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POLYNOMIAL RECONSTRUCTION OF SIGNED GRAPHS WHOSE LEAST EIGENVALUE IS CLOSE TO -2*

SLOBODAN K. SIMIĆ† AND ZORAN STANIƇ‡

Abstract. The polynomial reconstruction problem for simple graphs has been considered in the literature for more than forty years and is not yet resolved except for some special classes of graphs. Recently, the same problem has been put forward for signed graphs. Here, the reconstruction of the characteristic polynomial of signed graphs whose vertex-deleted subgraphs have least eigenvalue greater than -2 is considered.

Key words. Signed graph, Characteristic polynomial, Eigenvalues, Signed line graph, Exceptional graph, Numerical computation.

AMS subject classifications. 05C50, 65F15.

1. Introduction. Given a simple graph $G = (V(G), E(G))$ of order $n = |V(G)|$ and size $m = |E(G)|$, let $\sigma : E(G) \to \{1, -1\}$ be a mapping defined on the edge set of G. Then, $\dot{G} = (G, \sigma)$ is a *signed graph* (or *sigraph*), G is its *underlying graph* and σ is its *sign function* (or *signature*).

Signed graphs are usually viewed as weighted graphs whose edge weights are equal to ± 1. If so, we can say that the vertex set $V(\dot{G})$ of a signed graph $\dot{G}$ coincide with the vertex set of its underlying graph, while the edge set $E(\dot{G})$ is divided into two disjoint subsets E^+ and E^- (defined by σ) that contain positive and negative edges, respectively. If two vertices $u, v \in V(\dot{G})$ are joined by an edge, let $a_{uv} = \pm 1$ depending on whether uv belongs to E^+ or E^-; otherwise, $a_{uv} = 0$. The adjacency matrix $A(\dot{G})$ of $\dot{G}$ is then defined by $A(\dot{G}) = (a_{uv})$. Its characteristic polynomial

$$\Phi_{\dot{G}}(x) = \det(xI - A(\dot{G})) = x^n + a_{n-1}(\dot{G})x^{n-1} + \cdots + a_1(\dot{G})x + a_0(\dot{G})$$

is also called the *characteristic polynomial* of $\dot{G}$. The eigenvalues of $A(\dot{G})$ are real and

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comprise the spectrum of $\dot{G}$. We denote them by

$$\lambda_1 \ (= \lambda_1(\dot{G})),\ \lambda_2 \ (= \lambda_2(\dot{G})),\ \ldots,\ \lambda_n \ (= \lambda_n(\dot{G})),$$

and also assume that $\lambda_i \geq \lambda_j$ whenever $1 \leq i < j \leq n$. In particular, the least eigenvalue will be denoted by λ $(= \lambda(G))$.

For $U \subset V(G)$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge joining a vertex in U with a vertex in $V(G) \setminus U$. Then the signed graph $\dot{G}^U$ is said to be *switching equivalent* to $\dot{G}$. Clearly, the switching equivalence is an equivalence relation that preserves the eigenvalues [16]. Needles to add, all switching equivalent graphs are cospectral (share the same spectrum).

Throughout the paper, we will follow the standard graph-theoretic notation, and adapt it to be used for signed graphs. For example, a signed cycle with n vertices is denoted by $\dot{C}_n$. A cycle in a signed graph is said to be *balanced* (or *positive*) if it contains an even number of negative edges; otherwise it is *unbalanced* (or *negative*). Hence, the sign of a cycle $\dot{C}$ is the product of signs of its edges, i.e., it is equal to $\prod_{e \in E(\dot{C})} \sigma(e)$. A signed graph is said to be *balanced* if all its cycles are balanced; otherwise, it is *unbalanced.* It is easy to see that two switching equivalent graphs are either both balanced, or both unbalanced. In addition, a balanced signed graph shares the spectrum with its underlying graph.

Most of the standard concepts defined for graphs are directly extended to signed graphs. For example, this refers to connectedness and bipartiteness. If considering subgraphs of signed graphs, then their signed functions are the restrictions of the original ones to the corresponding edge subsets. Therefore, if H is a subgraph of G (not necessarily an induced one), then $\dot{H}$ stands for the resulting signed subgraph. If v is a vertex of G (or $\dot{G}$) then we write $G - v$ (resp. $\dot{G} - v$) for the corresponding vertex-deleted subgraph. Most of the standard graph invariants coincide for G and $\dot{G}$. For example, $\deg(v)$ is a degree of a vertex v (in G, or in $\dot{G}$). Let

$$\mathcal{P}(\dot{G}) \ = \ \left\{\Phi_{\dot{G}_1}, \Phi_{\dot{G}_2}, \ldots, \Phi_{\dot{G}_n}\right\}, \tag{1.1}$$

where $\dot{G}_i = \dot{G} - v_i$ $(1 \leq i \leq n)$, be the collection of characteristic polynomials of vertex-deleted subgraphs of $\dot{G}$. $\mathcal{P}(\dot{G})$ is also called the *polynomial deck* of $\dot{G}$. Here we consider the following problem.

