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MEAN MATRICES AND CONDITIONAL NEGATIVITY*

RAJENDRA BHATIA† AND TANVI JAIN‡

To R. B. Bapat on his sixtieth birthday

Abstract. In earlier papers R. Bhatia and H. Kosaki have shown that certain matrices associated with means are infinitely divisible. In this paper it is shown that many of them possess a stronger property: their Hadamard reciprocals have exactly one positive eigenvalue.

Key words. Positive definite matrix, conditionally negative definite matrix, infinitely divisible matrix, Hadamard inverse, means.

AMS subject classifications. 15A57, 15B57, 15A48

1. Introduction.

This note on some structured matrices is a continuation of earlier work [Bh2, BhK]. In those papers it was shown that certain special matrices are infinitely divisible. Here we show that many of them possess a stronger property.

Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix. Then A is said to be *positive semidefinite* (psd for short), if for all $x \in \mathbb{R}^n$ we have $\langle x, Ax\rangle \geq 0$. It is said to be *conditionally positive definite* (cpd for short), if $\langle x, Ax\rangle \geq 0$ for all x in the $(n-1)$-dimensional space $\mathcal{H}_1 = \left\{ x : \sum_{j=1}^{n} x_j = 0 \right\}$. If $-A$ is cpd, we say that A is *conditionally negative definite* (cnd for short).

Let A be a symmetric matrix, and suppose $a_{ij} \geq 0$ for all i, j. We say A is *infinitely divisible* if for all $r > 0$ the entrywise power (*Hadamard power*)

$$A^{\circ r} = \left[a_{ij}^{r}\right]$$

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is psd. If A is a symmetric matrix such that $a_{ij} > 0$ for all i, j, and A has exactly one positive eigenvalue, then we say that A is *in the class* $\mathcal{A}$. (Here we follow the notation and terminology in [BR].) Every cnd matrix with all entries positive is in the class $\mathcal{A}$.

The special classes defined above are important in diverse areas of mathematics. For example, cpd and infinitely divisible kernels are important in probability theory. A famous theorem of Schoenberg in distance geometry says that a symmetric matrix A with $a_{ii} = 0$ and $a_{ij} \geq 0$, is cnd if and only if there exist vectors $u_1, \ldots, u_n$ in some Euclidean space $\mathbb{R}^d$ satisfying $\|u_i - u_j\|^2 = a_{ij}$. We call such a matrix a *Euclidean distance matrix.* See [BR].

The following theorem due to R. Bapat [B] is the starting point for our discussion.

Theorem 1.1. *If* $A = [a_{ij}]$ *is any matrix in class* $\mathcal{A}$, *then its Hadamard inverse matrix* $[1/a_{ij}]$ *is infinitely divisible.*

In earlier papers [Bh2], [BhK] it was shown that several matrices arising naturally in the study of matrix means are infinitely divisible. Here we show that many of these, but not all, are Hadamard reciprocals of matrices in class $\mathcal{A}$.

We will repeatedly use some elementary facts. Let X' stand for the transpose of X. If X is a nonsingular matrix, then the transformation $A \mapsto X'AX$ is called a *congruence.* If A is psd, then so is every matrix congruent to it; and if $A \in \mathcal{A}$, then every matrix congruent to A lies in $\mathcal{A}$. This is a consequence of Sylvester's law of inertia [Bh1]. If A is cnd, then for every permutation matrix P, the matrix $P'AP$ is cnd. If $A_1, \ldots, A_k$ are cnd matrices, and $\alpha_1, \ldots, \alpha_k$ are nonnegative real numbers, then $\alpha_1 A_1 + \cdots + \alpha_k A_k$ is cnd. On the other hand positive linear combinations of matrices in $\mathcal{A}$ need not be in $\mathcal{A}$. The matrix E with all its entries equal to 1 is a psd matrix of rank 1. The space $\mathcal{H}_1$ is the null space of E. Consequently, E is both cpd and cnd.

Most of the examples we discuss arise as follows. Let $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a mean; i.e. a map that satisfies the following properties:

(i) $m(a, b) = m(b, a)$.
(ii) $\min(a, b) \leq m(a, b) \leq \max(a, b)$.
(iii) $m(\alpha a, \alpha b) = \alpha m(a, b)$ for all $\alpha > 0$.
(iv) $m(a, b)$ is an increasing function of a and b.
(v) $m(a, b)$ is a continuous function of a and b.

Let $\lambda_1, \ldots, \lambda_n$ be given positive numbers. We consider the $n \times n$ matrix $M =$

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$[m(\lambda_i, \lambda_j)]$. With the earlier works as motivation, we ask the following questions: If $m(a,b) \geq \sqrt{ab}$ for all a, b, then is the matrix M in class $\mathcal{A}$ for all $\lambda_1, \ldots, \lambda_n$? And if $m(a,b) \leq \sqrt{ab}$ for all a, b, then is the Hadamard reciprocal of M in class $\mathcal{A}$ for all $\lambda_1, \ldots, \lambda_n$? For several classes of means this is answered in the next section. Some related matrices are also considered.

