$ is such that $d(U) = n/2$. Hence, 0 is in $\Lambda_{n/2}(U)$ for
every such $U$. Then Theorem 1.2[3] then yields that $0$ is in $\Lambda_{n/2}(A)$. Hence, $d(A) \geq n/2$. We deduce from Proposition 2.1 that $d(A) = n/2$. This proves (a). The converse (a) $\Rightarrow$ (e) is proven by reversing the above arguments.

Since an $S_n$-matrix $A$ is uniquely determined by its eigenvalues up to unitary similarity, it is desirable to have an equivalent eigenvalue condition for $d(A) = n/2$ ($n$ even) in the preceding theorem. As the next proposition shows, such a condition may involve one or more inequalities of the eigenvalues.

**Proposition 2.4.** Let $A$ be an $S_2$-matrix with eigenvalues $\lambda_1$ and $\lambda_2$. Then $d(A) = 1$ if and only if $|\lambda_1 + \lambda_2| + |\lambda_1 \lambda_2| \leq 1$.

**Proof.** We need to show that 0 is in $W(A)$ if and only if the above inequality holds. Indeed, since $A$ is unitarily similar to the matrix

$$
\begin{bmatrix}
\lambda_1 & (1 - |\lambda_1|^2)^{1/2}(1 - |\lambda_2|^2)^{1/2} \\
0 & \lambda_2
\end{bmatrix}
$$

(cf. [5] Corollary 1.3), its numerical range equals the elliptic disc with foci $\lambda_1$ and $\lambda_2$ and the lengths of the minor and major axes equal to $(1 - |\lambda_1|^2)^{1/2}(1 - |\lambda_2|^2)^{1/2}$ and $|1 - \lambda_1 \lambda_2|$, respectively. Thus, 0 is in $W(A)$ if and only if $|\lambda_1| + |\lambda_2| \leq |1 - \lambda_1 \lambda_2|$, the latter being equivalent to $|\lambda_1 + \lambda_2| + |\lambda_1 \lambda_2| \leq 1$. $\square$

**3. Companion matrix.** We start with the following result on the nullity of the real part of a companion matrix.

**Theorem 3.1.** Let $A$ be an $n$-by-$n$ companion matrix.

(a) If $n$ is odd, then $\dim \ker \left( \text{Re}(e^{i\theta}A) \right) \leq 1$ for all real $\theta$.

(b) If $n$ is even, then $\dim \ker \left( \text{Re}(e^{i\theta}A) \right) \leq 2$ for all real $\theta$ and, moreover, $\dim \ker \left( \text{Re}(e^{i\theta}A) \right) \leq 1$ for all but at most $n$ many values of $\theta$ in $[0,2\pi)$.

Note that the assertion in (a) above does not hold for even $n$. For example, if $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, then

$$
\dim \ker \left( \text{Re}(e^{i\theta}A) \right) = \begin{cases} 2 & \text{if } e^{i\theta} = \pm 1, \\ 0 & \text{otherwise.} \end{cases}
$$

**Proof of Theorem 3.1.** Since $e^{i\theta}A$ is unitarily similar to a companion matrix for any real $\theta$ (cf. [4] Lemma 2.8)), we need only prove the assertion in (a) and the first
assertion in (b) for $\text{Re } A$ (instead of $\text{Re } (e^{i\theta} A)$). If $A$ is of the form \[ A = \begin{bmatrix} 0 & 1 & & & -\alpha_n \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots \\ & & 1 & -\alpha_3 \\ -\alpha_n & & \cdots & 1 & -\alpha_2 + 1 \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 \end{bmatrix}, \]

