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SOLUTIONS OF THE SYSTEM OF OPERATOR EQUATIONS $BXA = B = AXB$ VIA $*$-ORDER

MEHDI VOSOUGH† AND MOHAMMAD SAL MOSLEHIAN‡

Abstract. In this paper, some necessary and sufficient conditions are established for the existence of solutions to the system of operator equations $BXA = B = AXB$ in the setting of bounded linear operators on a Hilbert space, where the unknown operator $X$ is called the inverse of $A$ along $B$. After that, under some mild conditions, it is proved that an operator $X$ is a solution of $BXA = B = AXB$ if and only if $B \preceq AXA$, where the $*$-order $C \preceq D$ means $CC^* = DC^*, C^*C = C^*D$. Moreover, the general solution of the equation above is obtained. Finally, some characterizations of $C \preceq D$ via other operator equations, are presented.

Key words. $*$-Order, Moore–Penrose inverse, Matrix equation, Operator equation.

AMS subject classifications. 15A24, 15B48, 47A62, 46L05.

1. Introduction and preliminaries. Throughout the paper, $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces. We denote the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and write $\mathcal{B}(\mathcal{H})$ when $\mathcal{K} = \mathcal{H}$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and then we write $A \succeq 0$. We shall write $A > 0$ if $A$ is positive and invertible. An operator $A \in \mathcal{B}(\mathcal{H})$ is a generalized projection if $A^2 = A^*$. Let $\mathcal{S}(\mathcal{H}), \mathcal{Q}(\mathcal{H}), \mathcal{O}(\mathcal{H}), \mathcal{GP}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$, the set of all idempotents, the set of orthogonal projections and the set of all generalized projections on $\mathcal{H}$, respectively.

For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the range and the null space of $A$, respectively. The projection corresponding to a closed subspace $M$ of $\mathcal{H}$ is denoted by $P_M$. The symbol $A^{-}$ stands for an arbitrary generalized inner inverse of $A$, that is, an operator $A^{-}$ satisfying $AA^{-}A = A$. The Moore–Penrose inverse of a closed range operator $A$ is the unique operator $A^\dagger \in \mathcal{B}(\mathcal{H})$ satisfying the following equations:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$ 

Then, $A^*AA^\dagger = A^* = A^\dagger AA^*$, and we have the following properties:

$$\mathcal{R}(A^\dagger) = \mathcal{R}(A^*), \quad \mathcal{R}(A^\dagger A) = \mathcal{R}(A^* A), \quad \mathcal{N}(A^\dagger) = \mathcal{N}(A^*), \quad \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger),$$

$$\mathcal{R}(A) = \mathcal{R}(AA^\dagger), \quad P_{\mathcal{R}(A)} = AA^\dagger \quad \text{and} \quad P_{\mathcal{R}(A^\dagger)} = A^\dagger A. \quad \text{(1.1)}$$

For $A, B \in \mathcal{I}(\mathcal{H})$, $A \preceq B$ means $B - A \succeq 0$. The order $\preceq$ is said to be the Löwner order on $\mathcal{I}(\mathcal{H})$. If there exists $C \in \mathcal{I}(\mathcal{H})$ such that $AC = 0$ and $A + C = B$, then we write $A \preceq B$. The order $\preceq$ is said to be the logic order on $\mathcal{I}(\mathcal{H})$.

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For $A, B \in \mathcal{B}(\mathcal{H})$, let $A \preceq B$ mean

$$AA^* = BA^*, \quad A^*A = A^*B.$$ (1.2)

It is known that, for $A, B \in \mathcal{B}(\mathcal{H})$, $A \preceq B$ if and only if $A^*A \leq B$; see [6]. We denote by $A \wedge B$ the infimum (or the greatest lower bound) of $A$ and $B$ over the $*$-order and $A \vee B$ the supremum (or the least upper bound) of $A$ and $B$ over the $*$-order, if they exist; cf. [12].