After the papers of Bhatia and Kosaki cited earlier, Kosaki has carried out a very extensive and deep analysis of infinite divisibility of several matrices arising from means. See [K1, K2, K3].

General references for matrix analysis used here are [Bh1] and [HJ]. An exposition of min matrices in a lighter vein is given in [Bh3].

2. Examples.

2.1. *The arithmetic and harmonic means.*

Let $\lambda_1, \ldots, \lambda_n$ be positive numbers, and let $a_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)$. Then the matrix $A = [a_{ij}]$ can be expressed as $A = \frac{1}{2}(DE + ED)$, where D is the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. So, for all $x \in \mathcal{H}_1$, we have $\langle x, Ax \rangle = 0$. This shows that the matrix A is both cpd and cnd. Likewise, the matrix B with entries $b_{ij} = \frac{1}{2}\left(\lambda_i^{-1} + \lambda_j^{-1}\right)$ is both cpd and cnd.

A very special corollary of this is that the famous Hilbert matrix $\left[\frac{1}{i+j}\right]$ is infinitely divisible, a fact that is very well known.

2.2. *The maximum.*

Theorem 2.1. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. Then the matrix $M = [\max(\lambda_i, \lambda_j)]$ is cnd.*

Proof. First assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. In this case we have

$$M = \lambda_1 E + (\lambda_2 - \lambda_1)F_1 + (\lambda_3 - \lambda_2)F_2 + \cdots + (\lambda_n - \lambda_{n-1})F_{n-1}, \tag{2.1}$$

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where F_j, $1 \leq j \leq n-1$, is the matrix with all entries zero in the first j rows and j columns, and all other entries equal to 1. The matrix $\lambda_1 E$ is cnd. The matrices F_j are psd and the coefficients $(\lambda_{j+1} - \lambda_j)$, $1 \leq j \leq n-1$ are negative. So, the decomposition (2.1) shows that M is cnd.

If $\lambda_1, \ldots, \lambda_n$ are any positive numbers, then we can choose a permutation matrix P so that

$$P'MP = \left[\max\left(\lambda_i', \lambda_j'\right)\right],$$

where $\lambda_1' \geq \cdots \geq \lambda_n'$ is a decreasing rearrangement of $\lambda_1, \ldots, \lambda_n$. So M is cnd. ◻

Corollary 2.2. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. The matrix $[min(\lambda_i, \lambda_j)]$ is infinitely divisible.*

Proof. This follows from Theorems 1.1, 2.1 and the observation

$$\min(\lambda_i, \lambda_j) = \left(\max\left(\frac{1}{\lambda_i}, \frac{1}{\lambda_j}\right)\right)^{-1}.$$

◻

Corollary 2.3. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. Then the matrix $[|\lambda_i - \lambda_j|]$ is cnd, as is the matrix $[1 + |\lambda_i - \lambda_j|]$.*

Proof. The first statement follows from the relation

$$|\lambda_i - \lambda_j| = 2\max(\lambda_i, \lambda_j) - (\lambda_i + \lambda_j),$$

and the facts that $[\max(\lambda_i, \lambda_j)]$ is cnd and $[\lambda_i + \lambda_j]$ cpd. The second statement follows from the first, since the matrix E is cnd. ◻

As a consequence, the matrix

$$\left[\frac{1}{1 + |\lambda_i - \lambda_j|}\right]$$

is infinitely divisible. See Exercise 5.2.33 in [Bh1] for another proof of this.

2.3. *Heinz means.*

These means are defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}, 0 \leq \nu \leq 1.$$

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$H_0 = H_1$ is the arithmetic mean and $H_{1/2}$ is the geometric mean. Let

$$M = [H_\nu(\lambda_i, \lambda_j)] .$$

Note that

$$\begin{aligned} m_{ij} &= \frac{1}{2}\left(\lambda_i^\nu \lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^\nu\right) \\ &= \lambda_i^{1-\nu}\left(\frac{\lambda_i^{2\nu-1} + \lambda_j^{2\nu-1}}{2}\right)\lambda_j^{1-\nu}. \end{aligned}$$

So $M = DAD$, where $D = \operatorname{diag}(\lambda_1^{1-\nu}, \dots, \lambda_n^{1-\nu})$ and $a_{ij} = \frac{1}{2}\left(\lambda_i^{2\nu-1} + \lambda_j^{2\nu-1}\right)$. In Section 2.1 we have noted that the matrix A is cnd. So M, being congruent to A, is in the class $\mathcal{A}$.

2.4. *The logarithmic mean.*

This mean is defined as

$$\begin{aligned} &L(a,b) = \frac{a-b}{\log a - \log b}, \quad a \neq b, \\ &L(a,a) = a. \end{aligned}$$

We have also

$$L(a,b) = \int_0^1 H_\nu(a,b)\ d\nu.$$

Our discussion so far may lead us to speculate that the matrix $[L(\lambda_i, \lambda_j)]$ may be in $\mathcal{A}$ for all $\lambda_1, \dots, \lambda_n$. However, this fails in the most spectacular way. For $n \geq 3$, *no* such matrix is in $\mathcal{A}$.