Since $\text{Re } J_{n-1}$ is the $(n-1)$-by-$(n-1)$ leading principal submatrix of $\text{Re } A$, the eigenvalues of $\text{Re } J_{n-1}$ and $\text{Re } A$ interlace by Cauchy’s interlacing theorem (cf. [11, Theorem 4.3.17]). Hence, if $\dim \ker(\text{Re } A) \geq 2$ for odd $n$ (resp., $\dim \ker(\text{Re } A) \geq 3$ for even $n$), then 0 is an eigenvalue of $\text{Re } J_{n-1}$ with multiplicity at least one (resp., at least two). However, it is known that $\text{Re } J_{n-1}$ has eigenvalues $\cos(j\pi/n), 1 \leq j \leq n-1$ (cf. [9, p. 373]). For an odd (resp., even) $n$, none of these (resp., exactly one of these) is zero. Thus, the contradiction leads to $\dim \ker(\text{Re } A) \leq 1$ for odd $n$ (resp., $\dim \ker(\text{Re } A) \leq 2$ for even $n$).

To prove the second assertion in (b), for any real $\theta$, let $x_\theta = [x_1 \cdots x_n]^T$ in $\mathbb{C}^n$ be such that $\text{Re } (e^{i\theta} A)x_\theta = 0$. Carrying out the matrix multiplication, we obtain a system of $n/2$ equalities:

\[
e^{i\theta} x_2 - \alpha_n e^{-i\theta} x_n = 0,
\]
\[
e^{-i\theta} x_j + e^{i\theta} x_{j+2} - \alpha_n e^{-i\theta} x_n = 0, \quad j = 2, 4, \ldots, n-4,
\]

and
\[
e^{-i\theta} x_{n-2} + (-\alpha_2 e^{-i\theta} + e^{i\theta}) x_n = 0.
\]

It follows that
\[ x_2 = \alpha_n e^{-2i\theta} x_n, \]
\[ x_{j+2} = (\alpha_n e^{-2i\theta} x_n - x_j) e^{-2i\theta}, \quad j = 2, 4, \ldots, n-4, \]

and
\[ x_{n-2} = (\alpha_2 - e^{2i\theta}) x_n. \]

Equating the last two expressions of $x_{n-2}$ and then iteratively substituting $x_{n-4}, \ldots, x_4, x_2$ into the resulting equality, we obtain that $e^{i\theta}$ is a root of the equation $x_n p(z) = 0$ for all real $\theta$, where $p(z)$ is the polynomial $\sum_{j=0}^{n/2} (-1)^j a_{n-2j} z^{n-2j}$. If
θ is such that \( p(e^{i\theta}) \neq 0 \), then its corresponding \( x_n \) must equal zero. Our assumption \( \text{Re} (e^{i\theta}A)x_\theta = 0 \), where \( x_\theta = [x_1 \cdots x_{n-1} 0]^T \), yields that \( \text{Re} (e^{i\theta}J_{n-1})x_\theta = 0 \) with \( x_\theta \equiv [x_1 \cdots x_{n-1}]^T \). However, since \( \dim \ker (\text{Re} (e^{i\theta}J_{n-1})) = \dim \ker (\text{Re} J_{n-1}) = 1 \), we conclude that \( \dim \ker (\text{Re} (e^{i\theta}A)) \leq 1 \). \( \blacksquare \)

Using Theorem 3.1, we can now say something about the zero-dilation index of a companion matrix.

**Theorem 3.2.** If \( A \) is an \( n \)-by-\( n \) companion matrix, then \( d(A) \leq \lfloor n/2 \rfloor \). Moreover, if \( n \) is odd (resp., even), then \( d(A) = (n+1)/2 \) or \( (n-1)/2 \) (resp., \( d(A) = n/2 \)).

**Proof.** That \( d(A) \leq \lfloor n/2 \rfloor \) is a consequence of Theorem 3.1 and [4, Corollary 2.5].

Assume now that \( n \) is odd (resp., even) and \( A \) is of the form \( [I \mathbf{J}] \). Permuting rows and the corresponding columns of \( A \), we can transform \( A \) to

\[
A' \equiv \begin{bmatrix}
0_{(n-1)/2} & 0 \\
-a_n & -a_{n-2} & \cdots & -a_1 & -a_{n-1} & -a_{n-3} & \cdots & -a_2 \\
\vdots & & & & I_{(n-1)/2} & 0_{(n-1)/2} \\
0 & & & & I_{(n-1)/2} & 0_{(n-1)/2}
\end{bmatrix}
\]