Problem 1. Given two signed graphs $\dot{G}$ and $\dot{H}$ on at least three vertices, is it true that

$$\mathcal{P}(\dot{G}) = \mathcal{P}(\dot{H}) \Rightarrow \Phi_{\dot{G}} = \Phi_{\dot{H}}, \tag{1.2}$$

i.e., is the characteristic polynomial of a signed graph determined uniquely by its polynomial deck?

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If (1.2) does not hold for two signed graphs then $(\dot{G}, \dot{H})$ will be called a *counterexample pair*.

The same problem for simple graphs is called the *polynomial reconstruction problem* and it was posed by D. Cvetković in 1973. Although there are many positive particular results (see [3, 4, 6, 7, 12, 13, 15, 14]), the original problem in general case is still unsolved. So far, if $n > 2$, no counterexample exists in the literature.

Since

$$\Phi'_{\dot{G}}(x) = \sum_{i=1}^{n} \Phi_{\dot{G}_i}(x) \tag{1.3}$$

(see, for example, [5, p. 60]), we can readily determine the characteristic polynomial of a (signed) graph from its polynomial deck except for the constant term. Therefore, we can also deduce for $\dot{G}$ its order, size, degree sequences, etc. For more details see [16].

If we can deduce from the polynomial deck any root of the characteristic polynomial of $\dot{G}$ or its value at some point, then the reconstruction is unique. In particular, if at least one characteristic polynomial in the deck has repeated roots, the reconstruction is unique (by the Interlacing theorem – see [5, Theorem 0.10]). Since $\Phi_{\dot{G}}$ and $\Phi_{\dot{H}}$ differ only in the constant term, it follows that $\dot{G}$ and $\dot{H}$ have no common eigenvalues.

For Problem 1 (so with signed graphs), the counterexample pairs $(\dot{G}, \dot{H})$ were found in [16], with $\dot{G}$ and $\dot{H}$ being cycles of the same order but different signatures. This fact prompted us to pursue further counterexample pairs (if any) whose underlying graphs are not isomorphic. The class of signed graphs to be examined consists of signed graphs with least eigenvalue close to -2. More precisely, we will consider signed graphs whose vertex-deleted subgraphs have the least eigenvalue greater than -2 (in [15], we considered (simple) graphs whose least eigenvalues is greater than or equal to -2). The crucial reason for this restriction is that, in spite of big progress made recently in the domain of signed graphs, some important results such as computational ones are still missing (see also [8, 9]).

The rest of the paper is organized as follows. In Section 2, following [16] we give some basic observations, and also add some results to be used further on. In Section 3, we prove our main result – Theorem 3.1. The Appendix contains the description of our computational results.

2. Preliminaries. We first observe that at least one of signed graphs $\dot{G}$ and $\dot{H}$ from a counterexample pair is connected. Otherwise, their largest eigenvalues coincide, i.e., they have a common eigenvalue, a contradiction. For the polynomial

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reconstruction of disconnected signed graphs, the following theorem (see [7, 13], and [16] for signed graphs) is relevant.

THEOREM 2.1. *Let $\dot{H}$ be a disconnected signed graph whose components are $\dot{H}_1, \dot{H}_2, \dots, \dot{H}_p$. Then $\Phi_{\dot{H}}$ is reconstructible if $p > 2$, or if $p = 2$ and $n(\dot{H}_1) \neq n(\dot{H}_2)$. Otherwise, for $p = 2$ and $n(\dot{H}_1) = n(\dot{H}_2) = k$, let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the eigenvalues of $\dot{H}_1$, while $\mu_1 > \mu_2 > \cdots > \mu_{k-1}$ the eigenvalues of $\dot{H}_2 - u$, where u is an arbitrary vertex of $\dot{H}_2$. Then, the polynomial reconstruction is unique whenever the following inequalities do not hold:*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{k-1} > \lambda_k. \tag{2.1}$$

Here we give a simple observation based on the previous result. Although it will not be used in the sequel, it contains a useful property of a disconnected signed graph $\dot{H}$.