THEOREM 2.4. *Let $n \geq 3$ and let $\lambda_1, \dots, \lambda_n$ be any positive numbers. Then the matrix $[L(\lambda_i, \lambda_j)]$ has at least two positive eigenvalues.*

The proof of this theorem is given in the Appendix.

2.5. *The binomial means.*

This is the family of means defined as

$$B_\alpha(a,b) = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{1/\alpha}, \qquad -\infty \leq \alpha \leq \infty.$$

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The values at $\alpha = -\infty, 0, \infty$ are understood in the sense of limits. These are, respectively, $\min(a, b)$, $\sqrt{ab}$ and $\max(a, b)$.

It has been shown in [BhJ1] that for any positive numbers $\lambda_1, \dots, \lambda_n$ the matrix $[(\lambda_i + \lambda_j)^r]$ is cnd for $0 \le r \le 1$. If $n \ge 3$, then this matrix has at least two positive eigenvalues for all $r > 1$. Using this one can see that the matrix $[B_\alpha(\lambda_i, \lambda_j)]$ is cnd for $\alpha \ge 1$, but not for $0 < \alpha < 1$. The special cases $\alpha = 1$ and ∞ have been studied in Section 2.1 and 2.2.

It may be appropriate here to discuss a related matrix. Let x_i be real numbers with $|x_i| < 1$. Consider the matrix

$$S = \left[(1 - x_i\, x_j)^r\right], \qquad 0 < r < 1.$$

We have a power series expansion

$$(1 - x_i\, x_j)^r = \sum_{m=0}^{\infty} a_m (x_i\, x_j)^m,$$

in which $a_0 = 1$ and $a_m < 0$ for all $m \ge 1$. The matrices $[(x_i\, x_j)^m]$ are psd for all $m \ge 1$, and the matrix E is cnd. It follows that S is cnd for $0 < r < 1$.

The Cayley transformation

$$\lambda = \frac{1 - x}{1 + x}$$

is a bijection from the interval $(-1, 1)$ onto $\mathbb{R}_+$. With this substitution we see that

$$\frac{\lambda_i + \lambda_j}{2} = \frac{1 - x_i x_j}{(1 + x_i)(1 + x_j)}.$$

So, the matrices $[(\lambda_i + \lambda_j)^r]$ and $[(1 - x_i\, x_j)^r]$ are congruent.

2.6. *Lehmer means.*

These are defined as

$$L_p(a, b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}}, \qquad -\infty \le p \le \infty.$$

For $p = 0, \frac{1}{2}, 1$ these are the harmonic, geometric and arithmetic means, respectively. For $1 < p < \infty$, $L_p(a, b)$ is not a monotone function of a and b, and therefore is not

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a "mean" in the sense we have defined. However, it is customary to study the entire family together. For $p = \pm\infty$, $L_p(a,b)$ is $\max(a,b)$ and $\min(a,b)$, respectively.

THEOREM 2.5. *Let $\lambda_1,\dots,\lambda_n$ be positive numbers. If $n \le 3$, then the matrix $[L_p(\lambda_i,\lambda_j)]$ is in the class $\mathcal{A}$ for all $p \ge \frac{1}{2}$. For $n > 3$, the matrix $[L_p(\lambda_i,\lambda_j)]$ is in $\mathcal{A}$ if and only if p is in the set $\left[\frac{1}{2},\frac{3}{4}\right] \cup \{1\} \cup \left[\frac{3}{2},\infty\right]$.*

If $n \le 3$, then the matrix $\left[L_p(\lambda_i,\lambda_j)^{-1}\right]$ is in the class $\mathcal{A}$ for all $p \le \frac{1}{2}$. For $n > 3$, the matrix $\left[L_p(\lambda_i,\lambda_j)^{-1}\right]$ is in $\mathcal{A}$ if and only if p is in the set $\left[-\infty,-\frac{1}{2}\right] \cup \{0\} \cup \left[\frac{1}{4},\frac{1}{2}\right]$.

Proof. Let $x_1,\dots,x_n$ be distinct positive real numbers and let r be a real number. In our ongoing work [BhJ2] we have studied the eigenvalue distribution of the matrix

$$\left[\frac{x_i^r + x_j^r}{x_i + x_j}\right]. \tag{2.2}$$

We have shown that this matrix has exactly one positive eigenvalue if and only if r is in the set

$$\begin{cases} [-3,-1] \cup [1,3] & \text{if } n > 3 \\ (-\infty,-1] \cup [1,\infty) & \text{if } n \le 3. \end{cases}$$

(The proof of these statements use ideas similar to those in [BhFJ] and [BhJ1].) Let $p \ne 1$. The matrix under consideration

$$[L_p(\lambda_i,\lambda_j)] = \left[\frac{\lambda_i^p + \lambda_j^p}{\lambda_i^{p-1} + \lambda_j^{p-1}}\right] \tag{2.3}$$

reduces to the form in (2.2) with the substitution $x_i = \lambda_i^{p-1}$, and $r = \frac{p}{p-1}$. Thus for $n > 3$, the matrix in (2.3) is in the class $\mathcal{A}$ if and only if $\frac{p}{p-1} \in [1,3] \cup [-3,-1]$. Now $1 \le \frac{p}{p-1} \le 3$ if and only if $p \ge \frac{3}{2}$, and $-3 \le \frac{p}{p-1} \le -1$ if and only if $\frac{1}{2} \le p \le \frac{3}{4}$. We have already seen that for $p = 1$, the matrix (2.3) is in $\mathcal{A}$.