(\text{resp.,} \( A' \equiv \begin{bmatrix}
\begin{bmatrix} 0 & 1 & & & & & \\
:\ & : & & & & & \\
-\begin{bmatrix} a_n & a_{n-2} & \cdots & a_2 \\
\vdots & : & & & & & \\
0 & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{bmatrix} & \begin{bmatrix} I_{n/2} \\
\vdots & : & & & & & \\
-\begin{bmatrix} a_{n-1} & a_{n-3} & \cdots & a_1 \\
\vdots & : & & & & & \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\),

where the rows (resp., columns) of \( A' \) numbered 1, 2, \ldots, \( n \) are the rows (resp., columns) of \( A \) numbered 1, 3, \ldots, \( n \), 2, \ldots, \( n-1 \) (resp., 1, 3, \ldots, \( n-1 \), 2, \ldots, \( n \), respectively. This shows that \( d(A) = d(A') \geq (n+1)/2 \) or \( (n-1)/2 \) depending on whether \( a_1 = a_3 = \cdots = a_n = 0 \) or otherwise (resp., \( d(A) = d(A') \geq n/2 \)). Together with \( d(A) \leq \lfloor n/2 \rfloor \) for all companion matrices \( A \), we thus obtain \( d(A) = (n+1)/2 \) or \( (n-1)/2 \) (resp., \( d(A) = n/2 \)) as asserted. \( \blacksquare \)

The next result gives equivalent conditions for \( d(A) = (n+1)/2 \) when \( A \) is a companion matrix of odd size \( n \).

**Theorem 3.3.** Let \( A \) be an \( n \)-by-\( n \) companion matrix of the form \( [I \mathbf{J}] \). If \( n \) is odd, then the following conditions are equivalent:

(a) \( d(A) = (n+1)/2 \),
then $\lambda \leq 1$ for all real $\theta$.

(c) $\text{Re} (e^{i\theta} A)$ is noninvertible for all real $\theta$,

(d) $\dim \ker (\text{Re} (e^{i\theta} A)) = 1$ for all real $\theta$,

(e) $a_1 = a_3 = \ldots = a_n = 0$,

(f) $A$ and $-A$ are unitarily similar,

(g) $A$ is unitarily similar to a matrix of the form

$$\begin{bmatrix} 0 & \ast & \cdots & \ast \\ \ast & 0 & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \ast & 0 \end{bmatrix},$$

(h) the eigenvalues of $A$ are of the form $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ with $b_1, \ldots, b_{(n-1)/2}$ in $\mathbb{C}$.

In this case, $A$ is unitarily irreducible, meaning that it is not unitarily similar to the direct sum of two other matrices.

Proof. (a) $\Leftrightarrow$ (b). The proof is analogous to the one for (a) $\Leftrightarrow$ (b) of Theorem 2.2, except that, in proving (b) $\Rightarrow$ (a), we use Theorem 3.2 instead of Proposition 2.1.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (b). Note that (c) says that 0 is an eigenvalue of $\text{Re} (e^{i\theta} A)$ for all real $\theta$. Since $\text{Re} (e^{i\theta} J_{n-1})$ is the $(n-1)$-by-$(n-1)$ leading principal submatrix of $\text{Re} (e^{i\theta} A)$, their eigenvalues interlace by Cauchy’s interlacing theorem (cf. [11, Theorem 4.3.17]). The unitary similarity of $e^{i\theta} J_{n-1}$ and $J_{n-1}$ and [9, p. 373] yield that

$$\lambda_j (\text{Re} (e^{i\theta} J_{n-1})) = \lambda_j (\text{Re} J_{n-1}) = \cos (j\pi/n)$$

for $1 \leq j \leq n-1$. These together imply that $\lambda_{(n+1)/2} (\text{Re} (e^{i\theta} A)) = 0$ for all $\theta$, that is, (b) holds.