It is known that if $A \in \mathcal{B}(\mathcal{H}, H)$ has closed range, then by considering

$$\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \quad \text{and} \quad \mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*),$$

we can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$ (1.3)

where $A_1 : \mathcal{R}(A^*) \rightarrow \mathcal{R}(A)$ is invertible; see [7, Lemma 2.1]. Therefore, the Moore–Penrose generalized inverse of $A$ can be represented as

$$A^+ = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$ (1.4)

Many results have been obtained on the solvability of equations for matrices and operators on Hilbert spaces and Hilbert $C^*$-modules. In 1976, Mitra [11] considered the matrix equations $AX = B, AXB = C$ and the system of linear equations $AX = C, XB = D$. He got the necessary and sufficient conditions for existence and expressions of general Hermitian solutions. In 1966, the celebrated Douglas Lemma was established in [8]. It gives some conditions for the existence of a solution to the equation $AX = B$ for operators on a Hilbert space. Using the generalized inverses of operators, in 2007, Dajić and Koliha [4] got the existence of the common Hermitian and positive solutions to the system $AX = C, XB = D$ for operators acting on a Hilbert space. In 2008, Xu [17] extended these results to the adjointable operators. Several general operator equations and systems in some general settings such as Hilbert $C^*$-modules have been studied by some mathematicians; see, e.g., [9, 10, 13, 16].

The matrix equation $AXB = C$ is consistent if and only if $AA^*CB^{-1}B = C$ for some $A^*, B^{-1}$, and the general solution is $X = A^*CB^{-1} + Y - A^*AYBB^{-1}$, where $Y$ is an arbitrary matrix; see [11]. In 2010, Gonzalez [1] got some necessary and sufficient conditions for existence of a solution to the equation $AXB = C$ for operators on a Hilbert space.

Let $A, B$ or $C$ have closed range. Then, the operator equation $AXB = C$ is solvable if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$; see [1, Theorem 3.1]. Therefore, if $A$ or $C$ has closed range, then the equation $AXC = C$ is solvable if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, and $CXA = C$ is solvable if and only if $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$. Deng [5] investigated the equation $CAX = C = XAC$, which is essentially different from ours. In this paper, we first characterize the existence of solutions of the system of operator equations $BXA = B = AXB$ by means of $*$-order. After that, we generalize the solutions to the system of operator equations $BXA = B = AXB$ in a new fashion.
2. The existence of solutions of the system $BXA = B = AXB$. We start our work with the celebrated Douglas lemma.

**Lemma 2.1** (Douglas Lemma, [8]). Let $A, C \in \mathfrak{H}$. Then, the following statements are equivalent:

(a) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$.
(b) There exists $X \in \mathfrak{H}$ such that $AX = C$.
(c) There exists a positive number $\lambda$ such that $CC^* \leq \lambda^2 AA^*$.

If one of these conditions holds, then there exists a unique solution $\tilde{X} \in \mathfrak{H}$ of the equation $AX = C$ such that $\mathcal{R}(\tilde{X}) \subseteq \mathcal{R}(A^*)$ and $\mathcal{N}(\tilde{X}) = \mathcal{N}(C)$.

**Lemma 2.2.** Let $A, B \in \mathfrak{H}$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, then $B = B_1 \oplus 0$, where $B_1 \in \mathfrak{H}(\mathcal{R}(A^*), \mathcal{R}(A))$.

**Proof.** Let $A, B$ be operators from the decomposition $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ into the decomposition $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then, by Lemma 2.1, there exists $C \in \mathfrak{H}$ such that $B = AC$ and $\mathcal{N}(C) = \mathcal{N}(B)$. Since $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, so $\mathcal{R}(C^*) \subseteq \mathcal{R}(C^*) = \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) = \mathcal{N}(P_{\mathcal{N}(A)})$. Hence, $P_{\mathcal{N}(A)}C^* = 0$ and so $P_{\mathcal{N}(A)} = 0$. It follows from $\mathcal{N}(C) = \mathcal{N}(B)$ that $BP_{\mathcal{N}(A)} = 0$.

If $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, then a similar reasoning shows that $P_{\mathcal{N}(A^*)}B = 0$. Therefore, $P_{\mathcal{R}(A)}BP_{\mathcal{N}(A)} = P_{\mathcal{N}(A^*)}BP_{\mathcal{R}(A)} = 0$. Hence, $B = B_1 \oplus 0$, where $B_1 = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A^*)}$.

**Theorem 2.3.** Let $A \in \mathfrak{H}$ and $B \in \mathfrak{H}$. If $A$ has closed range, then the following statements are equivalent:

1. The system of operator equations $BXA = B = AXB$ is solvable.
2. $AA^1 BA^1 A = B$.
3. $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$.