By following the same arguments as above, we see that for $n \le 3$, the matrix (2.3) is in the class $\mathcal{A}$ for all $p \ge \frac{1}{2}$. This proves the first statement of the theorem. The second can be derived from this using the relation

$$L_{1-p}(a,b) = \frac{ab}{L_p(a,b)}.$$

□

2.7. *Power difference means.*

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These means are defined as

$$K_p(a,b) = \frac{p-1}{p} \quad \frac{a^p - b^p}{a^{p-1} - b^{p-1}}.$$

When $p = -1, \frac{1}{2}, 1, 2$, they give, respectively, the harmonic, geometric, logarithmic, and arithmetic means. When $p = \pm\infty$, they are the max and min, respectively.

THEOREM 2.6. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. Then the matrix $[K_p(\lambda_i, \lambda_j)]$ is in the class $\mathcal{A}$ if and only if p is in $\left[\frac{1}{2}, \frac{2}{3}\right] \cup [2, \infty]$. The matrix $\left[K_p(\lambda_i, \lambda_j)^{-1}\right]$ is in $\mathcal{A}$ if and only if $p \leq -1$.*

Proof. The matrix under consideration is

$$[K_p(\lambda_i, \lambda_j)] = \frac{p-1}{p} \left[\frac{(\lambda_i^{p-1})^{p/p-1} - (\lambda_j^{p-1})^{p/p-1}}{\lambda_i^{p-1} - \lambda_j^{p-1}} \right]. \tag{2.4}$$

The matrix

$$\mathcal{L}_r = \left[\frac{x_i^r - x_j^r}{x_i - x_j} \right], \qquad r \in \mathbb{R} \tag{2.5}$$

is the much studied Loewner matrix. It is a classical result that for $0 < r < 1$, this matrix is psd. It was shown by Bhatia and Holbrook [BhH] that this matrix is in the class $\mathcal{A}$ when $1 \leq r \leq 2$. Bhatia and Sano [BhS1] carried this further and showed that this matrix is cnd when $1 \leq r \leq 2$, and cpd when $2 \leq r \leq 3$. More generally, the eigenvalue distribution of this matrix was studied by Bhatia et al [BhFJ] for all real r. It follows from this work that the matrix $\mathcal{L}_r$ has exactly one positive eigenvalue if and only if $r \in [1,2] \cup [-3,-2]$, and it has exactly one negative eigenvalue if and only if $r \in [2,3] \cup [-2,-1]$. By the arguments given in Section 2.6 it follows that for $p > 1$ the matrix (2.5) is of the form $\frac{1}{r}\,\mathcal{L}_r$. Hence it is in $\mathcal{A}$ if and only if $\frac{p}{p-1} \in [1,2]$; i.e., if and only if $p \geq 2$.

Let $0 < p < 1$. Then $\frac{p-1}{p}$ is a negative number. So, in this case the matrix (2.5) is of the form $\frac{-1}{r}\,\mathcal{L}_r$. Hence it has exactly one positive eigenvalue if and only if $\mathcal{L}_r$ has exactly one negative eigenvalue. This happens if and only if $-2 \leq \frac{p}{p-1} \leq -1$; i.e., if and only if $\frac{1}{2} \leq p \leq \frac{2}{3}$. Hence the matrix (2.5) is in $\mathcal{A}$ if and only if $p \in [2,\infty] \cup \left[\frac{1}{2}, \frac{2}{3}\right]$. This completes the proof of the first assertion of the Theorem.

To prove the second assertion, note that

$$\left[\frac{1}{K_p(\lambda_i, \lambda_j)} \right] = \frac{p}{p-1} \quad \left[\frac{(\lambda_i^p)^{\frac{p-1}{p}} - (\lambda_j^p)^{\frac{p-1}{p}}}{\lambda_i^p - \lambda_j^p} \right]. \tag{2.6}$$

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Suppose $p < 0$. Then $\frac{p-1}{p}$ is positive. So, the matrix (2.6) is in $\mathcal{A}$ if and only if $1 \leq \frac{p-1}{p} \leq 2$; i.e., if and only if $p \leq -1$. Finally if $0 < p < 1$, then this matrix is in $\mathcal{A}$ if and only if $-2 \leq \frac{p-1}{p} \leq -1$; i.e., $\frac{1}{3} \leq p \leq \frac{1}{2}$. □

Next we consider a matrix obtained from a ratio of two means.

2.8. *The generalised Lehmer matrix.*

Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$, and consider the matrix

$$A = \left[\frac{\max(\lambda_i, \lambda_j)}{\min(\lambda_i, \lambda_j)}\right]. \tag{2.7}$$

The Hadamard reciprocal of A, for the special choice $\lambda_i = i$, is called the Lehmer matrix. The Lehmer matrix is infinitely divisible [Bh2]. The following theorem displays a stronger property.