COROLLARY 2.2. *If the inequalities* (2.1) *hold for some disconnected signed graph $\dot{H}$ with two components $\dot{H}_1$ and $\dot{H}_2$, then the spectra of both components $\dot{H}_1$ and $\dot{H}_2$ (and consequently the spectrum of $\dot{H}$) are reconstructible from the polynomial deck $\mathcal{P}(\dot{H})$.*

Proof. By the previous theorem, the spectrum of one component consists of the eigenvalues $\lambda_i(\dot{H} - u)$ $(i = 1, 3, \dots, n-1)$, where u is a fixed vertex of $\dot{H}$. If there is $v \in V(\dot{H})$ such that $\lambda_i(\dot{H} - u) \neq \lambda_i(\dot{H} - v)$ holds for at least one i $(i = 1, 3, \dots, n-1)$, then the spectrum of the other component consists of the eigenvalues $\lambda_i(\dot{H} - v)$ $(i = 1, 3, \dots, n-1)$. Otherwise, $\dot{H}_1$ and $\dot{H}_2$ share the same spectrum. ◻

We immediately obtain the next result.

COROLLARY 2.3. *Let $\dot{H}$ be a disconnected signed graph whose components are $\dot{H}_1$ and $\dot{H}_2$. If all roots of polynomials in the polynomial deck $\mathcal{P}(\dot{H})$ are greater than $c \in \mathbb{R}$, then $\lambda(\dot{H}_1), \lambda(\dot{H}_2) > c$.*

Proof. Assume to the contrary, and say that $\lambda(\dot{H}_1) \leq c$. Also let $v \in V(\dot{H}_2)$. Then $\lambda(\dot{H} - v) = \min\{\lambda(\dot{H}_1), \lambda(\dot{H}_2 - v)\} \leq \lambda(\dot{H}_1) \leq c$, a contradiction. ◻

There is another consequence of the last theorem.

COROLLARY 2.4. *Let $(\dot{G}, \dot{H})$ be a counterexample pair for Problem* 1. *If $\dot{H}$ is disconnected, then $\Phi_{\dot{H}}(x) > \Phi_{\dot{G}}(x)$ holds for any $x \in \mathbb{R}$.*

Proof. Since $\dot{H}$ is disconnected, there exists a vertex v of $\dot{G}$ such that $\lambda_1(G - v) = \lambda_1(\dot{H})$, which yields $\lambda_1(\dot{G}) > \lambda_1(\dot{H})$ (by the interlacing), and the rest immediately follows from the fact that $\Phi_{\dot{G}}$ and $\Phi_{\dot{H}}$ differ in a constant term. ◻

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More situations in which the polynomial reconstruction of signed graphs is unique (including those that concern disconnected signed graphs or signed graphs with pendant vertices) can be found in our previous paper [16]. We now focus our attention on signed graphs that are relevant to our investigations.

It is well known that within simple graphs the generalized line graphs (with the line graphs included) and the exceptional graphs are the only connected graphs whose least eigenvalue is not less than -2. For the definitions of these classes of graphs the reader is referred to [8]. Accordingly, root graphs of line graphs are simple graphs while root graphs of proper generalized line graphs (i.e., those which are not line graphs) can be interpreted as some special multigraphs (with petals as double edges). On the other hand, exceptional graphs are connected graphs with least eigenvalue not less than -2 but not generalized line graphs. Recall also that generalized line graphs can be represented in the root system D_n (for some $n \geq 4$), while exceptional graphs are representable in the root system E_8 (or, occasionally in subsystems E_6 and E_7). For more details about root systems see [8].

In what follows, we will give more relevant details about the signed counterpart of the above classes of graphs along with results needed in the next sections. To define *signed line graphs* we first introduce the following terminology and notation.

(a) Bi-directed signed graph. It is an ordered pair $\dot{G}_\eta = (\dot{G}, \eta)$, where

(2.2) $$\eta : V(G) \times E(G) \to \{+1, -1, 0\}$$

is an orientation satisfying the following three conditions:

- (i) $\eta(u, vw) = 0$ whenever $u \notin \{v, w\}$;
- (ii) $\eta(v, vw) = +1$ (resp. -1) if an arrow at v is going into (resp. out of) v (cf. Fig. 1);
- (iii) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$.

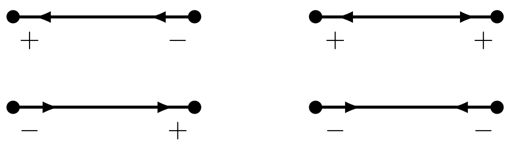

Figure 1: Bi-directed edges.

(b) Incidence matrix. It is an $n \times m$ matrix associated to $\dot{G}_\eta$ (denoted by $B(\dot{G}_\eta)$ or B_η) with $b_{ij} = \eta(v_i, e_j)$ for each $v_i \in V(G)$ and $e_j \in E(G)$. Usually, only $\dot{G} = (G, \sigma)$ is given, and then η is determined as explained in (a). Any row of the incidence matrix corresponding to vertex v_i contains $\deg(v_i)$ non-zero entries, each equal to $+1$

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or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1.

