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NOTES ON TWO RECENT RESULTS OF AUDENAERT*

YAZHOU HAN† AND JINGJING SHAO‡

Abstract. In this article, some inequalities for τ-measurable operators which are related to two recent results of Audenaert are proved.

Key words. Noncommutative L_p spaces, Von Neumann algebras, τ-Measurable operators.

AMS subject classifications. 47A63, 46L52.

1. Introduction. Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$.

Let $\mathbb{M}_n^+$ be the positive part of $\mathbb{M}_n$. In [1], Audenaert proved that if $A_i, B_i \in \mathbb{M}_n^+$ $(i = 1, 2, \ldots, n)$, such that $A_iB_i = B_iA_i$, then

$$\left|\left|\left|\sum_{i=1}^n A_iB_i\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^n A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^n A_i\right)\left(\sum_{i=1}^n B_i\right)\right|\right|\right|. \tag{1.1}$$

A special case of inequality (1.1) confirms a conjecture of Hayajneh and Kittaneh in [9] and answers a question of Bourin.

In another paper [2], Audenaert proved that for $X, Y \in \mathbb{M}_n$ and $0 \le q \le 1$,

$$|||XY^*|||^2 \le |||qX^*X + (1-q)Y^*Y|||\,|||(1-q)X^*X + qY^*Y|||. \tag{1.2}$$

As is explained in [2], inequality (1.2) interpolates between the Arithmetic-Geometric mean and Cauchy-Schwarz matrix norm inequalities. Very recently Lin [12] gave another proof of inequality (1.1) and (1.2).

Using the notion of the generalized singular numbers studied by Fack and Kosaki [7], we show that the inequality (1.1) and (1.2) hold for the norm on noncommutative L_p spaces. Our idea of proof follows the one given in [12].

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2. Preliminaries. Unless stated otherwise, $\mathcal{M}$ will always denote a semifinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful finite normalized trace τ. We refer to [14] for noncommutative integration. We denote the identity of $\mathcal{M}$ by 1. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, we define its distribution function by $\lambda_s(x) = \tau(e_s^{\perp}(|x|))$ and x will be called τ-measurable if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. The measure topology in $L_0(\mathcal{M})$ is the vector space topology defined via the neighbourhood base $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, where $N(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \tau(e_{(\varepsilon,\infty)}(|x|)) \leq \delta\}$ and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε, ∞). With respect to the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$-algebra.

Definition 2.1. Let $x \in L_0(\mathcal{M})$ and $t > 0$. The *t-th singular number (or generalized singular number) of* x, $\mu_t(x)$, is defined by

$$\mu_t(x) = \inf\left\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^{\perp}) \leq t\right\}.$$

If $x, y \in L_0(\mathcal{M})$, then we say that x is submajorized by y and write $x \prec y$ if and only if

$$\int_0^a \mu_t(x)dt \leq \int_0^a \mu_t(y)dt \quad \text{for all } a \geq 0.$$

We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively. For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all densely-defined closed operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} = \left(\int_0^\infty \mu_t(x)^p dt\right)^{\frac{1}{p}} < \infty.$$

As usual, we put $L^\infty(\mathcal{M};\tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ $(1 \leq p \leq \infty)$. For every $x \in L_0(\mathcal{M})$, there is a unique polar decomposition $x = u|x|$ where $|x| \in L_0(\mathcal{M})^+$ (the positive part of $L_0(\mathcal{M})$) and u is a partial isometry operator. Let $r(x) = u^*u$ and $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x, respectively. Note that $l(x)$ (resp., $r(x)$) is the least projection e of $\mathcal{B}(\mathcal{H})$ such that $ex = x$ (resp., $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. Let $\mathcal{M}^+ = \{x \in \mathcal{M} : x \geq 0\}$(i.e., the

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positive part of $\mathcal{M}$). We write $S(\mathcal{M})^+ = \{x \in \mathcal{M}^+ : \tau(s(x)) < \infty\}$. Let $S(\mathcal{M})$ be the linear span of $S(\mathcal{M})^+$. It is well known that $(S(\mathcal{M}), \|\cdot\|_p)$ is dense on $L^p(\mathcal{M})$. For further results about noncommutative L_p spaces, the reader is referred to [7, 14].