THEOREM 2.7. *The matrix A defined by (2.7) is in the class $\mathcal{A}$.*

Proof. Let X be any symmetric matrix with positive entries, and let

$$t^n + a_1 t^{n-1} + \cdots + a_n \tag{2.8}$$

be the characteristic polynomial of X. By the Descartes rule of signs, X has only one positive eigenvalue if and only if the coefficients of the polynomial (2.8) change signs only once. Since $a_1 = -\operatorname{tr} X < 0$, this condition is equivalent to saying that X is in $\mathcal{A}$ if and only if $a_k < 0$ for all $1 \leq k \leq n$. Since a_k equals $(-1)^k$ times the sum of $k \times k$ principal minors of X, this condition is satisfied if for each k all the $k \times k$ principal minors have the same sign, and all the $(k+1) \times (k+1)$ principal minors have the sign opposite to this.

Apply these considerations to A. All principal submatrices of A have the same form as A. Let A_k denote a $k \times k$ matrix of the form A; i.e. $A_k = \left[\frac{\max(\lambda_i,\lambda_j)}{\min(\lambda_i,\lambda_j)}\right]$ for some $0 < \lambda_1 < \cdots < \lambda_k$. If we show that for each such matrix $\operatorname{sgn}(\det A_k) = (-1)^{k-1}$, it would follow that A is in the class $\mathcal{A}$. This is a consequence of the LU decomposition of A that is obtained in Lemma 2.8. We shall resume the proof of the Theorem after proving the Lemma.

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Writing out the entries of A shows

$$A = \begin{bmatrix} 1 & \lambda_2/\lambda_1 & \lambda_3/\lambda_1 & \cdots & \lambda_n/\lambda_1 \\ \lambda_2/\lambda_1 & 1 & \lambda_3/\lambda_2 & \cdots & \lambda_n/\lambda_2 \\ \lambda_3/\lambda_1 & \lambda_3/\lambda_2 & 1 & \cdots & \lambda_n/\lambda_3 \\ \vdots & & & \ddots & \\ \lambda_n/\lambda_1 & \lambda_n/\lambda_2 & . & \cdots & 1 \end{bmatrix}, \tag{2.9}$$

i.e., $a_{ij} = \lambda_j/\lambda_i$ if $1 \le i \le j \le n$. We prove the following:

Lemma 2.8. *The matrix A in (2.11) can be factored as $A = LU$, where L is the lower triangular matrix with entries*

$$\ell_{ij} = \begin{cases} a_{ij} & \textit{for all} \quad 1 \le j \le i \le n \\ 0 & \textit{for} \qquad\qquad j > i, \end{cases} \tag{2.10}$$

and U is the upper triangular matrix with entries

$$u_{ij} = \begin{cases} a_{ij} & \textit{for} \quad i = 1, \quad 1 \le j \le n \\ -\frac{(\lambda_i^2 - \lambda_{i-1}^2)\lambda_j}{\lambda_i\,\lambda_{i-1}^2} & \textit{for} \quad 2 \le i \le j \\ 0 & \textit{for} \quad i > j. \end{cases} \tag{2.11}$$

Proof. Let i, j be any two indices with $i \ge j$. If L and U are as in (2.10) and (2.11), then the (i, j) entry of LU is equal to

$$\begin{aligned} \sum_{k=1}^{n} \ell_{ik}\, u_{kj} &= \sum_{k=1}^{j} \ell_{ik}\, u_{kj} \\ &= \ell_{i1}\, u_{1j} + \sum_{k=2}^{j} \ell_{ik}\, u_{kj} \\ &= \frac{\lambda_i}{\lambda_1}\frac{\lambda_j}{\lambda_1} - \sum_{k=2}^{j} \frac{\lambda_i}{\lambda_k}\frac{(\lambda_k^2 - \lambda_{k-1}^2)\lambda_j}{\lambda_k\,\lambda_{k-1}^2} \\ &= \frac{\lambda_i\,\lambda_j}{\lambda_1^2} + \lambda_i\,\lambda_j \sum_{k=2}^{j} \left(\frac{1}{\lambda_k^2} - \frac{1}{\lambda_{k-1}^2}\right) \\ &= \frac{\lambda_i\,\lambda_j}{\lambda_1^2} + \lambda_i\,\lambda_j \left(\frac{1}{\lambda_j^2} - \frac{1}{\lambda_1^2}\right) \\ &= \frac{\lambda_i}{\lambda_j} = a_{ij}. \end{aligned}$$

We have shown that for $i \ge j$ the (i, j) entry of LU agrees with that of A. A similar calculation shows that this happens also when $i < j$. ∎

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As a consequence of Lemma 2.8 we have

$$\det\, A = (-1)^{n-1} \prod_{i=2}^{n} \frac{\lambda_i^2 - \lambda_{i-1}^2}{\lambda_{i-1}^2}. \tag{2.12}$$

Hence $\operatorname{sgn}(\det\, A) = (-1)^{n-1}$. This completes the proof of Theorem 2.7. $\square$

Corollary 2.9. *Let $x_1, \ldots, x_n$ be any real numbers. Then the matrix $\left[e^{|x_i - x_j|}\right]$ is in the class $\mathcal{A}$.*

The *LU* factoring of the Lehmer matrix is obtained in the paper [KS].