(c) Signed line graph. Based on above facts, even in the case that multiple edges exist in $\dot{G}$, we easily obtain that

$$B_\eta^\top B_\eta = 2I + A(L(\dot{G}_\eta)) \tag{2.3}$$

holds, where $L(\dot{G}_\eta)$ is in fact the signed line graph of a bi-directed signed graph $\dot{G}_\eta$. This definition can also be found in [2]. It is noteworthy to say here that $L(\dot{G}_\eta)$ has $L(G)$ as its underlying graph, while the sign of the edge $ef \in L(\dot{G})$ $(e, f \in E(G))$ in the resulting signed graph is equal to $\sigma(ef) = \eta(v, e)\eta(v, f)$ if v is the unique common vertex of the edges e and f in G; if the edges e and f have two vertices in common (which may occur if G is a multigraph) then the signs are summed up leading to either a zero sign edge (so no edge), or two parallel edges. Of course, this is a matrix combinatorial definition of the line graph of a signed graph (tailored for spectral graph theory). Needless to add, generalized line graphs (and line graphs) are very naturally included in the class of signed line graphs. Observe, signed line graphs are just signed graphs representable in root system D_n for some $n \geq 4$.

(d) Exceptional signed graphs. These graphs are connected signed graphs as well as signed line graphs whose least eigenvalue is not less than -2. They are also represented (like their unsigned counterparts) by the subset of the root system E_8, but not by a subset of the root system D_n, for any $n \geq 4$. It turns out (see [10]) that any minimal exceptional signed graph is switching equivalent to one of 32 exceptional signed graphs on 6 vertices, or to one of 233 exceptional signed graphs on 7 vertices, or to one of 1242 exceptional signed graphs on 8 vertices. We denote minimal exceptional signed graphs on n $(6 \leq n \leq 8)$ vertices by $\mathcal{E}_n$.

The following result is taken from [10] and restated here in somewhat nicer form which reflects the matrix definition of signed line graphs adopted in the above. Here we put focus on root (signed) graphs and the property of being balanced; exceptional signed graphs are also included. Signed graphs from items (i) and (ii) below can be found in [1].

THEOREM 2.5. *Let $\dot{G}$ be a connected signed graph on n vertices with $\lambda(\dot{G}) > -2$. Then $\dot{G}$ belongs to one of the following classes:*

(i) $\mathcal{A}_1$ *– signed graphs that are switching equivalent to the line graph of a (signed) tree;*

(ii) $\mathcal{A}_2$ *– signed graphs that are switching equivalent to the line graph of an unbalanced unicyclic signed graph of girth at least* 2 *(so* 2*-cycles, i.e., double edges are allowed);*

(iii) $\mathcal{A}_3$ *- minimal exceptional signed graphs, that is those belonging to the set*

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$\bigcup_{n=6}^{8} \mathcal{E}_n$.

The converse is also true.

In the next theorem we describe the signed graphs whose vertex-deleted subgraphs have the property stated in the above theorem.

THEOREM 2.6. *Let $\dot{G}$ be a connected signed graph with $\lambda(\dot{G}-v) > -2$ for each $v \in V(\dot{G})$. Then $\dot{G}$ belongs to one of the classes $\mathcal{A}_1 - \mathcal{A}_3$ (from Theorem 2.5) or to one of the following classes:*

- *(i) $\mathcal{B}_1$ – signed graphs that are switching equivalent to a balanced cycle (of length at least three);*
- *(ii) $\mathcal{B}_2$ – signed graphs that are switching equivalent to the signed line graph of a signed graph $L(\dot{B})$, where $\dot{B}$ is a bicyclic graph consisting of two unbalanced cycles (of lengths at least two) and a path joining them;*
- *(iii) $\mathcal{B}_3$ – signed graphs which are one-vertex extensions of signed graphs from the class $\mathcal{A}_3$.*

The converse now holds only for (i) and (ii).

If in (ii) of the above theorem, the path under consideration is of length zero, then the corresponding (signed) graph is called a *double unbalanced infinity graph*; otherwise, if its length is non-zero it is called a *double unbalanced dumbbell*. Prefix double is used to emphasize that both cycles are unbalanced.

Proof. Clearly, all signed graphs from Theorem 2.5 are directly included in this theorem (by interlacing). So, we next consider signed graphs with least eigenvalue not exceeding -2.

Signed line graphs with least eigenvalue equal to -2 (see (i) and (ii)) must be minimal with respect to this property. Therefore, they can be obtained as one-vertex extensions of the signed graphs belonging to classes $\mathcal{A}_1$ and $\mathcal{A}_2$, followed by discarding their superfluous vertices (i.e., vertices corresponding in root graphs to light edges; for more details see [1]).