**Proof.** ((1) $\implies$ (2)) : Using (1.1) and $B = BXA$, we get that $\mathcal{R}(B) \subseteq \mathcal{R}(A) = \mathcal{R}(A^*) \subseteq \mathcal{R}(A)$. Hence, by Lemma 2.1, there exists $C \in \mathfrak{H}$ such that $B = A^1 AC^*$. Hence, $B = CA^1 A$. Applying (1.1) and $AXB = B$, we derive that $\mathcal{R}(B) \subseteq \mathcal{R}(A) = \mathcal{R}(A^1 A)$. Thus, by Lemma 2.1, there exists $C \in \mathfrak{H}$ such that $B = AA^1 C$. It follows that $AA^1 BA^1 A = AA^1 (AA^1 C)A^1 A = AA^1 C A^1 A = BA^1 A = (CA^1 A)A^1 A = CA^1 A = B$.

((2) $\implies$ (3)) : Let $AA^1 BA^1 A = B$. Then, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. It follows from $B = B^* = (AA^1 BA^1 A)^* = A^1 ABAA^1$ and (1.1) that $\mathcal{R}(B) \subseteq \mathcal{R}(A^1) = \mathcal{R}(A^*)$.

((3) $\implies$ (1)) : Let $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$. Upon applying Lemma 2.2, $B = B_1 \oplus 0$, where $B_1 = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A)}$. Since $A$ has closed range, so by using (1.3) and (1.4) we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A^1 = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $AA^1 B = B$ and $BA^1 A = B$. Thus $X = A^1$ is a solution of the system $BXA = B = AXB$.

**Proposition 2.4.** Let $A, B, X \in \mathfrak{H}$. Then,

$\mathcal{R}(A) \subseteq \mathcal{R}(B), \quad \mathcal{N}(B) \subseteq \mathcal{N}(A) \quad \text{and} \quad BXA = B = AXB$

if and only if

$\mathcal{N}(B) = \mathcal{N}(A), \quad \mathcal{R}(B) = \mathcal{R}(A) \quad \text{and} \quad AXA = A.$
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**Proof.** ($\Rightarrow$) : Suppose that $\mathcal{R}(A) \subseteq \mathcal{R}(B), \mathcal{N}(B) \subseteq \mathcal{N}(A)$ and $BXA = B = AXB$. It follows from $BXA = B$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ that $\mathcal{N}(A) \subseteq \mathcal{N}(B) \subseteq \mathcal{N}(A)$. Hence, $\mathcal{N}(A) = \mathcal{N}(B)$. It follows from $AXB = B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ that $\mathcal{R}(A) = \mathcal{R}(B)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(B)$. Moreover, $(I - AX)B = 0$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Hence, we derive that $(I - AX)A = 0$. So, $AXA = A$.

($\Leftarrow$) : Suppose that $\mathcal{N}(B) = \mathcal{N}(A), \mathcal{R}(B) = \mathcal{R}(A)$ and $AXA = A$. Hence, $(I - AX)A = 0 \Rightarrow \mathcal{R}(A) \subseteq \mathcal{N}(I - AX) \Rightarrow \mathcal{R}(B) \subseteq \mathcal{N}(I - AX) \Rightarrow B = AXB,

$A(I - XA) = 0 \Rightarrow \mathcal{R}(I - XA) \subseteq \mathcal{N}(A) \Rightarrow \mathcal{R}(I - XA) \subseteq \mathcal{N}(B) \Rightarrow B = BXA$. □

3. System of operator equations $BXA = B = AXB$ via $\ast$-order. We know that $(\mathcal{B}(\mathcal{H}), \leq^\ast)$ is a partially ordered set; see [2]. Let $G_1, G_2 \in \mathcal{B}(\mathcal{H})$ be invertible and $G_1 \leq A, G_2 \leq A$. Then, $G_1G_1^\ast = AG_1^\ast$ and $G_2G_2^\ast = AG_2^\ast$. Hence, we obtain $G_1 = G_2 = A$. This fact leads us to consider the characterizations of $A \leq B$. Now we state the necessary and sufficient conditions in which the common $\ast$- lower or $\ast$- upper bounds of $A$ and $B$ exist.

We need the following essential lemma.