2.9. *Stolarsky means***.**

This is the family defined as

$$S_\gamma(a,b) = \left(\frac{a^\gamma - b^\gamma}{\gamma(a-b)}\right)^{1/(\gamma-1)} = \left(\frac{1}{b-a}\int_a^b t^{\gamma-1}dt\right)^{1/(\gamma-1)},$$

$-\infty < \gamma < \infty$. For $\gamma = -1, 0, 1, 2$, this yields the geometric, the logarithmic, the identric, and the arithmetic means, respectively. It was shown in [BhK] that for every choice of positive numbers $\lambda_1, \ldots, \lambda_n$, the matrix $\left[\frac{1}{S_\gamma(\lambda_i,\lambda_j)}\right]$ is infinitely divisible if $\gamma \geq -1$, and the matrix $[S_\gamma(\lambda_i, \lambda_j)]$ is infinitely divisible if $\gamma \leq -1$.

This family does not yield readily to our analysis. We have some evidence for the following:

Conjecture 1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive numbers. The matrix $[S_\gamma(\lambda_i, \lambda_j)]$ is in the class $\mathcal{A}$ if $\gamma \geq 2$ but this is not always so when $0 \leq \gamma < 2$.*

Conjecture 1 is known to be true for some special values of γ.

(i) S_0 is the logarithmic mean, and we have studied this in Section 2.4.
(ii) $S_{1/2}$ is the same as the binomial mean $B_{1/2}$ which we have studied in Section 2.5.
(iii) S_2 is the arithmetic mean discussed in Section 2.1.

We show that the statement of Conjecture 1 is true when $\gamma = 3$ or 4. Note that

$$S_3(\lambda_i, \lambda_j) = \left(\frac{1}{3}\left(\lambda_i^2 + \lambda_i\lambda_j + \lambda_j^2\right)\right)^{1/2}.$$

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The following theorem includes the assertion that $[S_3(\lambda_i, \lambda_j)]$ is a cnd matrix.

THEOREM 2.10. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. Then for $-2 \leq t \leq 2$, and $0 \leq r \leq 1/2$, the matrix*

$$A = \left[\left(\lambda_i^2 + t\ \lambda_i\ \lambda_j + \lambda_j^2 \right)^r \right] \tag{2.13}$$

is cnd.

Proof. Let $|x| < 1$ and $0 < r < 1$. Then we have the binomial expansion

$$(1-x)^r = \sum_{m=0}^{\infty} a_m\ x^m,$$

where $a_0 = 1$ and $a_m < 0$ for all $m \geq 1$.

We have the identity

$$\lambda_i^2 +\ t\ \lambda_i \lambda_j + \lambda_j^2 = (\lambda_i + \lambda_j)^2 \left(1 - \frac{(2-t)\lambda_i \lambda_j}{(\lambda_i + \lambda_j)^2} \right).$$

If $-2 < t < 2$, then

$$\frac{(2-t)\lambda_i \lambda_j}{(\lambda_i + \lambda_j)^2} < \frac{4\ \lambda_i \lambda_j}{(\lambda_i + \lambda_j)^2} \leq 1.$$

So for $0 < r < 1$, we have

$$\begin{aligned}
&\left[\left(\lambda_i^2 + t\ \lambda_i \lambda_j + \lambda_j^2 \right)^r \right] \\
&= \left[(\lambda_i + \lambda_j)^{2r} \left\{ 1 + \sum_{m=1}^{\infty} \frac{a_m (2-t)^m \lambda_i^m \lambda_j^m}{(\lambda_i + \lambda_j)^{2m}} \right\} \right], \\
&= \left[(\lambda_i + \lambda_j)^{2r} \right] + \sum_{m=1}^{\infty} a_m (2-t)^m \left[\frac{\lambda_i^m \lambda_j^m}{(\lambda_i + \lambda_j)^{2(m-r)}} \right].
\end{aligned} \tag{2.14}$$

The Cauchy matrix $\left[\frac{1}{\lambda_i + \lambda_j} \right]$ is infinitely divisible. So, for each $m \geq 1$, the last matrix on the right-hand side of (2.14) is psd. The coefficients a_m are all negative. So, the last infinite sum represents a negative definite matrix. Further, if $0 \leq r \leq 1/2$, then the matrix $\left[(\lambda_i + \lambda_j)^{2r} \right]$ is cnd by the results in [BhJ1]. Hence the matrix in (2.15) is cnd. ∎

We remark that in the special case $t = -2$, the matrix A in (2.13) reduces to

$$A = \left[|\lambda_i - \lambda_j|^{2r} \right]. \tag{2.15}$$

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It is known [BR] that this matrix is cnd for $0 \le r \le 1$. Likewise in the case $t = 0$, we have

$$A = \left[\left(\lambda_i^2 + \lambda_j^2 \right)^r \right], \tag{2.16}$$

and this matrix is also cnd for $0 \le r \le 1$; [BhJ1]. This raises the question whether for $-2 \le t \le 0$, the matrix A in (2.13) is cnd for $0 \le r \le 1$.

