Electronic Journal of Linear Algebra

Volume 31 *Volume 31: (2016)* Article 39

2016

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Recommended Citation
Beasley, LeRoy B.. (2016), "Preservers of term ranks and star cover numbers of symmetric matrices", *Electronic Journal of Linear Algebra*, Volume 31, pp. 549-564.
DOI: https://doi.org/10.13001/1081-3810.3231

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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PRESERVERS OF TERM RANKS AND STAR COVER NUMBERS OF SYMMETRIC MATRICES*

LEROY B. BEASLEY†

Abstract. Let $\mathcal{S}_n(\mathbb{S})$ denote the set of symmetric matrices over some semiring, $\mathbb{S}$. A line of $A \in \mathcal{S}_n(\mathbb{S})$ is a row or a column of A. A star of A is the submatrix of A consisting of a row and the corresponding column of A. The term rank of A is the minimum number of lines that contain all the nonzero entries of A. The star cover number is the minimum number of stars that contain all the nonzero entries of A. This paper investigates linear operators that preserve sets of symmetric matrices of specified term rank and sets of symmetric matrices of specific star cover numbers. Several equivalences to the condition that T preserves the term rank of any matrix are given along with characterizations of a couple of types of linear operators that preserve certain sets of matrices defined by the star cover number that do not preserve all term ranks.

Key words. Semiring, Semimodule, Upper ideal, Linear preserver, Term rank, Star cover number.

AMS subject classifications. 05C50, 15A86.

1. Introduction. Let A be a matrix with entries from an algebraic structure with a zero element. The term rank of A is the minimum number of rows and or columns of A that contain all of the nonzero entries of A. As the nature of the nonzero entries is not important when studying the term rank, just whether they are zero or nonzero, it is usual to restrict the study to $(0,1)$-matrices since the term rank of A is the term rank of the support of A. In this paper we will mostly restrict our attention to Boolean matrices, that is $(0,1)$-matrices with arithmetic the same as for real numbers except that $1+1=1$.

In this article, we consider linear operators on sets of matrices that leave certain sets defined by term rank invariant. The definition of a linear operator on sets of matrices where the entries are from any semiring is similar to the definition over the real or complex numbers. The study of the invariants of maps has been an ongoing topic of research for over a century. When the operators are between sets of matrices, that study was begun by Frobenius in 1897 [4] when he classified linear operators that preserve the determinant function. Since that time much research has been published on preserver problems. See [6, 9] for an excellent survey.

*Received by the editors on February 23, 2016. Accepted for publication on June 3, 2016. Handling Editor: Zejun Huang.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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Recently, the study of linear operators on matrix spaces over sets which are not fields has become established. In particular, there has been a lot of activity in the study of linear operators on spaces of $(0,1)$-matrices. This study is related to maps on directed or undirected graphs and bipartite graphs, and so has importance in combinatorics, computing, etc. The underlying set of scalars in this case is usually Boolean in that addition acts much like union of sets and multiplication much like intersection.

Unlike the investigation of preservers of sets of matrices over fields, preservers of sets of Boolean matrices usually require more hypothesis than to just assume that an operator preserves the set. Defining an operator to be O at O and the image of all other matrices to be a fixed element in the specified set produces a linear operator that preserves that set. Clearly this operator is not very interesting. Thus, an additional hypothesis is needed. This additional hypothesis is commonly that the operator is bijective. Another condition also used, and the one we are using in this article, is that the operator "strongly" preserves the set, that is the image of an element in the set is in the set, while the image of an element not in the set is not in the set. This condition appears not only in research on matrices over discrete semirings, but also in research on preservers over real and complex matrices. See for example [8] and [10].

2. Preliminaries. A *semiring* is a system, $(\mathbb{S},+,\times)$, where $\mathbb{S}$ is a nonempty set, $(\mathbb{S},+)$ is an Abelian monoid (identity 0), $(\mathbb{S},\times)$ is a monoid (identity 1), $\times$ distributes over $+$, and $0\times s = s\times 0 = 0$ for all $s \in \mathbb{S}$. Usually $\mathbb{S}$ denotes the system and $\times$ is denoted by juxtaposition. If $(\mathbb{S},\times)$ is Abelian then we say $\mathbb{S}$ is *commutative.* If 0 is the only element of $\mathbb{S}$ that has an additive inverse then $\mathbb{S}$ is *antinegative.* Note that all rings with unity are semirings, but none are antinegative. Algebraic terms like *unit* and *zero divisor* are defined for semirings as if $\mathbb{S}$ were a ring.

In this paper, unless specified differently, we will assume that $\mathbb{S}$ is commutative, antinegative and with no zero divisors. Of special interest to us is the *binary Boolean semiring* $\mathbb{B} = \{0,1\}$ with arithmetic the same as for the reals except that $1+1=1$.

A *semimodule over* $\mathbb{S}$ is a triple $(\mathcal{K},+,\bullet)$, where $\mathcal{K}$ is a nonempty set, $(\mathcal{K},+)$ is an Abelian monoid (identity O) and $\bullet$ is a scalar product. Recall that $\bullet$ is a scalar product if for all $\alpha,\beta \in \mathbb{S}$ and $A,B \in \mathcal{K}$: $\alpha\bullet A \in \mathcal{K}$; $\alpha\bullet(\beta\bullet A) = (\alpha\beta)\bullet A$; $0\bullet A = O$; $1\bullet A = A$; $(\alpha+\beta)\bullet A = \alpha\bullet A + \beta\bullet A$; and $\alpha\bullet(A+B) = \alpha\bullet A + \alpha\bullet B$. Thus, a semimodule satisfies all the properties of a vector space except the one that requires additive inverses. As with semirings, $\mathcal{K}$ will denote the system and $\bullet$ is denoted by juxtaposition. The semimodules of interest in this article are $\mathcal{S}_n(\mathbb{S})$, the symmetric $n\times n$ matrices with entries in the semiring $\mathbb{S}$, $\mathcal{S}_n^{(0)}(\mathbb{S})$, the subset of $\mathcal{S}_n(\mathbb{S})$ all of whose members have zero diagonals, and in particular $\mathcal{S}_n(\mathbb{B})$, symmetric matrices with entries in $\mathbb{B}$, and $\mathcal{S}_n^{(0)}(\mathbb{B})$, the subset of $\mathcal{S}_n(\mathbb{B})$ all of whose members have zero

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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diagonals. An element of a basis is called a *base element*. In $\mathcal{S}_n(\mathbb{S})$ base elements are multiples of *diagonal cells* (diagonal matrices with a "1" in the i^{th} location and zeros elsewhere, denoted $E_{i,i}$), and *digons* (symmetric matrices containing exactly two nonzero entries, they being "1's" in the (i, j) and (j, i) locations, denoted $D_{i,j}$). The base elements of $\mathcal{S}_n^{(0)}(\mathbb{S})$ are just the digons since all members have only zeros on the main diagonal. We let J_n denote the $n \times n$ matrix of all ones, K_n denote the $n \times n$ matrix with a zero diagonal and all other entries ones, O_n denote the $n \times n$ matrix of all zeros, and I_n denote the $n \times n$ identity matrix. We usually drop the subscripts unless the context is not obvious, and write J, K, O, I.