**Lemma 3.1.** [18, Lemma 2.1]. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ denote the closure of a space $\mathcal{M}$. Then,

(a) $AA^\ast = BA^\ast \iff A = BP_{\mathcal{M}(A)}B \iff A = BQ$ for some $Q \in \mathcal{P}(\mathcal{H})$;
(b) $A^\ast A = A^\ast B \iff A = P_{\mathcal{M}(A)}B \iff A = PB$ for some $P \in \mathcal{P}(\mathcal{H})$;
(c) $A \leq B \iff B = A + P_{\mathcal{N}(A^\ast)}BP_{\mathcal{N}(A)}$;
(d) $A \leq B \iff A = P_{\mathcal{M}(A)}B = BP_{\mathcal{M}(A)} = P_{\mathcal{M}(A)}BP_{\mathcal{M}(A)}$;
(e) $A \leq B \iff A = A_1 \oplus 0, B = A_1 \oplus B_1$;

where $A_1 \in \mathcal{B}(\mathcal{M}(A^\ast), \mathcal{M}(A)), B_1 \in \mathcal{B}(\mathcal{N}(A), \mathcal{N}(A^\ast))$ and $A \oplus B$ means the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The following Lemma is a version of Lemma 2.1 when the operator $A$ has closed range.

**Lemma 3.2.** [4, Theorem 2]. Let $A \in \mathcal{B}(\mathcal{H})$ have closed range. Then, the equation $AX = C$ has a solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $AA^\ast C = C$, and this if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. In this case, the general solution is $X = A^\dagger C + (I - A^\dagger A)T$, where $T \in \mathcal{B}(\mathcal{H})$ is arbitrary.

**Proposition 3.3.** Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

(a) If $A$ has closed range and $B \leq A$, then $X = A^\dagger$ is a solution of the system $BXA = B = AXB$.
(b) If $B$ has closed range and $B \leq A$, then $X = B^\dagger$ is a solution of the system $BXA = B = AXB$.

**Proof.** (a) Let $A$ be a closed range operator and $B \leq A$. It follows from Lemma 3.1 (d) that $B = AP_{\mathcal{M}(B^\ast)}$ and $B = P_{\mathcal{M}(B)}A$. Hence, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(B^\ast) \subseteq \mathcal{R}(A^\ast)$. It follows from $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and Lemma 3.2 that $AA^\dagger B = B$. It follows from $\mathcal{R}(B^\ast) \subseteq \mathcal{R}(A^\ast)$ and Lemma 3.2 that $BA^\dagger A = \left((A^\dagger A)^\ast B^\ast\right)^\ast = \left(A^\ast A^\dagger A^\ast A^\ast B^\ast\right)^\ast = B$. Hence, $X = A^\dagger$ is a solution of the system of operator equations $BXA = B = AXB$.

(b) Let $B$ be a closed range operator and $B \leq A$. It follows from Lemma 3.1 that $B = AP_{\mathcal{M}(B^\ast)}$ and $B = P_{\mathcal{M}(B)}A$. Applying (1.1), we conclude that $AB^\dagger B = B$ and $BB^\dagger A = B$. Hence, $X = B^\dagger$ is a solution of the system $BXA = B = AXB$. □
Proposition 3.4. Let $A, B, X \in \mathcal{B}(\mathcal{H})$. If $A \preceq B$ and $BXA = B = AXB$, then $\mathcal{N}(B) = \mathcal{N}(A)$, $\mathcal{R}(B) = \mathcal{R}(A)$ and $AXA = A$.

Proof. Let $A \preceq B$ and $BXA = B = AXB$. Applying Lemma 3.1 (d), we have $A = P_{\mathcal{R}(A)}B = BP_{\mathcal{R}(A^*)}$. Hence, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Using Proposition 2.4,

$$\mathcal{N}(B) = \mathcal{N}(A), \quad \mathcal{R}(B) = \mathcal{R}(A) \quad \text{and} \quad AXA = A.$$ 

Remark 3.5. Note that the converse of Proposition 3.4 is not true, in general. Set $A^\dagger, A^*, A$ instead of $A, B, X$. If $A \in \mathcal{B}(\mathcal{H})$ has closed range, then, by (1.1), we have $\mathcal{R}(A^*) = \mathcal{R}(A^\dagger), \mathcal{N}(A^*) = \mathcal{N}(A^\dagger)$ and $A^\dagger AA^\dagger = A^\dagger$ but not $A^\dagger \preceq A^*$. Indeed, if $A^\dagger \preceq A^*$, then by utilizing Lemma 3.1 (d), we have $A^\dagger = P_{\mathcal{R}(A^\dagger)}A^*$. It follows from $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$ that $A^\dagger = P_{\mathcal{R}(A^\dagger)}A^* = A^*$.