Next, consider the matrix $[S_4 (\lambda_i, \lambda_j)]$. Note that

$$S_4(\lambda_i, \lambda_j) = \left(\frac{\lambda_i^4 - \lambda_j^4}{4(\lambda_i - \lambda_j)} \right)^{1/3} = \left(\frac{(\lambda_i + \lambda_j)^3 - 2(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2)}{4} \right)^{1/3}.$$

So, the following theorem contains the assertion that $[S_4 (\lambda_i, \lambda_j)]$ is a cnd matrix.

THEOREM 2.11. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers. Then for $-1 \le t \le 3$ and $0 \le r \le 1/3$, the matrix*

$$A = \left[\left\{ (\lambda_i + \lambda_j)^3 - (3 - t) \left(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2 \right) \right\}^r \right], \tag{2.17}$$

is cnd.

Proof. If $-1 < t \le 3$, then

$$(3 - t) \frac{\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2}{(\lambda_i + \lambda_j)^3} < \frac{4 \, \lambda_i \lambda_j}{(\lambda_i + \lambda_j)^2} \le 1.$$

So the entries of the matrix A can be written as

$$\begin{aligned} a_{ij} &= \left\{ (\lambda_i + \lambda_j)^3 \left(1 - \frac{(3 - t)(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2)}{(\lambda_i + \lambda_j)^3} \right) \right\}^r \\ &= (\lambda_i + \lambda_j)^{3r} \left(1 + \sum_{m=1}^{\infty} \frac{a_m (3 - t)^m \lambda_i^m \lambda_j^m}{(\lambda_i + \lambda_j)^{2m}} \right), \end{aligned}$$

where the coefficients a_m are negative. Hence,

$$A = \left[(\lambda_i + \lambda_j)^{3r} \right] + \sum_{m=1}^{\infty} a_m (3 - t)^m \left[\frac{\lambda_i^m \lambda_j^m}{(\lambda_i + \lambda_j)^{2m - 3r}} \right]. \tag{2.18}$$

By the same arguments as in the proof of Theorem 2.10, the infinite series on the right-hand side of (2.18) represents a negative definite matrix, and the matrix $\left[(\lambda_i + \lambda_j)^{3r} \right]$ is cnd. Hence, the matrix A is cnd. $\square$

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3. Remarks.

We have seen here many examples of cnd matrices A with positive diagonal. In each case, replacing the diagonal by zero we get cnd matrices with zero diagonal. By the theorem of Schoenberg cited earlier, all these are Euclidean distance matrices. It is also known [BR] that if A is in the class $\mathcal{A}$, then $[\log a_{ij}]$ is a cnd matrix. So, more examples of distance matrices can be obtained from the matrices studied here.

Appendix A. Proof of Theorem 2.4.

It suffices to prove the statement of the Theorem for $n = 3$. The general case follows from this by Cauchy's interlacing principle. Let $\lambda_1, \lambda_2, \lambda_3$ be positive numbers and let L be the 3×3 matrix $L = [L(\lambda_i, \lambda_j)]$. Since L has at least one positive eigenvalue, if we show that det $L < 0$, then it will follow that it has exactly one more positive eigenvalue. The inequality det $L < 0$ translates to

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3 + 2L(\lambda_1,\lambda_2)L(\lambda_2,\lambda_3)L(\lambda_1,\lambda_3)\\&< \lambda_1 L(\lambda_1,\lambda_2)^2 + \lambda_2 L(\lambda_1,\lambda_3)^2 + \lambda_3 L(\lambda_1,\lambda_2)^2 \qquad \text{(A.1)}\end{aligned}$$

Make the substitution $\lambda_1 = \mathrm{e}^x$, $\lambda_2 = \mathrm{e}^y$, $\lambda_3 = \mathrm{e}^z$ and then divide both sides of the inequality so obtained by e^{x+y+z}. This shows that the assertion of the inequality (A.1) is equivalent to saying that for all real numbers x, y, z, we have

$$\begin{aligned}&1 + 2\frac{\sinh(x-y)\sinh(y-z)\sinh(z-x)}{(x-y)(y-z)(z-x)}\\&< \frac{\sinh^2(x-y)}{(x-y)^2} + \frac{\sinh^2(y-z)}{(y-z)^2} + \frac{\sinh^2(z-x)}{(z-x)^2}\end{aligned}$$

Assume $x > y > z$, and put $x - y = \alpha$, $y - z = \beta$. Then $x - z = \alpha + \beta$, and the above inequality says that for all $\alpha, \beta > 0$

$$\begin{aligned}&1 + 2\frac{\sinh\alpha\ \sinh\beta\ \sinh(\alpha+\beta)}{\alpha\beta(\alpha+\beta)}\\&< \frac{\sinh^2\alpha}{\alpha^2} + \frac{\sinh^2\beta}{\beta^2} + \frac{\sinh^2(\alpha+\beta)}{(\alpha+\beta)^2}. \qquad \text{(A.2)}\end{aligned}$$