Given $x, y \in L_0(\mathcal{M})$ and $0 < p < \infty$, from Theorem 4.2 of [7], we have

$$\int_0^t \mu_s(xy)^p ds \le \int_0^t \mu_s(y)^p \mu_s(x)^p ds, \quad t > 0.$$

Let $0 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $x \in L^q(\mathcal{M})$ and $y \in L^r(\mathcal{M})$, then the usual Hölder inequality implies that

$$\left(\int_0^\infty \mu_s(xy)^p ds\right)^{\frac{1}{p}} \le \left(\int_0^\infty (\mu_s(y)\mu_s(x))^p ds\right)^{\frac{1}{p}} \le \left(\int_0^\infty \mu_s(x)^q ds\right)^{\frac{1}{q}} \left(\int_0^\infty \mu_s(y)^r ds\right)^{\frac{1}{r}}.$$

That is

$$\|xy\|_p \le \|x\|_q \|y\|_r. \tag{2.1}$$

3. Main result.

We start this section by two simple lemmas.

LEMMA 3.1. *Let $x_1, x_2, \ldots, x_n \in L_0(\mathcal{M})^+$ and $1 \le p < \infty$.*

(1) *If f is any nonnegative convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$\left\|\sum_{i=1}^n f(x_i)\right\|_p \le \left\|f\left(\sum_{i=1}^n x_i\right)\right\|_p. \tag{3.1}$$

(2) *If f is any nonnegative concave function on $[0, \infty)$, then*

$$\left\|f\left(\sum_{i=1}^n x_i\right)\right\|_p \le \left\|\sum_{i=1}^n f(x_i)\right\|_p. \tag{3.2}$$

Proof. (1) If $f(\sum_{i=1}^n x_i) \notin L^p(\mathcal{M})$, then $\|f(\sum_{i=1}^n x_i)\|_p = \infty$. Thus, the inequality (3.1) is clear. If $f(\sum_{i=1}^n x_i) \in L^p(\mathcal{M})$, it follows from Theorem 5.3 (i) of [6] that

$$\int_0^t \mu_s\left(\sum_{i=1}^n f(x_i)\right) ds \le \int_0^t \mu_s\left(f\left(\sum_{i=1}^n x_i\right)\right) ds, \quad t > 0.$$

Then Theorem 2.1 of [4] tells us that

$$\int_0^t \mu_s\left(\sum_{i=1}^n f(x_i)\right)^p ds \le \int_0^t \mu_s\left(f\left(\sum_{i=1}^n x_i\right)\right)^p ds, \quad t > 0.$$

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Hence, $\| \sum_{i=1}^n f(x_i)\|_p \leq \|f(\sum_{i=1}^n x_i)\|_p$.

(2) The proof can be done similarly to (1) by using Theorem 5.3(ii) of [6]. The details are omitted. □

The matrix version of Lemma 3.1 appears in [11].

Let $x, y, z \in L_0(\mathcal{M})$. The block matrix $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is positive partial transpose (i.e., PPT) if $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ and $\begin{pmatrix} x & z^* \\ z & y \end{pmatrix} \geq 0$.

Lemma 3.2. *Let $x, y \in S(\mathcal{M})^+$ and $z \in S(\mathcal{M})$. If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT and $1 \leq p < \infty$, then $\|z^*z\|_p \leq \|xy\|_p$.*

Proof. From the fact $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT, we deduce $z = x^{\frac{1}{2}}k_1y^{\frac{1}{2}}$ and $z^* = x^{\frac{1}{2}}k_2y^{\frac{1}{2}}$, where k_1, k_2 are contraction operators. By Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned}
\int_0^t \mu_s(z^*z)^p ds &= \int_0^t \mu_s\left(x^{\frac{1}{2}}k_2y^{\frac{1}{2}}x^{\frac{1}{2}}k_1y^{\frac{1}{2}}\right)^p ds \\
&= \int_0^t \mu_s\left(y^{\frac{1}{2}}x^{\frac{1}{2}}k_2y^{\frac{1}{2}}x^{\frac{1}{2}}k_1\right)^p ds \\
&\leq \int_0^t \mu_s\left(y^{\frac{1}{2}}x^{\frac{1}{2}}k_2y^{\frac{1}{2}}x^{\frac{1}{2}}\right)^p ds \\
&\leq \int_0^t \mu_s\left(\left(y^{\frac{1}{2}}x^{\frac{1}{2}}\right)^2\right)^p ds, \quad t > 0.
\end{aligned}$$

It follows from Theorem 2 of [10] that

$$\begin{aligned}
\int_0^t \mu_s(z^*z)^p ds &\leq \int_0^t \mu_s\left(\left(y^{\frac{1}{2}}x^{\frac{1}{2}}\right)^2\right)^p ds \\
&\leq \int_0^t \mu_s(yx)^p ds \\
&= \int_0^t \mu_s(xy)^p ds, \quad t > 0.
\end{aligned}$$

This completes the proof. □

The matrix version of Lemma 3.2 appears in [13].

Now, using Lemma 3.1 and Lemma 3.2, we get our first main result of this note.