Theorem 3.6. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $B \preceq A$. Then, the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{B}(\mathcal{H})$ of the system $BXA = B = AXB$.

(b) $X \preceq AXA$.

Proof. ((a) $\implies$ (b)) : Let $X \in \mathcal{B}(\mathcal{H})$ is a solution of the system $BXA = B = AXB$. Hence, $B -BXA = 0$ and $B - AXB = 0$. It follows from the assumption $B \preceq A$ and Lemma 3.1 (d) that $B = P_{\mathcal{R}(B)}A$ and $B = AP_{\mathcal{R}(B^*)}$. Hence,

$$P_{\mathcal{R}(B)}(B - AXA) = B - P_{\mathcal{R}(B)}AXA = B -BXA = 0$$

and

$$(B - AXA)P_{\mathcal{R}(B^*)} = B - AXAP_{\mathcal{R}(B^*)} = B -AXB = 0.$$ 

Therefore, $B \preceq AXA$.

((b) $\implies$ (a)) : Suppose that $B \preceq AXA$. Applying Lemma 3.1 (d), we infer that $P_{\mathcal{R}(B)}(B - AXA) = 0$ and $(B - AXA)P_{\mathcal{R}(B^*)} = 0$. It follows from the assumption $B \preceq A$ and Lemma 3.1 (d) that $B = P_{\mathcal{R}(B)}A$ and $B = AP_{\mathcal{R}(B^*)}$, whence

$$B -BXA = B - P_{\mathcal{R}(B)}AXA = P_{\mathcal{R}(B)}(B - AXA) = 0$$

and

$$B - AXB = B - AXAP_{\mathcal{R}(B^*)} = (B - AXA)P_{\mathcal{R}(B^*)} = 0.$$ 

Therefore, $X$ is a solution of the system $BXA = B = AXB$. 

Let $A, B \in \mathcal{B}(\mathcal{H})$ have closed ranges. It follows from Proposition 3.3 that $A^\dagger$ and $B^\dagger$ are solutions of the system $BXA = B = AXB$. Therefore, we are interested in the study of the following system of operator equations:

$$BXA = B = AXB,$$  \hspace{1cm} (3.5)

$$BAX = B = XAB.$$  \hspace{1cm} (3.6)
Let $A, B \in \mathbb{B}(\mathcal{H})$. An operator $C \in \mathbb{B}(\mathcal{H})$ is said to be an inverse of $A$ along $B$ if it fulfills one of the equations (3.5) or (3.6). If $A \in \mathbb{B}(\mathcal{H})$ is invertible, then $X = A^{-1}$ is a solution of the system $XA = I = AX$. Hence, $A^{-1}$ is an inverse of $A$ along $I$, where $I$ is the identity of $\mathbb{B}(\mathcal{H})$.

Let $A \in \mathbb{B}(\mathcal{H})$ have closed range. Using (1.1), we have $AA^\dagger A = A = AA^\dagger A$. Hence, $A^\dagger$ satisfies Eq. (3.5). Therefore, $A^\dagger$ is the inverse of $A$ along $A$.

It follows from (1.1) that $A^*AA^\dagger = A^*A^\dagger A^*$. Hence, $A^\dagger$ satisfies Eq. (3.6). Therefore, $A$ is the inverse of $A$ along $A^*$.

**Lemma 3.7.** [11, Theorem 2.1]. Let $C \in \mathbb{B}(\mathcal{H})$ and $A, B \in \mathbb{B}(\mathcal{H})$ have closed ranges. Then, the equation $AXB = C$ has a solution $X \in \mathbb{B}(\mathcal{H})$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A), \mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, and this if and only if $AA^\dagger CB^\dagger B = C$. In this case, $X = A^\dagger CB^\dagger + U - A^\dagger AUBB^\dagger$, where $U \in \mathbb{B}(\mathcal{H})$ is arbitrary.

In the next result, we provide a general solution of the system $BXA = B = AXB$.

**Theorem 3.8.** Let $A, B \in \mathbb{B}(\mathcal{H})$ have closed ranges and $B \lesssim A$. Then, the general solution of the system of operator equations $BXA = B = AXB$ is

$$X = A^\dagger BB^\dagger + A^\dagger \left[ B(I - AA^\dagger) + (A - B)^\dagger S \right] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B)$$

$$-A^\dagger B(I - AA^\dagger)(A - B)^\dagger BB^\dagger - A^\dagger (A - B)S(A - B)^\dagger BB^\dagger$$

$$-A^\dagger ATBB^\dagger + A^\dagger AT(A - B)^\dagger (A - B)BB^\dagger,$$

where $S, T \in \mathbb{B}(\mathcal{H})$.