A mapping $T : \mathcal{K} \to \mathcal{L}$ of semimodules $\mathcal{K}$ and $\mathcal{L}$ over $\mathbb{S}$ is said to be *linear* if given any $A, B \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{S}$ then $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$. A linear operator $T : \mathcal{K} \to \mathcal{K}$ is said to *preserve* a subset $\mathcal{X} \subseteq \mathcal{K}$ if $X \in \mathcal{X}$ implies that $T(X) \in \mathcal{X}$. We say that T *strongly preserves* $\mathcal{X}$ if $X \in \mathcal{X}$ if and only if $T(X) \in \mathcal{X}$. Further, T preserves a function f if T preserves the preimage of each image element.

Let $A, B \in \mathcal{S}_n(\mathbb{S})$. We say that A *dominates* B, $A \sqsupseteq B$ or $B \sqsubseteq A$, if $a_{i,j} = 0$ implies that $b_{i,j} = 0$.

Let A be a matrix. A *line* of A is a row or column of A. The *term rank* of A, $\tau(A)$, is the fewest number of lines of A that contain all the nonzero entries of A. A *line matrix* is a matrix of term rank one. A *full line matrix* is a term rank 1 matrix which has a row or column of all ones. Let $R_i(A)$ denote the i^{th} row and $C_j(A)$ denote the j^{th} column of A. A *double star* is a square matrix whose i^{th} row and column contain some ones and all other entries are zero. Let $S_i(A)$ denote the double star of A containing $R_i(A)$ and $C_i(A)$. For convenience we let $R_i = R_i(J)$, the full i^{th} row matrix, $C_j = C_j(J)$, the full j^{th} column matrix, and $S_i = S_i(J)$ the full i^{th} double star matrix. Let "*set of star-free lines*" mean that no two lines in the set are dominated by a double star.

A *cover* of A is a set of lines that contain all the nonzero entries of A. If that cover is a set of s double stars and t star-free lines, we say it is an (s, t)-cover. So the term rank of A is the minimum number of lines in any cover. An (s, t)-cover is a *minimal cover* if $2s + t = \tau(A)$. A *proper* (s, t)*-cover* of A is an (s, t)-minimal cover such that given any other (s', t')-minimal cover we have that $s \geq s'$. It follows that replacing with zeros all the nonzero entries in A that are dominated by a double star dominating one of the t star-free lines reduces the term rank by exactly one. For example:

Example 2.1. Let $A = \begin{bmatrix} J_3 & J_3 & J_3 \\ J_3 & Z & O \\ J_3 & O & O \end{bmatrix}$, where $Z = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Notice that $\{S_1, S_2, S_3, R_4, R_5, R_6\}$ is a cover of A. It is proper since replacing all the ones in row

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016
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and column i by zeros, for any $i = 4, 5, 6$, reduces the term rank by just one. For example for $i = 5$, the matrix arrived at by replacing all the ones in the 5^{th} row and 5^{th} column by zeros is

$$\begin{bmatrix} 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&1&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0 \end{bmatrix}.$$

This matrix has term rank 8, one less than the term rank of A.

Let $B = \begin{bmatrix} J_3 & J_3 & J_3 \\ J_3 & Y & O \\ J_3 & O & O \end{bmatrix}$, where $Y = \begin{bmatrix} 1&1&0\\1&1&1\\0&1&0 \end{bmatrix}$. As above we have that $\{S_1, S_2, S_3, R_4, R_5, R_6\}$ is a cover of A, but it is not proper since replacing all the ones in row and column 5 by zeros reduces the term rank by two. The matrix arrived at by replacing all the ones in row and column 5 by zeros is

$$\begin{bmatrix} 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0 \end{bmatrix}.$$

A proper cover of A would be a $(4, 1)$-cover consisting of $\{S_1, S_2, S_3, S_5, R_4\}$.

Note that the condition "replacing of all of the nonzero entries dominated by a double star covering one of the t starfree lines with zeros reduces the term rank by at most 1" is satisfied vacuously in any proper $(s, 0)$-cover. With this in mind, one can see that every matrix in $\mathcal{S}_n(\mathbb{B})$ has a proper cover. Further, the star-free lines in a proper cover can always be taken to be all rows, or all columns, we will always take rows.

A *star cover* of A is a sum of double stars that cover all the nonzero entries of A, and the *star cover number* of A, $s(A)$, is the minimum number of stars in any star cover of A.

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A publication of the International Linear Algebra Society
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Let $\mathbb{S}$ be any semiring. If $A \in \mathcal{S}_n(\mathbb{S})$ the *support* of A is the matrix $\overline{A} \in \mathcal{S}_n(\mathbb{B})$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$. Let $T : \mathcal{S}_n(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ be a linear operator. We define the corresponding linear operator $\overline{T} : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ by $\overline{T}(\overline{D_{i,j}}) = \overline{T(D_{i.j})}$ for all i, j and extend linearly. Note that $D_{i,i}$ denotes the diagonal cell $E_{i,i}$.