Finally, signed graphs from (iii) are related to exceptional graphs. Their least eigenvalue is either equal to -2 (then they are minimal signed exceptional graphs with least eigenvalue equal to -2), or less than -2 (then they are minimal forbidden subgraphs for the property $\lambda \geq -2$). In the latter case, all their vertex deleted subgraphs can be either all signed line graphs (so such minimal signed graphs have at most six vertices, see [17, Corollary 3.2]), or at least one of these (signed) subgraphs has an exceptional component (then such minimal signed graphs have at most ten vertices, see [17, Theorem 4.1]; but more precisely, at most nine vertices if we require that their vertex-deleted subgraphs have the least eigenvalue strictly greater than

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-2). This completes the proof. ◻

With notation from Theorem 2.5 we next have the following lemma.

LEMMA 2.7. *If $\dot{G}$ is a signed graph on n vertices, then:*

(i) $\Phi_{\dot{G}}(-2) = (-1)^n(n+1)$, *if* $\dot{G} \in \mathcal{A}_1$,
(ii) $\Phi_{\dot{G}}(-2) = (-1)^n 4$, *if* $\dot{G} \in \mathcal{A}_2$,
(iii) $\Phi_{\dot{G}}(-2) = (-1)^n(9-n)$, *if* $\dot{G} \in \mathcal{A}_3$.

Proof. First, (i) and (ii) are proved in [10] (cf. proof of Theorem 6), and as well in [2]. On the other hand, here we resolve (iii) by a computer search (see part (a) in the Appendix). ◻

Let

$$\mu(\dot{G}) = |\Phi_{\dot{G}}(-2)| \text{ and } \mu^*(\dot{G}) = \min_{v \in V(\dot{G})} |\Phi_{\dot{G}-v}(-2)|.$$

If $n \geq 4$ is the order of $\dot{G}$, then from Lemma 2.7 we also have

$$\mu(\dot{G}) = \begin{cases} n+1, & \text{if } \dot{G} \in \mathcal{A}_1, \\ 4, & \text{if } \dot{G} \in \mathcal{A}_2, \\ 3, & \text{if } \dot{G} \in \mathcal{E}_6, \\ 2, & \text{if } \dot{G} \in \mathcal{E}_7, \\ 1, & \text{if } \dot{G} \in \mathcal{E}_8. \end{cases} \tag{2.4}$$

We next examine some features related to polynomial decks of signed graphs from classes $\mathcal{A}_1 - \mathcal{A}_3$ and $\mathcal{B}_2 - \mathcal{B}_3$.

LEMMA 2.8. *If $\dot{G}$ is a signed graph on n vertices, where $n \geq 4$, then:*

$$\mu^*(\dot{G}) = \begin{cases} n, & \text{if } \dot{G} \in \mathcal{A}_1, \\ 4, & \text{if } \dot{G} \in \mathcal{A}_2, \\ 4 \text{ or } 6, & \text{if } \dot{G} \in \mathcal{E}_6, \\ 3, & \text{if } \dot{G} \in \mathcal{E}_7, \\ 2, & \text{if } \dot{G} \in \mathcal{E}_8. \end{cases} \tag{2.5}$$

Proof. (i) Here, $\dot{G} = L(\dot{T})$, where $\dot{T}$ is a (signed) tree. Then $\dot{G} - v = L(\dot{T} - e)$, where e is the edge corresponding to vertex v. Since e is a bridge in $\dot{T}$, assume that the two subtrees of $\dot{T} - e$ are of sizes p and q, where $p + q + 1 = n$. Then by (2.4), $|\Phi_{\dot{G}-v}(-2)| = (p+1)(q+1)$. If p or q is equal to 0 (so if e is a pendant edge) then $|\Phi_{\dot{G}-v}(-2)| = n$; otherwise if p and q are non-zero, then $(p+1)(q+1) = pq + (p+q+1) \geq (p+q-1) + (p+q+1) = 2(p+q) = 2(n-1)$, and we are done.

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(ii) Here, $\dot{G}=L(\dot{U})$, where $\dot{U}$ is unbalanced and unicyclic. Then $\dot{G}-v=L(\dot{U}-e)$, where e is the edge corresponding to vertex v. By (2.4), we have: if e is a pendant edge in $\dot{U}$ then $\dot{G}-v\in\mathcal{A}_2$, and so $|\Phi_{\dot{G}-v}(-2)|=4$; if e is an edge belonging to a (unique) cycle of $\dot{U}$ then $\dot{G}-v\in\mathcal{A}_1$ and so $|\Phi_{\dot{G}-v}(-2)|=n\geq 5$; otherwise, e is a bridge in $\dot{U}$, so it consists of two component one belonging to $\mathcal{A}_1$ (having $k>0$ edges) and the other belonging to $\mathcal{A}_2$, and thus $|\Phi_{\dot{G}-v}(-2)|=4(k+1)>4$, and we are done.