(c) $\Leftrightarrow$ (d) follows by Theorem 3.1 (a).

(a) $\Rightarrow$ (e). Note that, for any $n$-by-$n$ matrix $A$ ($n$ odd), $d(A) = (n+1)/2$ implies that 0 is an eigenvalue of $A$ (cf. [2 Proposition 2.2]). Hence, if $A$ is of the form (3.1), then $a_n = 0$, and, for any real $\theta$,

$$\text{Re} (e^{i\theta} A) = \frac{1}{2} \begin{bmatrix} 0 & e^{i\theta} & 0 & \cdots & \cdots & 0 \\ e^{-i\theta} & 0 & e^{i\theta} & \cdots & \cdots & -\overline{a}_{n-1} e^{-i\theta} \\ 0 & e^{-i\theta} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & e^{i\theta} & -\overline{a}_{3} e^{-i\theta} \\ \vdots & \vdots & \ddots & \ddots & 0 & -\overline{a}_{2} e^{-i\theta} + e^{i\theta} \\ 0 & -a_{n-1} e^{i\theta} & \cdots & -a_{3} e^{i\theta} & -a_{2} e^{i\theta} + e^{-i\theta} & -2\text{Re} (a_1 e^{i\theta}) \end{bmatrix}.$$ 

Let $A_{n-2}$ denote the $(n-2)$-by-$(n-2)$ submatrix of $A$ obtained by deleting the first two rows and columns of $A$. Then

$$\det (\text{Re} (e^{i\theta} A)) = \frac{1}{2} (e^{-i\theta} e^{i\theta} \det (\text{Re} (e^{i\theta} A_{n-2})))$$

and

$$\det (\text{Re} (e^{i\theta} A_{n-2})) = \frac{1}{2} (e^{i\theta} e^{-i\theta} \det (\text{Re} (e^{i\theta} A_{n-2})))$$

are both equal to $0$. Therefore, $\det (\text{Re} (e^{i\theta} A)) = 0$ for all real $\theta$, implying that $\text{Re} (e^{i\theta} A)$ is noninvertible for all real $\theta$.
via expanding by minors along the first column of $\text{Re}(e^{i\theta}A)$ and then along the first row of the resulting minor. Since (a) and (c) are proven to be equivalent, from (c) we have $\det(\text{Re}(e^{i\theta}A)) = 0$, and hence, $\det(\text{Re}(e^{i\theta}A_{n-2})) = 0$ for all real $\theta$, which in turn implies, from the equivalence of (a) and (b), that $d(A_{n-2}) = (n - 1)/2$. Thus, $a_{n-2} = 0$ by [2, Proposition 2.2]. By induction, we obtain $a_j = 0$ for $j = n - 4, n - 6, \ldots, 1$, successively.

(e) $\Rightarrow$ (f). For $A$ of the form (1.1) with odd $n$, $-A$ is unitarily similar to

\[
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
a_n & -a_{n-1} & a_{n-2} & \cdots & a_3 & -a_2 & a_1
\end{bmatrix}
\]

(cf. [7, Lemma 2.8]). Under (e), the latter matrix is exactly $A$.

(f) $\Leftrightarrow$ (g) follows by [13, Theorem 2.3] and Theorem 3.2.

(f) $\Rightarrow$ (h). The proof is the same as the one for (d) $\Rightarrow$ (e) of Theorem 2.2.

(h) $\Rightarrow$ (e). Under the assumption in (h), the characteristic polynomial of $A$ is $z(z^2 - b_1) \cdots (z^2 - b_{(n-1)/2})$. Since this is the same as $z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, we conclude that $a_1, a_3, \ldots, a_n$, the odd-indexed coefficients, are all equal to zero.

(f) $\Rightarrow$ (a). This is seen by [3, Corollary 2.6] since $A$ and $-A$ are unitarily similar and $\dim \ker(\text{Re}(e^{i\theta}A)) \leq 1$ for all real $\theta$ by Theorem 3.1 (a).

Finally, the unitary irreducibility of $A$ follows from (h) and [7, Theorem 1.1].