For most of this article we consider matrices over the binary Boolean semiring. In order to generalize to other semirings, we appeal to the following easily established theorem:

THEOREM 2.2. *Let $T : \mathcal{S}_n(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ be a linear operator. Then for any k,*

- *T preserves term rank k if and only if $\overline{T}$ preserves term rank k;*
- *T preserves star cover number k if and only if $\overline{T}$ preserves star cover number k.*

Proof. Since the term rank and the star cover number of a matrix depend only on the zero/nonzero pattern of the matrix, for any $A \in \mathcal{S}_n(\mathbb{S})$, $\tau(A) = \tau(\overline{A})$ and $s(A) = s(\overline{A})$. By the definition of $\overline{T}$, since $\mathbb{S}$ is antinegative, $T(A)$ and $\overline{T}(\overline{A})$ have the same zero/nonzero pattern. Thus, $\tau(T(A)) = \tau(\overline{T}(\overline{A}))$ and $s(T(A)) = s(\overline{T}(\overline{A}))$. That is, T preserves term rank k if and only if $\overline{T}$ preserves term rank k and T preserves star cover number k if and only if $\overline{T}$ preserves star cover number k. □

For basic facts and definitions of linear algebraic concepts we refer the reader to Horn and Johnson, [5].

A linear operator $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ is a (P, P^t)-*operator* if and only if there is an $n \times n$ permutation matrix, P, such that for any $X \in \mathcal{S}_n(\mathbb{B})$, $T(X) = P^t X P$. A linear operator $T : \mathcal{S}_n(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ is a (P, P^t, B)-*operator* if and only if there are two matrices, an $n \times n$ permutation matrix P and a matrix $B \in \mathcal{S}_n(\mathbb{S})$ all of whose entries are nonzero, such that $T(X) = P^t(X \circ B)P$ for any $X \in \mathcal{S}_n(\mathbb{S})$ where $A \circ B = (a_{i,j} b_{i,j})$ is the Hadamard product of A and B.

Let $\mathcal{U}$ be a subset of $\mathcal{S}_n(\mathbb{B})$. Then $\mathcal{U}$ is said to be an *upper ideal* of $\mathcal{S}_n(\mathbb{B})$ if $A + X \in \mathcal{U}$ for every $A \in \mathcal{U}$ and $X \in \mathcal{S}_n(\mathbb{B})$. Let $\mathcal{W}$ be a subset of semimodule $\mathcal{S}_n(\mathbb{B})$. We say that $\mathcal{W}$ *separates base elements* if, given any two distinct base elements E and F, there is some $N \in \mathcal{S}_n(\mathbb{B})$ such that $N + E \in \mathcal{W}$ and $N + F \notin \mathcal{W}$. In this case we say that $\mathcal{W}$ separates E from F.

The following lemmas from [1] will be used.

LEMMA 2.3. [1, Lemma 7] *Let $\Psi : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator and let $\mathcal{U}$ be an upper ideal of $\mathcal{S}_n(\mathbb{B})$. If Ψ strongly preserves $\mathcal{U}$ and $\mathcal{U}$ separates base elements,*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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then Ψ is bijective on the set of all base elements.

LEMMA 2.4. [1, Lemma 8(3)] *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a bijective linear operator. For $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$, T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star if and only if T is a (P, P^t)-operator.*

Note that the expression "T preserves star matrices" is equivalent to "T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star".

LEMMA 2.5. *Let $1 \leq k \leq n$, and $A \in \mathcal{S}_n(\mathbb{B})$ with $\tau(A) = k$. Then, for any even $\ell \leq k$ there is a submatrix of A in $\mathcal{S}_n(\mathbb{B})$ with term rank ℓ.*

Proof. If $\tau(A) = 3, 4$ then one easily sees that A has a submatrix of term rank 2. We proceed by induction. Consider a proper (s, t)-cover of A. For $\ell = 2s' \leq 2s$, the submatrix of A dominated by s' of the double stars has term rank ℓ. If $\ell > 2s$, replace all the ones dominated by a double star corresponding to a starfree line of A by zeros. The remaining matrix has term rank precisely $2s + t - 1$ since the cover was proper. (See Example 2.1.) By induction this matrix, and hence A, has a submatrix of term rank ℓ. □

For any q, define $\mathcal{U}_q = \{A \in \mathcal{S}_n(\mathbb{B}) : \tau(A) \geq q\}$.

LEMMA 2.6. *Let $3 \leq q \leq n$. Then $\mathcal{U}_q$ is an upper ideal which separates base elements.*

Proof. Since $\tau(A+B) \geq \tau(A)$ for any B, $\mathcal{U}_q$ is an upper ideal. We now show that $\mathcal{U}_q$ separates base elements. Given two distinct base elements there are five cases, (up to symmetric permutation). The table below lists the five cases together with the matrix N which separates E from F.

$E = E_{n,n}$	$F = E_{1,1}$	$N = I_{q-1} \oplus O$
$E = D_{1,n}$	$F = E_{1,1}$	$N = I_{q-1} \oplus O$
$E = D_{n-1,n}$	$F = E_{1,1}$	$N = I_{q-2} \oplus O$
$E = D_{1,n}$	$F = D_{1,2}$	$N = 0 \oplus I_{q-2} \oplus O$
$E = D_{1,n}$	$F = D_{2,3}$	$N = 0 \oplus I_{q-2} \oplus O$

In each case $\tau(N+E) = q$ and $\tau(N+F) \leq q-1$, so that $N+E \in \mathcal{U}$ while $N+F \notin \mathcal{U}$. □

LEMMA 2.7. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ and $3 \leq k \leq n$. If T strongly preserves term rank k then T is bijective on the set of base elements.*

Proof. If T strongly preserves term rank k for $3 \leq k \leq n$, then T strongly preserves $\mathcal{U}_k$. By Lemma 2.6, $\mathcal{U}_k$ separates base elements, and hence, by Lemma 2.3, T is bijective on the set of base elements. □

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A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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One should observe that if $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ is bijective on the set of base elements, then T is bijective on all of $\mathcal{S}_n(\mathbb{B})$, since each member of $\mathcal{S}_n(\mathbb{B})$ is a unique linear combination of base elements.