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THEOREM 3.3. *Let $1 \le p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. For $i = 1, 2, \dots, n$, let $x_i \in L^q(\mathcal{M})^+$, $y_i \in L^r(\mathcal{M})^+$ such that $x_i y_i = y_i x_i$. Then*

$$\left\| \sum_{i=1}^n x_i y_i \right\|_p \le \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \le \left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \right\|_p .$$

Proof. By inequality (2.1), we obtain

$$\sum_{i=1}^n x_i y_i \in L^p(\mathcal{M}), \; \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \in L^p(\mathcal{M}).$$

Since $x_i y_i = y_i x_i$, we have $x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} = y_i^{\frac{1}{2}} x_i^{\frac{1}{2}}$. Thus, $x_i y_i \ge 0$ and $x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \ge 0$. From Lemma 3.1, we deduce

$$\left\| \sum_{i=1}^n x_i y_i \right\|_p = \left\| \sum_{i=1}^n \left(x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \le \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p .$$

Let $0 < s \le \infty$. It is well known that $S(\mathcal{M})$ is dense in $L^s(\mathcal{M})$. Hence, there exist

$$\{x_{i,k}\}_{k=1}^\infty, \, \{y_{i,k}\}_{k=1}^\infty, \, \{z_{i,k}\}_{k=1}^\infty \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x_{i,k}\|_q \to 0, \;\; \|y_i - y_{i,k}\|_r \to 0, \;\; \|z_i - z_{i,k}\|_{2p} \to 0, \;\; k \to \infty.$$

Note that

$$\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix} = \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \ge 0.$$

It is easy to see that $\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix}$ is PPT. Hence,

$$\begin{pmatrix} \sum_{i=1}^n x_{i,k} & \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ \sum_{i=1}^n y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & \sum_{i=1}^n y_{i,k} \end{pmatrix}$$

is PPT. It follows from Lemma 3.2 that

$$\left\| \left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right)^2 \right\|_p \le \left\| \left(\sum_{i=1}^n x_{i,k} \right) \left(\sum_{i=1}^n y_{i,k} \right) \right\|_p . \tag{3.3}$$

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On the other hand, inequality (2.1) implies that

$$\begin{aligned}\left\|x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right\|_{2p} &\le \left\|x_i^{\frac{1}{2}}\left(y_i^{\frac{1}{2}}-y_{i,k}^{\frac{1}{2}}\right)\right\|_{2p}+\left\|\left(x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right)y_i^{\frac{1}{2}}\right\|_{2p}\\ &\le \left\|x_i^{\frac{1}{2}}\right\|_{2q}\left\|y_i^{\frac{1}{2}}-y_{i,k}^{\frac{1}{2}}\right\|_{2r}+\left\|x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right\|_{2q}\left\|y_i^{\frac{1}{2}}\right\|_{2r}\\ &= \|x_i\|_q^{\frac{1}{2}}\left\|y_i^{\frac{1}{2}}-y_{i,k}^{\frac{1}{2}}\right\|_{2r}+\left\|x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right\|_{2q}\|y_i\|_r^{\frac{1}{2}}.\end{aligned}$$

Put $g(t)=t^{\frac{1}{2}}$, then g is nonnegative and operator monotone. According to Theorem 1.1 in [5], we obtain

$$\int_0^t \mu_s\left(x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right)ds \le \int_0^t \mu_s\left(|x_i-x_{i,k}|^{\frac{1}{2}}\right)ds,\quad t>0.$$

From the fact $2q>1$ and Theorem 2.1 of [4] and Lemma 2.5(iv) of [7], we get

$$\int_0^t \mu_s\left(x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right)^{2q}ds \le \int_0^t \mu_s(x_i-x_{i,k})^q ds,\quad t>0.$$

This implies that

$$\left\|x_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}\right\|_{2q}\le \|x_i-x_{i,k}\|_q^{\frac{1}{2}}\to 0,\ \ k\to\infty.$$

Similarly,

$$\left\|y_i^{\frac{1}{2}}-y_{i,k}^{\frac{1}{2}}\right\|_{2r}\le \|y_i-y_{i,k}\|_r^{\frac{1}{2}}\to 0,\ \ k\to\infty.$$

Thus, $\left\|x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right\|_{2p}\to 0$ as $k\to\infty$, and so

$$\left\|\sum_{i=1}^n x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-\sum_{i=1}^n x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right\|_{2p}\to 0,\ \ k\to\infty.$$

Therefore, $\left\|\sum_{i=1}^n x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right\|_{2p}\to\left\|\sum_{i=1}^n x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\right\|_{2p}$, $k\to\infty$. Hence,

$$\left\|\left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right)^2\right\|_p\to\left\|\left(\sum_{i=1}^n x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\right)^2\right\|_p,\ \ k\to\infty. \tag{3.4}$$