(iii) Assume first that $\dot{G}\in\mathcal{E}_6$. Now, $\dot{G}-v$ is a signed line graph for any $v\in V(\dot{G})$. If $\dot{G}-v$ is disconnected it is easy to check that $|\Phi_{\dot{G}-v}(-2)|>6$. Otherwise, if $\dot{G}-v$ is connected we are done by (2.4).

Assume now that $\dot{G}\in\mathcal{E}_n$, with $n\in\{7,8\}$. If $\dot{G}-v$ is a signed line graph for any $v\in V(\dot{G})$, then $\dot{G}$ is a signed line graph as well, a contradiction. Indeed, signed line graphs are characterized by a collection of minimal forbidden (induced) subgraphs, and such graphs have at most six vertices [17]. At least one of these subgraphs has an exceptional graph component on less than $n-1$ vertices, or is an exceptional on $n-1$ vertices. In the former case, there exists another vertex in $\dot{G}$, say w, adjacent to v but not being in the component in question such that $\dot{G}-w$ has a larger exceptional component. Therefore, for at least one v, $\dot{G}-v$ is exceptional on $n-1$ vertices. Hence, the (lower) bounds that were claimed are attained. Moreover, they are the best possible since they cannot be made smaller in neither the connected candidates nor the disconnected ones (This easily follows by inspecting all possibilities and using (2.4).), and the proof is complete. ∎

3. Main result. Throughout this section $\dot{G}$ denotes a signed graph on n vertices whose vertex-deleted subgraphs have the least eigenvalue greater than -2. $\dot{H}$ denotes a signed graph (if any) which acts as a counterexample to our reconstruction problem (Problem 1 from Section 1). $(\dot{G},\dot{H})$ is referred to as a counterexample pair.

Since at least one of the signed graphs $\dot{G}$ and $\dot{H}$ is connected, without loss in generality, let $\dot{G}$ be connected. We also use the following correspondence between the vertices of $\dot{G}$ and $\dot{H}$: if $v\in V(\dot{G})$ and $v'\in V(\dot{H})$ then these two vertices will be called *partners* if $\Phi_{\dot{G}-v}(x)=\Phi_{\dot{H}-v'}(x)$. Note that the degrees of partners are the same if $n>2$.

Since switching equivalent signed graphs share the same polynomial deck [16], the polynomial reconstruction of a signed graph is unique if and only if the polynomial reconstruction of any switching equivalent signed graph is unique.

Our main result reads as follows.

THEOREM 3.1. *Let $\dot{G}$ be a signed graph of order $n>2$ whose vertex-deleted subgraphs have the least eigenvalue greater than -2. Then the polynomial reconstruction*

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is unique, unless the underlying graph of $\dot{G}$ is a cycle. If $(\dot{G}, \dot{H})$ is a counterexample pair, then both underlying graphs are cycles (of the same order) one being balanced while the other unbalanced.

We now give some preparatory facts on putative counterexample pairs which will enable us to give a more organized proof.

Fact 1. For signed graphs on at most six vertices, the theorem has been verified by examining all candidates $(\dot{G}, \dot{H})$ by a computer search (see part (b) in the Appendix). Moreover, there are no further counterexamples even if we consider all signed graphs on up to six vertices.

Fact 2. In view of results from Section 2, we still have to examine all instances of $\dot{G}$ from the classes $\mathcal{A}_1 - \mathcal{A}_3$ and $\mathcal{B}_1 - \mathcal{B}_3$. In addition, some reductions are possible.

First, signed graphs whose underlying graph is a cycle may be ignored in our further considerations. Namely, they give rise to counterexample pairs already found in [15]. Therefore, class $\mathcal{A}_2$ can be reduced to $\mathcal{A'}_2$ by discarding unbalanced cycles, while class $\mathcal{B}_1$ need not to be considered at all.

Next, the class $\mathcal{B}_2$ can be reduced to $\mathcal{B'}_2$ since double unbalanced dumbbells may be ignored. Indeed, for any such graph there is a cut-vertex which upon deletion gives rise to a signed cycle in the obtained subgraph, and thus to multiple roots in the corresponding polynomial in the polynomial deck. In other words, only double unbalanced infinite graphs remain to be considered.

Finally, recall that classes $\mathcal{A}_3$ and $\mathcal{B}_3$ are finite (since we are dealing with exceptional graphs). Therefore, they can be examined by exhaustive computer search in order to avoid too involved case analysis. According to results reported in part (c) of the Appendix, we can say that the theorem holds if $\dot{G}$ belongs to both of these two classes.

Therefore, further on we have to deal, in respect to $\dot{G}$, with class $\mathcal{A}_1$ and two subclasses $\mathcal{A'}_2$ and $\mathcal{B'}_2$.

We next distinguish between two cases depending on the connectivity of $\dot{H}$.

Case 1. $\dot{G}$ and $\dot{H}$ are connected.