The study of term rank and star cover numbers was initiated in [2, 3]. The following theorems summarize those results:

Theorem 2.8. [2, Corollary 3.3] *Let $T : \mathcal{S}_n^{(0)}(\mathbb{S}) \to \mathcal{S}_n^{(0)}(\mathbb{S})$ be a linear operator. Then the following are equivalent:*

1. *T preserves term rank;*
2. *T is bijective and preserves term rank 2 or 3;*
3. *T preserves term rank 2 and $\overline{T(K)} = K$;*
4. *T preserves term ranks 2 and k, $3 \le k \le n$;*
5. *T preserves term ranks 3 and k, $4 \le k \le n$;*
6. *T is a (P, P^t, B)-operator.*

Theorem 2.9. [3, Theorem 3.2.4] *If $T : \mathcal{S}_n^{(0)}(\mathbb{S}) \to \mathcal{S}_n^{(0)}(\mathbb{S})$ is a linear operator, then the following are equivalent:*

1. *T preserves the star cover number;*
2. *T preserves star cover number 1 and $\overline{T(K)} = K$;*
3. *T preserves star cover numbers 1 and 2;*
4. *T is a (P, P^t, B)-operator.*

Results of M. H Lim on this topic are:

Theorem 2.10. [7, Lemma 5.1] *If $m \le n$ and $T : \mathcal{S}_m(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ is a linear operator, then the following are equivalent:*

1. *T preserves term rank 1 and term rank k for some $k, 2 \le k \le m-1$;*
2. *T preserves term rank 2 and term rank k for some $k, 3 \le k \le m$;*
3. *there exist two matrices, an $n\times m$ submatrix P of an $n\times n$ permutation matrix and an $m\times m$ matrix B with all nonzero entries, such that $T(A) = P(B\circ A)P^t$ for all $A \in \mathcal{S}_m(\mathbb{S})$. (For $m = n$, T is a (P, P^t, B)-operator.)*

The following example shows that condition 1 in Theorem 2.10 above can not be improved to include $k = m$:

Example 2.11. Let $m \le n$ and $T : \mathcal{S}_m(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ be defined by $T(E_{i,i}) = E_{i,i}, i = 1, \ldots, m$ and for $i \ne j$, $T(D_{i,j}) = J_m \oplus O$, extended linearly. Then T preserves both term rank 1 and term rank m, but clearly there does not exist an $n \times m$ submatrix P of an $n \times n$ permutation matrix or an $m \times m$ matrix B with all nonzero entries such that $T(A) = P(B \circ A)P^t$ for all $A \in \mathcal{S}_m(\mathbb{S})$.

In this paper, we continue the study of preservers of term rank and star cover

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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numbers of symmetric matrices.

In the next section, we extend the Boolean results of Theorem 2.8 to the full semimodule $\mathcal{S}_n(\mathbb{B})$ and give further results. In Section 4, we extend the Boolean results of Theorem 2.9 to the full semimodule $\mathcal{S}_n(\mathbb{B})$. In Section 5, we have these results extended to arbitrary commutative, antinegative semirings without zero divisors.

3. Term rank preservers.

LEMMA 3.1. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ and $3 \leq k \leq n-1$ with k being odd. If T preserves term ranks k and $k+1$ then T strongly preserves $\mathcal{U}_{k+1} = \{A \in \mathcal{S}_n(\mathbb{B}) : \tau(A) \geq k+1\}$.*

Proof. Let $A \in \mathcal{S}_n(\mathbb{B})$ with $\tau(A) \leq k$, then $\tau(T(A)) \leq k$, for otherwise, there is some matrix B such that $\tau(A+B) = k$. Then $k+1 \leq \tau(T(A)) \leq \tau(T(A+B)) = k$, a contradiction.

Suppose that $\tau(A) \geq k+1$ and $\tau(T(A)) \leq k$. Since T preserves term rank $k+1$, $\tau(A) \geq k+2$. Since k is odd, then by Lemma 2.5, A, and hence $T(A)$, has a submatrix of term rank $k+1$. Thus, $\tau(T(A)) \geq k+1$, a contradiction. Thus, T strongly preserves $\mathcal{U}_{k+1}$. ☐

LEMMA 3.2. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ and $3 \leq k \leq n-1$ with k being odd. If T preserves term ranks k and $k+1$ then T is bijective.*

Proof. By Lemma 3.1, T strongly preserves $\mathcal{U}_{k+1}$, and by Lemma 2.6, $\mathcal{U}_{k+1}$ separates base elements. By Lemma 2.3 T is bijective on the set of base elements of $\mathcal{S}_n(\mathbb{B})$, but since $\mathcal{S}_n(\mathbb{B})$ is finite and every element in $\mathcal{S}_n(\mathbb{B})$ can be represented as a unique sum of base elements, T is bijective on $\mathcal{S}_n(\mathbb{B})$. ☐

THEOREM 3.3. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$. If T strongly preserves term rank k for $2 \leq k \leq n$, or if T preserves term ranks k and $k+1$ for $3 \leq k \leq n-1$ with k being odd, then T is a (P, P^t)-operator.*

Proof. Let $k = 2$ and suppose that T strongly preserves term rank 2. Suppose that T does not preserve term rank 1. Then, without loss of generality, $T(E_{1,1}) = O$ or $\tau(T(E_{1,1})) \geq 2$. If $T(E_{1,1}) = O$, then $T(E_{2,2}) = T(E_{1,1} + E_{2,2})$, an impossibility since $\tau(E_{2,2}) = 1$ and $\tau(E_{1,1}+E_{2,2}) = 2$. If $\tau(T(E_{1,1})) \geq 2$, then $\tau(T(E_{1,1})) \geq 3$ since T strongly preserves term rank 2. But then, $T(E_{1,1}+E_{2,2}) = T(E_{1,1})+T(E_{2,2})$. Note that adding $T(E_{2,2})$ to $T(E_{1,1})$ does not lower the term rank. Thus, $T(E_{1,1} + E_{2,2})$ has term rank at least 3, a contradiction. Thus, T preserves term ranks 1 and 2. By Theorem 2.10, T is a (P, P^t)-operator. Now, assume that $k \geq 3$.

By Lemmas 2.7 and 3.2, T is bijective on the set of base elements. Thus, by virtue

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A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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of Lemma 2.4 we only need to show that T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star.

Suppose that there is some pair of base elements not dominated by a double star whose image is dominated by a double star. Without loss of generality, either the two base elements are $E_{1,1}$ and $D_{2,3}$ or $D_{1,2}$ and $D_{3,4}$.