Let us first consider the special case $\alpha = \beta$. The argument is easier and illuminates the later discussion. In this case the inequality (A.2) reduces to

$$2\ \frac{\sinh^2\alpha}{\alpha^2}\left(\frac{\sinh(2\alpha)}{2\alpha} - 1\right) < \frac{\sinh^2(2\alpha)}{(2\alpha)^2} - 1;$$

i.e., to

$$2\frac{\sinh^2\alpha}{\alpha^2} < \frac{\sinh(2\alpha)}{2\alpha} + 1. \qquad \text{(A.3)}$$

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Use the power series expansions

$$\frac{\sinh(2\alpha)}{2\alpha} = \sum_{n=1}^{\infty} \frac{(2\alpha)^{2n-2}}{(2n-1)!} \tag{A.4}$$

and

$$\frac{\sinh^2\,\alpha}{\alpha^2} = \sum_{n=1}^{\infty} \frac{2(2\alpha)^{2n-2}}{(2n)!} \tag{A.5}$$

to see that

$$\begin{aligned} 1 &+ \frac{\sinh(2\alpha)}{2\alpha} - 2\frac{\sinh^2\,\alpha}{\alpha^2} \\ &= 1 + \sum_{n=1}^{\infty} (2\alpha)^{2n-2}\Big(\frac{1}{(2n-1)!} - \frac{4}{(2n)!}\Big) \\ &= \sum_{n=3}^{\infty} \frac{(2\alpha)^{2n-2}}{(2n-1)!}\Big(1 - \frac{4}{2n}\Big). \end{aligned}$$

This is positive, and that establishes (A.3).

Now we prove the inequality (A.1) for the general case. Using the power series (A.5) and expanding $(\alpha+\beta)^n$ by the binomial theorem, we can express the right hand side of (A.1) as

$$3 + \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k+1)!}\Big(\sum_{j=1}^{k-1} a(k,j)(\alpha^{j-1}\beta^{2k-1-j} + \alpha^{2k-1-j}\beta^{j-1}) + a(k,k)\alpha^{k-1}\beta^{k-1}\Big)$$

where

$$a(k,j) = \begin{cases} 2k+1 & k \geq 2,\ j = 1 \\ \frac{2k+1}{2}\binom{2k-2}{j-1} & k \geq 2,\ 2 \leq j \leq k. \end{cases} \tag{A.6}$$

Further since

$$\sinh\,\alpha\ \sinh\,\beta\ \sinh(\alpha+\beta) = \frac{\sinh(2(\alpha+\beta)) - \sinh(2\alpha) - \sinh(2\beta)}{2},$$

using the power series expansion (A.4), the left hand side of (A.1) can be expressed as

$$3 + \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k+1)!}\Big(\sum_{j=1}^{k-1} b(k,j)(\alpha^{j-1}\beta^{2k-1-j} + \alpha^{2k-1-j}\beta^{j-1}) + b(k,k)\alpha^{k-1}\beta^{k-1}\Big),$$

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where

$$b(k,j)=\begin{cases}2k+1 & k\geq 2,\ j=1\\ \binom{2k+1}{j}-\binom{2k+1}{j-1}+\cdots\pm\binom{2k+1}{1} & k\geq 2,\ 2\leq j\leq k.\end{cases}$$

Using the identity

$$\sum_{i=0}^{m}(-1)^i\binom{r}{i}=(-1)^m\binom{r-1}{m},$$

we have

$$b(k,j)=\begin{cases}\binom{2k}{j}-1 & \text{if } j \text{ is even}\\ \binom{2k}{j}+1 & \text{if } j \text{ is odd.}\end{cases}$$

Since α and β are positive, (A.1) will be established if we show that $a(k,j)\geq b(k,j)$ for all $k\geq 2$ and $2\leq j\leq k$, and $a(k,j)>b(k,j)$ for at least one such k,j.

Let $k\geq 2$ and $2\leq j\leq k$. Then straightforward computations yield

$$a(k,j)-b(k,j)=\begin{cases}\binom{2k-2}{j-1}\frac{p(k)}{2j(2k-j)}+1 & \text{if } j \text{ is even}\\ \binom{2k-2}{j-1}\frac{p(k)}{2j(2k-j)}-1 & \text{if } j \text{ is odd,}\end{cases}$$

where

$$p(k)=4(j-2)k^2-2(j-2)(j+1)k-j^2.$$

When $j=2$, $p(k)$ is the constant -4 and $a(k,2)=b(k,2)$. It is easy to see that $a(k,j)\geq b(k,j)$ for $j\geq 3$, and we will have $a(k,j)>b(k,j)$ for all even $j>3$ if we can show that $p(k)\geq j$.

Let $j\geq 3$. In this case $p(k)-j$ is quadratic polynomial in k with exactly one positive zero. When $k=j$, $p(j)-j=j(j-3)(2j-1)$ which is nonnegative. Also $p(k)-j$ is positive for large k. Hence the positive zero of $p(k)-j$ is no bigger than j, and therefore $p(k)\geq j$.

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