We first observe that these two signed graphs do not belong to the same (sub)classes – otherwise, we are immediately done by (2.4) regardless of their polynomial decks.

Claim 1. *Polynomial reconstruction is unique if $\dot{G}$ and $\dot{H}$ are both connected.*

Proof. Assume first that $\dot{G} \in \mathcal{A}_1$. Then $\dot{H} \in \mathcal{A'}_2 \cup \mathcal{B'}_2$. If so, then $\mathcal{P}(\dot{G}) \neq \mathcal{P}(\dot{H})$

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since $\mu^*(\dot{H}) = n \geq 7$ and $\mu^*(\dot{G}) = 4$ (see (2.5)).

Next, assume that $\dot{G} \in \mathcal{A}'_2$. Then $\dot{H} \in \mathcal{B}'_2$. But then $\mathcal{P}(\dot{G}) \neq \mathcal{P}(\dot{H})$ since there exists $v \in V(\dot{G})$ such that $|\Phi_{\dot{G}-v}(-2)| \neq 4$, while $|\Phi_{\dot{H}-v}(-2)| = 4$ for each $v \in V(\dot{H})$.

Finally, there is no need to consider the case when $\dot{G} \in \mathcal{B}'_2$ (no relevant $\dot{H}$ exists), and the proof follows. ◻

Case 2. $\dot{G}$ is connected, while $\dot{H}$ disconnected.

We have $\dot{H} = \dot{H}_1 \cup \dot{H}_2$, and also both components of $\dot{H}$ are of the same order, i.e. k $(= \frac{n}{2})$ – see Theorem 2.1. So, n is even and also $n \geq 7$ (see Fact 1), and therefore $n \geq 8$ (or $k \geq 4$). We also have:

$$\dot{G} \in \mathcal{A}_1 \cup \mathcal{A}'_2 \cup \mathcal{B}'_2.$$

Then $\mu^*(\dot{G}) = 2k$ if $\dot{G} \in \mathcal{A}_1$, or $\mu^*(\dot{G}) = 4$ otherwise (see (2.5)).

By Corollary 2.3, if $c = -2$, we obtain that

$$\dot{H}_1, \dot{H}_2 \in \mathcal{A}_1 \cup \mathcal{A}'_2 \cup \mathcal{A}_3.$$

Then, we easily get

$$\mu^*(\dot{H}) = \min\{\mu(\dot{H}_1)\mu^*(\dot{H}_2), \mu(\dot{H}_2)\mu^*(\dot{H}_1)\}. \tag{3.1}$$

We next prove the following two claims.

Claim 2. *The polynomial reconstruction is unique if $\dot{H}_1, \dot{H}_2 \in \mathcal{A}_1 \cup \mathcal{A}'_2$.*

Proof. Since $\mu^*(\dot{G}) \in \{2k, 4\}$ (see above) while $\mu^*(\dot{H}) \in \{k(k+1), 4k, 16\}$, we are done for each k, but possibly $k = 8$. If $k = 8$, then there exists $v \in V(\dot{G})$ such that $|\Phi_{\dot{G}-v}(-2)| \neq 4$, while $|\Phi_{\dot{H}-v}(-2)| = 4$ holds for each $v \in V(\dot{H})$. So, we are done. ◻

Claim 3. *The polynomial reconstruction is unique if $\dot{H}_1 \in \mathcal{A}_1 \cup \mathcal{A}'_2 \cup \mathcal{A}_3$ and $\dot{H}_2 \in \mathcal{A}_3$.*

Proof. Observe first that $\mu(\dot{H}_1) \geq \mu^*(\dot{H}_1)$ and $\mu^*(\dot{H}_2) > \mu(\dot{H}_2)$ (These follow from (2.4) and (2.5).). Therefore, from (3.1) we get $\mu^*(\dot{H}) = \mu(\dot{H}_2)\mu^*(\dot{H}_1)$. Clearly, $k \in \{6, 7, 8\}$, and thus $\mu(\dot{H}_2) = 9 - k$. So, $\mu^*(\dot{H}) = (9 - k)\mu^*(\dot{H}_1)$. Recall that $\mu^*(\dot{G}) \in \{2k, 4\}$. Since $\mu^*(\dot{G}) = \mu^*(\dot{H})$ must hold, we now easily obtain the four possibilities to be further examined. Namely,

(i) $\dot{G} \in \mathcal{A}_1$, $\dot{H}_1 \in \mathcal{A}'_2$, $\dot{H}_2 \in \mathcal{E}_6$,
(ii) $\dot{G} \in \mathcal{A}_1$, $\dot{H}_1, \dot{H}_2 \in \mathcal{E}_6$,
(iii) $\dot{G} \in \mathcal{A}_1$, $\dot{H}_1 \in \mathcal{A}_1$, $\dot{H}_2 \in \mathcal{E}_7$,
(iv) $\dot{G} \in \mathcal{A}'_2$, $\dot{H}_1 \in \mathcal{A}'_2$, $\dot{H}_2 \in \mathcal{E}_8$.