In the first case, $E_{1,1}+D_{2,3}$ has term rank 3 and $T(E_{1,1}+D_{2,3})$ has term rank 2. Here, $E_{1,1}+D_{2,3}+D_{4,5}+\cdots+D_{k-1,k}$ has term rank k but since T maps base elements to base elements, (by Lemma 2.3), $T(D_{4,5}+\cdots+D_{k-1,k})$ has term rank at most $k-3$ and $T(E_{1,1}+D_{2,3})$ has term rank 2. Thus, $T(E_{1,1}+D_{2,3}+D_{4,5}+\cdots+D_{k-1,k}) = T(E_{1,1}+D_{2,3})+T(D_{4,5}+\cdots+D_{k-1,k})$ has term rank at most $2+(k-3)=k-1$, a contradiction.

In the second case, $D_{1,2}+D_{3,4}$ has term rank 4 and $T(D_{1,2}+D_{3,4})$ has term rank 2. Here, $D_{1,2}+D_{3,4}+D_{5,6}+\cdots+D_{k,k+1}$ has term rank $k+1$ but since T maps base elements to base elements, (by Lemma 2.3), $T(D_{5,6}+\cdots+D_{k,k+1})$ has term rank at most $k-3$ and $T(D_{1,2}+D_{3,4})$ has term rank 2. Thus, $T(D_{1,2}+D_{3,4}+D_{5,6}+\cdots+D_{k,k+1}) = T(D_{1,2}+D_{3,4})+T(D_{5,6}+\cdots+D_{k,k+1})$ has term rank at most $2+(k-3)=k-1$, a contradiction.

Thus, T maps pairs of base elements not dominated by a double star to pairs of base elements not dominated by a double star. Since T is bijective, T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star. □

LEMMA 3.4. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ and $n \geq 3$. If T preserves term rank 2 and $T(J) = J$ then the image of any double star is dominated by a double star.*

Proof. Suppose without loss of generality that $T(S_1)$ is not dominated by a double star. Since $\tau(T(S_1)) = 2$, we must have that $T(S_1) \sqsubseteq J_2 \oplus O$. Since $T(J) = J$, there is some base element D such that $T(D) \sqsupseteq E_{3,3}$. Then either $D = E_{j,j}$ or $D = D_{i,j}$ for some i, j. Let $X = T(D_{1,j})$, so that $X \sqsubseteq J_2 \oplus O$ and has term rank 2. Then, $T(D_{1,j}+D) \sqsupseteq X+E_{3,3}$, a contradiction since $\tau(D_{1,j}+D) = 2$, while $\tau(X+E_{3,3}) = 3$. Thus, the image of any star is dominated by a double star. □

LEMMA 3.5. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T preserves term ranks 3 and k for some $k \in \{4, 5, \ldots, n\}$, then T is bijective.*

Proof. Suppose that the image of a double star has term rank 3. (It can't have term rank greater than three for if it did one could add the image of a diagonal cell to get a matrix of term rank three whose image has term rank greater than three.) Without loss of generality we may assume that $\tau(T(S_1)) = 3$ and that either $T(S_1) \sqsubseteq J_3 \oplus O$ or $T(S_1) \sqsubseteq S_1 + E_{2,2}$. Since $\tau(S_1 + E_{i.i}) = 3$, for $i \neq 1$, in the first

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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case, $T(E_{i,i}) \sqsubseteq J_3 \oplus O$ for all i and $T(I_k \oplus O) \sqsubseteq J_3 \oplus O$, and in the second case $T(E_{i,i}) \sqsubseteq S_1 + E_{2,2}$ and $T(I_k \oplus O) \sqsubseteq S_1 + E_{2,2}$. In both cases a term rank k matrix is mapped to a term rank 3 matrix, a contradiction. Now suppose that a term rank 2 matrix not dominated by a double star is mapped to a term rank 3 matrix. We may assume that $T(E_{1,1} + E_{2,2} + D_{1,2})$ is a term rank three matrix. Then, either $T(E_{1,1} + E_{2,2} + D_{1,2}) \sqsubseteq J_3 \oplus O$ or $T(E_{1,1} + E_{2,2} + D_{1,2}) \sqsubseteq S_1 + E_{2,2}$. In the first case, $T(E_{i,i}) \sqsubseteq J_3 \oplus O$ for all i and $T(I_k \oplus O) \sqsubseteq J_3 \oplus O$, and in the second case $T(E_{i,i}) \sqsubseteq S_1 + E_{2,2}$ and $T(I_k \oplus O) \sqsubseteq S_1 + E_{2,2}$. In both cases a term rank k matrix is mapped to a term rank 3 matrix, a contradiction. Since any term rank one matrix is a submatrix of a term rank two matrix, it follows that if $\tau(A) \leq 2$ then $\tau(T(A)) \leq 2$.

Suppose that some matrix of term rank at least 4 is mapped to a matrix of term rank less than three. Then by Lemma 2.5, there is a matrix A of term rank 4 that is mapped to a matrix of term rank at most 2. Now, there is a term rank two matrix A_1 such that $\tau(A + A_1) = 6$, a term rank two matrix A_2 such that $\tau(A + A_1 + A_2) = 8$, etc. So, if k is odd, then $k - 1$ is even and there are $\frac{(k-1)-4}{2}$ term rank 2 matrices $A_1, A_2, \ldots, A_q$, where $q = \frac{(k-1)-4}{2}$, such that $\tau(A+A_1+A_2+\cdots+A_q) = k-1$. Further there is a term rank one matrix A_{q+1} such that $\tau(A+A_1+A_2+\cdots+A_q+A_{q+1}) = k$. But then we must have $k = \tau(T(A + A_1 + A_2 + \cdots + A_q + A_{q+1})) \leq \tau(T(A)) + [\tau(T(A_1)) + \tau(T(A_2)) + \cdots + \tau(T(A_q)) + \tau(T(A_{q+1}))] \leq 2 + (q + 1)2 = k - 1$, a contradiction. If k is even, there are $q = \frac{k-4}{2}$ term rank two matrices $A_1, A_2, \ldots, A_q$ such that $\tau(A+A_1+A_2+\cdots+A_q) = k$. In this case, $k = \tau(T(A+A_1+A_2+\cdots+A_q)) \leq \tau(T(A))+[\tau(T(A_1))+\tau(T(A_2))+\cdots+\tau(T(A_q))] \leq 2+(q)2 = k-2$, a contradiction. Thus, T strongly preserves $\mathcal{U}_3$. By Lemma 2.6, $\mathcal{U}_3$ separates base elements and by Lemma 2.7, T is bijective. □