As was remarked in Section 1, it is unknown whether, for an $n$-by-$n$ ($n$ odd) noninvertible companion matrix $A$, the equality $W(A) = -W(A)$ would imply $d(A) = (n + 1)/2$ or, equivalently, that $A$ and $-A$ are unitarily similar. Our final result shows that this is indeed the case for $n = 3$. For larger values of (odd) $n$, we suspect that this may not be true.

**Proposition 3.4.** Let $A$ be a $3$-by-$3$ companion matrix. Then $d(A) = 2$ if and only if $A$ is noninvertible and $W(A) = -W(A)$.

For the proof of the sufficiency, we make use of the Kippenhahn polynomial of a matrix. Recall that the **Kippenhahn polynomial** of an $n$-by-$n$ matrix $A$ is the degree-$n$ real homogeneous polynomial $p_A(x, y, z) = \det(x \text{Re} A + y \text{Im} A + z I_n)$ in $x, y$ and $z$, where $\text{Im} A = (A - A^*)/(2i)$ is the imaginary part of $A$. It is known that the numerical range of $A$ equals the convex hull of the real points of the dual curve of
p_A(x, y, z) = 0 in the sense that \( W(A) = \{a + ib : a, b \text{ real and } ax + by + z = 0 \}^\triangleleft \), where, for any subset \( \triangle \) of the complex plane, \( \triangle^\wedge \) denotes its convex hull (cf. [14, Theorem 10]).

**Proof of Proposition 3.4.** If \( d(A) = 2 \), then \( A \) is noninvertible and \( W(A) \) is an elliptic disc with foci \( \pm b (b \in \mathbb{C}) \) by [4, Lemma 3.4] and Theorem [5.3](h). In particular, we have \( W(A) = -W(A) \).

For the converse, assume that

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_3 & -a_2 & -a_1
\end{bmatrix}
\]

is noninvertible and \( W(A) = -W(A) \). We readily have \( a_3 = 0 \). It remains to show that \( a_1 = 0 \). Two cases are considered separately:

(i) Suppose \( p_A \) is irreducible. Since \( -A \) is unitarily similar to the companion matrix

\[
A' \equiv \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -a_2 & -a_1
\end{bmatrix},
\]

the equality \( W(A) = W(-A) = W(A') \) together with the irreducibility of \( p_A \) yields that \( A = A' \) (cf. [7, Corollary 2.5]). It follows that \( a_1 = 0 \).

(ii) Suppose \( p_A \) is reducible. Then either \( p_A = p_1p_2 \), where \( p_1 \) (resp., \( p_2 \)) is a degree-2 irreducible (resp., degree-1) homogeneous polynomial in \( x, y \) and \( z \), or \( p_A = q_1q_2q_3 \), where the \( q_j \)'s are all of degree 1. The latter would imply that \( A \) is normal and hence unitary (cf. [7, Corollary 1.2]), which contradicts the noninvertibility of \( A \). Thus, we must have \( p_A = p_1p_2 \). The dual curves of \( p_1(x, y, z) = 0 \) and \( p_2(x, y, z) = 0 \) are an ellipse and a single point, respectively. That \( W(A) = -W(A) \) implies that \( W(A) \) can only be an elliptic disc centered at 0. If \( b_1 \) and \( b_2 \) are the foci of the ellipse \( \partial W(A) \), then they are eigenvalues of \( A \) satisfying \( b_1 + b_2 = 0 \) (cf. [14, Theorem 11]). If \( b_1 = b_2 = 0 \), then \( W(A) \) is a circular disc centered at 0, which implies that \( A = J_3 \) (cf. [7, Theorem 2.9]). Hence, in this case, we have \( a_1 = a_2 = a_3 = 0 \). On the other hand, if \( b_1 = -b_2 \neq 0 \), then the eigenvalues of \( A \) consist of 0 and \( \pm b_1 \), in which case, we readily have \( a_1 = 0 \). Hence, \( d(A) = 2 \) by Theorem [5.3](h).

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