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For case (i), we have $\Phi_{\dot{G}}(-2) = 13$ and $\Phi_{\dot{H}}(-2) = 12$ (both follow from (2.4)), but this is impossible by Corollary 2.4. Case (ii) is discarded in the same way (since $\Phi_{\dot{G}}(-2) = 13$ and $\Phi_{\dot{H}}(-2) = 9$).

Consider case (iii). Since $\dot{G}$ is the signed line graph of a tree, for any vertex $v \in V(\dot{G})$, $\dot{G} - v$ has at most two components which are signed line graphs of trees. Therefore, using (2.4), we compute $|\Phi_{\dot{G}-v}(-2)| \in \{14, 26, 36, 44, 50, 54, 56\}$. Observe for a moment that, among these values, only 56 is a multiple of 8. In addition, the possibility $|\Phi_{\dot{G}-v}(-2)| = 56$ occurs only if $\dot{G}-v$ has two components, one with six vertices and the other with seven. So, there are at most two vertices v for which 8 divides $|\Phi_{\dot{G}-v}(-2)|$. On the contrary, for each $v' \in V(\dot{H}_2)$ (so for more than two vertices) we have $|\Phi_{\dot{H}-v'}(-2)| = |\Phi_{\dot{H}_1}(-2)\Phi_{\dot{H}_2-v'}(-2)| = 8|\Phi_{\dot{H}_2-v'}(-2)|$, a contradiction.

Finally, case (iv) is discarded since $\Phi_{\dot{G}}(-2) = \Phi_{\dot{H}}(-2)$. $\square$

Summarizing the partial results from the above facts and claims we arrive at the proof of Theorem 3.1.

At this point we add some concluding remarks. In this paper we have proved that the polynomial reconstruction is unique for all signed graphs except (signed) cycles whose vertex-deleted subgraphs have all eigenvalues greater than -2. The next natural step is to consider that the eigenvalues in question are greater than or equal to -2. It may be possible to resolve this question in a similar manner, but with more involved discussion. Besides, computational work to be involved for this aim is expected to be more time consuming. Among others, all minimal signed graphs for the property that the least eigenvalue is greater than or equal to -2 will be needed. It is theoretically known that they have at most ten vertices, and thus too much time will be needed for their identification and further computations.

Appendix. Here we give some computational results related to our reconstruction problem.

(a) The value of the characteristic polynomials of minimal exceptional signed graphs in -2. Recall first that all switching equivalence classes of exceptional signed graphs with least eigenvalue greater than -2 are given in [10]. So, we use them as inputs for our computations to get the result as in Lemma 2.7(iii), i.e., that $\Phi_{\dot{G}}(-2) = (-1)^n(9-n)$, where $\dot{G} \in \mathcal{E}_n, 6 \leq n \leq 8$.

(b) Polynomial reconstruction of connected signed graphs with at most six vertices. First we have generated all connected signed graphs with at most six vertices. For this aim we have used McKay's library of programs *nauty* [11] for generating connected non-isomorphic simple graphs with a prescribed number of vertices. These graphs are taken as underlying graphs for signed graphs being generated. For each of them all possible signatures are assigned. Among the signed graphs obtained in

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TABLE 3.1
Connected signed graphs with at most six vertices.

n	3	4	5	6
# underlying graphs	2	6	21	112
# signed graphs	3	12	79	1123

this way, we have singled out the representatives of switching equivalent classes. The computational results are summarized in Table 1.

For each of the above signed graphs we have verified that the polynomial reconstruction is unique except for signed cycles (cf. Theorem 3.1).

(c) Polynomial reconstruction of one-vertex extensions of minimal exceptional signed graphs. Using the procedure explained in [10], we have generated the representatives of switching equivalence classes of all minimal exceptional signed graphs (32 on six vertices, 233 on seven vertices, and 1242 on eight vertices). For each of the representatives we have constructed all possible one-vertex extensions with all the possible signatures assigned, under the restriction that the first added edge is always positive. Clearly, in this way we have obtained a set of signed graphs which contains at least one representative of the required switching equivalence classes. The total number of such signed graphs on seven, eight, and nine vertices is 11 648, 254 669, and 4 073 760, respectively.

From the above sets of graphs we have singled out only those whose polynomial decks satisfy the restrictions of the problem we are considering. In this way the total number of signed graphs we are interested in is reduced to 918 on seven vertices, 7134 on eight vertices, and 5902 on nine vertices.

For each of the above signed graphs, we have verified that the polynomial reconstruction is unique.

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