THEOREM 3.6. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then the following are equivalent:*

1. *T preserves term rank;*
2. *T preserves term rank 1 and term rank k for some k, $2 \leq k \leq n - 1$;*
3. *T is bijective and preserves term rank k for some k, $2 \leq k \leq n$;*
4. *T preserves term rank 2 and $T(J) = J$;*
5. *T preserves term ranks 2 and k for some k, $3 \leq k \leq n$;*
6. *T preserves term ranks 3 and k for some k, $4 \leq k \leq n$;*
7. *T preserves term ranks k and $k + 1$ for some k, $3 \leq k \leq n - 1$ with k being odd;*
8. *T strongly preserves term rank k for some k, $2 \leq k \leq n$;*
9. *T is a (P, P^t)-operator.*

Proof. That condition 9) implies conditions 1) – 8) is obvious. Conditions 1), 2), and 5) imply condition 9) by Theorem 2.10. Conditions 7) and 8) imply condition 9)

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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by Theorem 3.3. We now proceed by showing that 3) $\Rightarrow$ 8), 4) $\Rightarrow$ 2) for $k = 2$, and 6) $\Rightarrow$ 3). With this accomplished, the theorem will be established.

3) $\Rightarrow$ 8) Suppose that T is bijective and preserves term rank 2. Since the set of matrices of term rank 2 is a subset of the finite set $\mathcal{S}_n(\mathbb{B})$, T strongly preserves term rank k. Thus, 8) holds.

4) $\Rightarrow$ 2) for $k = 2$. Suppose that T preserves term rank 2 and $T(J) = J$. We shall show that T preserves term rank 1. Note that the sum of any $n-1$ double stars is not J. By Lemma 3.4, the image of a star is dominated by a star. Let $\eta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be defined by $\eta(i) = j$ if $T(S_i) \sqsubseteq S_j$. Then, $T(E_{i,i}) = T(J \setminus \sum_{j=1, j\neq i}^{n} S_i) \sqsupseteq J \setminus \sum_{j=1,j\neq i}^{n} S_{\eta(i)}$. That is $T(E_{i,i})$ dominates a diagonal cell. Since $T(S_i) \sqsubseteq S_{\eta(i)}$, we must have that $T(E_{i,i}) \sqsubseteq E_{\eta(i),\eta(i)}$. If $T(E_{i,i})$ dominates a digon then the image of any other diagonal cell is dominated by the two double stars dominating $E_{\eta(i),\eta(i)}$ and that digon, so that $T(J) \neq J$, a contradiction. Thus, T preserves term rank 1.

6) $\Rightarrow$ 3). By Lemma 3.5, T is bijective and preserves term rank 3. ∎

4. Preservers of star cover numbers. As in the previous section, we investigate strong preservers of cover numbers and then establish the generalization of Theorem 2.9 to the semimodule $\mathcal{S}_n(\mathbb{B})$.

Recall that the star cover number of a matrix A is the minimum number of double stars in a star cover of A, and is denoted $s(A)$.

Let $\mathcal{S}_k = \{A \in \mathcal{S}_n(\mathbb{B}) \mid s(A) \geq k\}$.

LEMMA 4.1. *Let $n \geq 3$ and $2 \leq k \leq n-1$. Then, $\mathcal{S}_k$ is an upper ideal which separates base elements.*

Proof. That $\mathcal{S}_k$ is an upper ideal is easily seen. To show that $\mathcal{S}_k$ separates base elements, up to symmetric permutation of rows and columns, we have five cases. The following table lists those five cases, giving the base elements E and F, and the matrix N that separates E from F:

$E = E_{1,1}$	$F = E_{n,n}$	$N = O \oplus I_{k-1}$
$E = E_{1,1}$	$F = D_{1,2}$	$N = 0 \oplus I_{k-1} \oplus O$
$E = E_{1,1}$	$F = D_{n-1,n}$	$N = O \oplus I_{k-1}$
$E = D_{1,2}$	$F = D_{1,n}$	$N = O \oplus I_{k-1}$
$E = D_{1,2}$	$F = D_{3,4}$	$N = O_2 \oplus I_{k-1} \oplus O$

The proof is complete. ∎

LEMMA 4.2. *Let $n \geq 3$, $2 \leq k \leq n-1$, and $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T strongly preserves $s(A) = k$ then T is bijective on the set of base*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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elements.

Proof. If T strongly preserves $s(A) = k$, then it is easily seen that T strongly preserves $\mathcal{S}_k$. By Lemma 4.1, $\mathcal{S}_k$ separates base elements, and so by Lemma 2.3, T is bijective on the set of base elements. □

Suppose that a star cover of matrix A with star cover number k consists of double stars $D_1 + D_2 + \cdots + D_k$. Then clearly the entries of A covered by the sum of any $\ell \leq k$ of the $D_i's$ has star cover number ℓ. Thus, if B is dominated by A, $s(B) \leq s(A)$, and $s(T(B)) \leq s(T(A))$. And if A is dominated by C, $s(A) \leq s(C)$, and $s(T(A)) \leq s(T(C))$. These facts are used in the next lemma.

LEMMA 4.3. *Let $n \geq 3$, $2 \leq k \leq n-2$, and $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T preserves the set of matrices of star cover number k and the set of matrices of star cover number $k+1$, then T is strongly preserves $\mathcal{S}_{k+1}$, and hence is bijective on the set of base elements.*

Proof. Suppose that T preserves the set of matrices of star cover number k and the set of matrices of star cover number $k+1$, and suppose that for some A with $s(A) < k$, $s(T(A)) \geq k+1$. Then there exists B dominating A whose star cover number is k. Then $k = s(B) \geq s(T(A)) \geq k+1$, a contradiction. Now suppose that there is some matrix A whose star cover number is at least $k+1$ and whose image is at most k. Then there is some matrix C that is dominated by A, whose star cover number is $k+1$. Then $k+1 = s(T(C)) \leq s(T(A)) \leq k$, a contradiction. It follows that T strongly preserves $\mathcal{S}_{k+1}$. □

THEOREM 4.4. *Let $n \geq 3$ and $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T strongly preserves star cover number k for $2 \leq k \leq n-1$, or if T preserves star cover numbers k and $k+1$ for $2 \leq k \leq n-2$, then T is a (P, P^t)-operator.*

Proof. By Lemmas 4.2 and 4.3, T is bijective on the set of base elements. Thus, by virtue of Lemma 2.4 we only need show that T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star.