By an argument similar to the one presented above, we obtain

$$\|x_iy_j-x_{i,k}y_{j,k}\|_p\to 0,\ \ k\to\infty.$$

It follows that

$$\left\|\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)-\left(\sum_{i=1}^n x_{i,k}\right)\left(\sum_{i=1}^n y_{i,k}\right)\right\|_p\to 0,\ \ k\to\infty,$$

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which tells us that

$$\left\| \left(\sum_{i=1}^{n} x_{i,k} \right) \left(\sum_{i=1}^{n} y_{i,k} \right) \right\|_p \to \left\| \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right) \right\|_p, \quad k \to \infty. \tag{3.5}$$

Combing (3.3) and (3.4) with (3.5), we have

$$\left\| \left(\sum_{i=1}^{n} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \le \left\| \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right) \right\|_p. \quad \square$$

Theorem 3.3 includes a special case as follows.

Corollary 3.4. *Let $1 \le p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $s, t > 0$. For $t, s > 0$, let $x \in L^{tq}(\mathcal{M})^+$, $y \in L^{sr}(\mathcal{M})^+$. Then*

$$\|x^{t+s} + y^{t+s}\|_p \le \|(x^t + y^t)(x^s + y^s)\|_p.$$

Proof. If we replace n, x_1, y_1, x_2, y_2 by $2, x^t, x^s, y^t, y^s$, respectively, in Theorem 3.3, we deduce that

$$\|x^{t+s} + y^{t+s}\|_p \le \|(x^t + y^t)(x^s + y^s)\|_p. \quad \square$$

Lemma 3.5. *Let $x, y \in S(\mathcal{M})^+$ and $0 \le q \le 1$. If $1 \le p < \infty$, then*

$$\|x^q y^{1-q}\|_p \le \|qx + (1-q)y\|_p.$$

Proof. The result follows immediately from Theorem 3.3 of [8]. $\square$

Now, using Lemma 3.5, we get our another main result of this note.

Theorem 3.6. *Let $1 \le p < \infty$ and $0 \le q \le 1$. For all $x, y \in L^{2p}(\mathcal{M})$, we have*

$$\|xy^*\|_p^2 \le \|qx^*x + (1-q)y^*y\|_p \|(1-q)x^*x + qy^*y\|_p.$$

Proof. Since $S(\mathcal{M})$ is dense in $L^s(\mathcal{M})(0 < s \le \infty)$, then there exist

$$\{x_i\}_{k=1}^{\infty}, \{y_i\}_{k=1}^{\infty} \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x\|_{2p} \to 0, \quad \|y_i - y\|_{2p} \to 0, \quad k \to \infty.$$

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By inequality (2.1), Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned}\|x_i y_i^*\|_p^2 &= \left(\int_0^\infty \mu_t(y_i x_i^* x_i y_i^*)^{\frac{p}{2}} dt\right)^{\frac{2}{p}} = \left(\int_0^\infty \mu_t(x_i^* x_i y_i^* y_i)^{\frac{p}{2}} dt\right)^{\frac{2}{p}} \\ &= \left(\int_0^\infty \mu_t((y_i^* y_i)^q (x_i^* x_i)^{1-q} (x_i^* x_i)^q (y_i^* y_i)^{1-q})^{\frac{p}{2}} dt\right)^{\frac{2}{p}} \\ &= \|(y_i^* y_i)^q (x_i^* x_i)^{1-q} (x_i^* x_i)^q (y_i^* y_i)^{1-q}\|_{\frac{p}{2}} \\ &\leq \left\|(y_i^* y_i)^q (x_i^* x_i)^{1-q}\right\|_p \left\|(x_i^* x_i)^q (y_i^* y_i)^{1-q}\right\|_p\end{aligned}$$

for all $0 \leq q \leq 1$. It follows from Lemma 3.5 that

$$\|(y_i^* y_i)^q (x_i^* x_i)^{1-q}\|_p \leq \|q(y_i^* y_i) + (1-q)(x_i^* x_i)\|_p$$

and

$$\|(x_i^* x_i)^q (y_i^* y_i)^{1-q}\|_p \leq \|q(x_i^* x_i) + (1-q)(y_i^* y_i)\|_p.$$

Hence,

$$\|x_i y_i^*\|_p^2 \leq \|q(y_i^* y_i) + (1-q)(x_i^* x_i)\|_p \|q(x_i^* x_i) + (1-q)(y_i^* y_i)\|_p.$$

A similar argument to the proof of Theorem 3.3 shows that

$$\|xy^*\|_p^2 \leq \|q(y^* y) + (1-q)(x^* x)\|_p \|q(x^* x) + (1-q)(y^* y)\|_p. \quad \square$$

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