Suppose that there is some pair of base elements not dominated by a double star whose image is dominated by a double star.

Without loss of generality, either the two base elements are $E_{1,1}$ and $E_{2,2}$, $E_{1,1}$ and $D_{2,3}$, or $D_{1,2}$ and $D_{3,4}$.

In the first case, $s(E_{1,1} + E_{2,2}) = 2$ and $s(T(E_{1,1} + E_{2,2})) = 1$. Then $s(E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1}) = k+1$, whereas $s(T(E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1})) \leq k$, a contradiction.

In the second case, $s(E_{1,1} + D_{2,3}) = 2$ and $s(T(E_{1,1} + D_{2,3})) = 1$. Here, $s(E_{1,1} +$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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$D_{2,3} + E_{4,4} + \cdots + E_{k+1,k+1}) = k$ but since T maps base elements to base elements, (by Lemma 4.2), $s(T(E_{4,4}+\cdots+E_{k+1,k+1})) \leq k-2$ and $s(T(E_{1,1}+D_{2,3})) = 1$. Thus, $s(T(E_{1,1}+D_{2,3}+E_{4,4}+\cdots+E_{k+1,k+1})) = s(T(E_{1,1}+D_{2,3})+T(E_{4,4}+\cdots+E_{k+1,k+1}))$ has star cover number at most $1 + (k-2) = k-1$, a contradiction.

In the third case, $s(D_{1,2} + D_{3,4}) = 2$ and $s(T(D_{1,2} + D_{3,4})) = 1$. Here, $s(D_{1,2} + D_{3,4} + E_{5,5} + \cdots + E_{k+2,k+2}) = k$ but since T maps base elements to base elements, (by Lemma 4.2), $s(T(E_{5,5} + \cdots + E_{k+2,k+2})) \leq k-2$ and $s(T(D_{1,2} + D_{3,4})) = 1$. Thus, $s(T(D_{1,2} + D_{3,4} + E_{5,5} + \cdots + E_{k+2,k+2})) = s(T(D_{1,2} + D_{3,4}) + T(E_{5,5} + \cdots + E_{k+2,k+2})) \leq 1 + (k-2) = k-1$, a contradiction.

Thus, T maps pairs of base elements not dominated by a double star to pairs of base elements not dominated by a double star. and since T is bijective, T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star. ◻

Lemma 4.5. *Let $n \geq 3$. If T preserves star cover numbers 1 and k for some k, $2 \leq k \leq n-1$, then T preserves S_k strongly.*

Proof. Let $A \in S_k$ so that $s(A) \geq k$. Suppose that $T(A) \notin S_k$ so that $s(T(A)) \leq k-1$. Then there is some matrix $B \sqsubseteq A$ such that $s(B) = k$, but then $s(T(B)) = k$, and hence, $k = s(T(B)) \leq s(T(A)) \leq k-1$, a contradiction. Thus, T preserves S_k.

Now let $A \notin S_k$. Then $s(A) \leq k-1$, and hence, A is dominated by at most $k-1$ full double stars. Since T preserves star cover number 1, $s(T(A)) \leq k-1$. That is $T(A) \notin S_k$. We now have that T strongly preserves S_k. ◻

Corollary 4.6. *Let $n \geq 3$. If T preserves star cover numbers 1 and k for some k, $2 \leq k \leq n-1$, then T is a (P, P^t)-operator.*

Proof. By Lemma 4.5, T strongly preserves S_k and $2 \leq k \leq n-1$, so by Lemma 2.3 T is bijective, since by Lemma 4.1 S_k separates base elements. By Lemma 2.4, T is a (P, P^t)-operator. ◻

Theorem 4.7. *If $n \geq 3$ and $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ is a linear operator, then the following are equivalent:*

1. *T preserves the star cover number;*
2. *T strongly preserves star cover number k for some k, $2 \leq k \leq n-1$;*
3. *T preserves star cover numbers k and $k+1$ for some $2 \leq k \leq n-2$;*
4. *T preserves star cover numbers 1 and k for some k, $2 \leq k \leq n-1$;*
5. *T is a (P, P^t)-operator.*

Proof. That condition 5) implies conditions 1)-4) is obvious. Clearly 1) implies 3), and 2), 3) and 4) imply 5) by Theorem 4.4 and Corollary 4.6. ◻

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A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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The following example shows that the restriction in condition 4 of the theorem is necessary.

EXAMPLE 4.8. Consider $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ defined by $T(E_{i,i}) = S_i$, $i = 1, \ldots, n$ and $T(D_{i,j}) = D_{i,j}$, $1 \leq i < j \leq n$. Then, the image of any matrix of star cover number 1 that dominates a diagonal cell, is a full double star centered on that diagonal cell. The image of any matrix of star cover number 1 that does not dominate a diagonal cell, is that matrix. So T preserves star cover number 1. Now, if A has star cover number n, then A dominates the identity I_n. Thus, the image of any matrix of star cover number n is J, the sum of the n full star matrices. Thus, T preserves star cover number n. However, clearly, T is not a (P, P^t)-operator.

We can however address the linear preservers of star cover numbers 1 and n:

LEMMA 4.9. *Let $n \geq 3$. If T preserves star cover number 1 and $T(J) \sqsupseteq I_n$, then there exists a permutation matrix P and a set of star matrices in $\mathcal{S}_n(\mathbb{S})$, $\{\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_n\}$ with $\hat{S}_i \sqsubseteq S_i$ such that $T(X) = P^t(X + \sum_{i=1}^n x_{i,i}\hat{S}_i)P$ for all $X \in \mathcal{S}_n(\mathbb{S})$.*

Proof. Suppose that $T(D_{i,j}) \circ I_n \neq O$. Then, without loss of generality, by permuting, we may assume that $T(D_{1,2}) \sqsupseteq E_{1,1}$. Now, since $D_{1,2} \sqsubseteq S_1$ and $D_{1,2} \sqsubseteq S_2$, $T(S_1) \sqsupseteq E_{1,1}$ and $T(S_2) \sqsupseteq E_{1,1}$. But the only full double star that dominates $E_{1,1}$ is S_1. Thus, $T(S_1 + S_2) \sqsubseteq S_1$. Consider $T(J) = T(S_1 + S_2 + S_3 + \cdots + S_n) = T(S_1 + S_2) + T(S_3) + \cdots + T(S_n)$. Since $T(S_1 + S_2)$ is dominated by one double star, and $T(S_i)$, $i = 3, \ldots, n$ is dominated by one full double star, the star cover number of $T(J)$ must be at most $n - 1$, a contradiction since any matrix that dominates I_n has star cover number n. Thus, $T(D_{i,j}) \circ I_n = O$ for any digon $D_{i,j}$.

Since $T(J) \sqsupseteq I_n$, there must be n base elements $D_1, D_2, \ldots, D_n$ such that $T(D_1 + D_2 + \cdots + D_n) \sqsupseteq I_n$, and no digon can be one of these base elements, thus D_i is a diagonal cell. Since T preserves star cover number 1, the image of any diagonal cell can dominate at most one diagonal cell. It follows that T induces a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $T(E_{i,i}) \sqsupseteq E_{\sigma(i),\sigma(i)}$. Further we have that $T(S_i) \sqsubseteq S_{\sigma(i)}$ since T preserves star cover number 1.

Consider $T(D_{i,j})$. Since $D_{i,j} \sqsubseteq S_i$ and $D_{i,j} \sqsubseteq S_j$, $T(D_{i,j})$ is dominated by the intersection of $S_{\sigma(i)}$ and $S_{\sigma(j)}$. Since $T(D_{i,j})$ cannot be zero, we have that $T(D_{i,j}) = D_{\sigma(i),\sigma(j)}$.

Let $\hat{S}_{\sigma(i)} = T(E_{i,i})$, and let P be the permutation matrix corresponding to the permutation σ. Then, for any $X \in \mathcal{S}_n(\mathbb{S})$ it follows that $T(X) = P^t(X + \sum_{i=1}^n x_{i,i}\hat{S}_i)P$. ◻

COROLLARY 4.10. *Let $n \geq 3$. If T preserves star cover numbers 1 and n,*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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then there exists a permutation matrix P *and a set of star matrices in* $\mathcal{S}_n(\mathbb{S})$, $\{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_n\}$ *with* $\hat{S}_i \sqsubseteq S_i$ *such that* $T(X) = P^t(X + \sum_{i=1}^n x_{i,i}\hat{S}_i)P$ *for all* $X \in \mathcal{S}_n(\mathbb{S})$.

Proof. If T preserves star cover number n, since any matrix of star cover number n dominates the identity, we have that $T(J) \sqsupseteq I_n$. The corollary now follows from Lemma 4.9. ☐

THEOREM 4.11. *If* $n \geq 3$ *and* $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ *is a linear operator, then the following are equivalent:*

1. *T preserves star cover number 1 and* $T(J) \sqsupseteq I_n$*;*
2. *T preserves star cover numbers 1 and n;*
3. *There exist a permutation matrix P and a set of star matrices in* $\mathcal{S}_n(\mathbb{B})$, $\{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_n\}$ *with* $\hat{S}_i \sqsubseteq S_i$ *such that* $T(X) = P^t(X + \sum_{i=1}^n x_{i,i}\hat{S}_i)P$ *for all* $X \in \mathcal{S}_n(\mathbb{B})$.

Proof. That 1) or 2) imply 3) follows from Lemma 4.9 and Corollary 4.10.

Clearly, for any permutation matrix P, the star cover number of P^tXP is the same as the star cover number of X. Further, if $x_{i,i} \neq 0$, then any star cover of X must include the star S_i, so that the star cover number of $X + \hat{S}_i$ is the star cover number of X. Thus, $s(X) = s(X + \sum_{i=1}^n x_{i,i}\hat{S}_i)$. Note also that $P^t(J + \sum_{i=1}^n x_{i,i}\hat{S}_i)P = J$. We have now shown that 3) implies both 1) and 2). ☐

5. Summary. Let $\mathbb{S}$ be an arbitrary antinegative, commutative semiring with no zero divisors. By virtue of Theorem 2.2, the generalization of Theorems 3.3 and 4.7 can be summarized as:

THEOREM 5.1. *Let* $T : \mathcal{S}_n(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$. *The following are equivalent:*

1. *T preserves term rank;*
2. *T preserves term rank 1 and term rank k for some* $k, 2 \leq k \leq n-1$*;*
3. *T is bijective and preserves term rank k for any* $2 \leq k \leq n$*;*
4. *T preserves term rank 2 and* $T(J) = J$*;*
5. *T preserves term ranks 2 and k,* $3 \leq k \leq n$*;*
6. *T preserves term ranks 3 and k,* $4 \leq k \leq n$*;*
7. *T preserves term ranks k and $k+1$, for* $3 \leq k \leq n-1$, *k odd;*
8. *T strongly preserves term rank k, for* $2 \leq k \leq n$*;*
9. *T preserves the star cover number;*
10. *T strongly preserves star cover number k for some k,* $2 \leq k \leq n-1$*;*
11. *T preserves star cover number 1 and* $T(J) = J$*;*
12. *T preserves star cover numbers 1 and 2;*
13. *T is a* (P, P^t, B)*-operator.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 549-564, August 2016

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The generalization of Theorem 4.11 is:

THEOREM 5.2. *If* $n \geq 3$ *and* $T : \mathcal{S}_n(\mathbb{S}) \to \mathcal{S}_n(\mathbb{S})$ *is a linear operator, then the following are equivalent:*

1. *T preserves star cover number 1 and $T(J) \sqsupseteq I_n$;*
2. *T preserves star cover numbers 1 and n;*
3. *There exists a permutation matrix P, a matrix $B \in \mathcal{S}_n(\mathbb{S})$ with no zero entries, and a set of star matrices in $\mathcal{S}_n^{(0)}(\mathbb{S})$, $\{\hat{\hat{S}}_1, \hat{\hat{S}}_2, \ldots, \hat{\hat{S}}_n\}$ with $\hat{\hat{S}}_i \sqsubseteq S_i$ such that $T(X) = P^t((X \circ B) + \sum_{i=1}^n x_{i,i}\hat{\hat{S}}_i)P$ for all $X \in \mathcal{S}_n(\mathbb{S})$.*

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