**Proof.** Let $A, B$ have closed ranges. It follows from the assumption $B \lesssim A$ and Lemma 3.1 (d) that $B = AP_{\mathcal{R}(B)}$. Hence, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Using Lemma 3.2, we have $AA^\dagger B = B$. It follows from $AA^\dagger BB^\dagger B = B$ and Lemma 3.7 that the equation $AXB = B$ is solvable. In this case, the general solution is

$$X = A^\dagger BB^\dagger + W - A^\dagger AWBB^\dagger,$$  \hspace{1cm} (3.7)

where $W \in \mathbb{B}(\mathcal{H})$ is arbitrary. If $X$ satisfies the equation $BXA = B$, then

$$B(A^\dagger BB^\dagger + W - A^\dagger AWBB^\dagger)A = B.$$

It follows from the assumption $B \lesssim A$ and Lemma 3.1 (d) that $B = P_{\mathcal{R}(B)}A$. Applying (1.1), $BB^\dagger A = B$. Hence,

$$BA^\dagger B + BWA - BA^\dagger AWB = B.$$

Therefore, $B(A^\dagger B + WA - A^\dagger AWB) = B$. So, $A^\dagger B + WA - A^\dagger AWB$ is a solution of the equation $BX = B$. Utilizing Lemma 3.2 again, we have

$$A^\dagger B + WA - A^\dagger AWB = B^\dagger B + (I - B^\dagger B)S,$$  \hspace{1cm} (3.8)

where $S \in \mathbb{B}(\mathcal{H})$ is arbitrary. Multiply the left hand side of Eq. (3.8) by $A$, to get

$$AA^\dagger B + AW - AA^\dagger AWB = AB^\dagger B + A(I - B^\dagger B)S.$$
It follows from the assumption $B \preceq A$ and Lemma 3.1 (d) that $B = AP_{\mathcal{R}(B^*)}$. Applying (1.1), $AB^\dagger B = B$. We derive that

$$AA^\dagger B + AW A - AW B = B + (A - B)S.$$ 

Now, we get $AW(A - B) = B(I - AA^\dagger) + (A - B)S$. So, $W$ is a solution of the equation $AX(A - B) = B(I - AA^\dagger) + (A - B)S$. Using Lemma 3.7, we get that

$$W = A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B),$$

where $T \in \mathcal{B}(\mathcal{H})$ is arbitrary. By putting $W$ in Eq. (3.7), we reach

$$X = A^\dagger BB^\dagger + A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B)$$

$$- A^\dagger A(A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B)$$

$$+ T - A^\dagger AT(A - B)^\dagger (A - B)BB^\dagger$$

$$= A^\dagger BB^\dagger + A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B)$$

$$- A^\dagger A^\dagger A^\dagger (A - B)^\dagger BB^\dagger - A^\dagger (A - B)S(A - B)^\dagger BB^\dagger$$

$$- A^\dagger ATBB^\dagger + A^\dagger AT(A - B)^\dagger (A - B)BB^\dagger.$$ 

\textbf{Theorem 3.9.} Let $A, B \in \mathcal{B}(\mathcal{H})$ where $A$ has closed range. If the system $BX A = B = AX B$ is solvable, then the system $XB = A^\dagger B, BX = BA^\dagger$ is solvable. Conversely, if $B \preceq A$ and the system $XB = A^\dagger B, BX = BA^\dagger$ is solvable, then the system $BX A = B = AX B$ is solvable.

\textbf{Proof.} ($\Rightarrow$) : Let $\tilde{X}$ be a solution of the system $BX A = B = AX B$. It follows from $B = \tilde{A}X B$ that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Using Lemma 3.2, $AA^\dagger B = B$. It follows from (1.1) that

$$P_{\mathcal{R}(A^*)} \tilde{X} AA^\dagger B = (A^\dagger A)\tilde{X} (AA^\dagger) B = (A^\dagger A)\tilde{X} (AA^\dagger) B = A^\dagger (A\tilde{X} B) = A^\dagger B.$$ 

So, $P_{\mathcal{R}(A^*)} \tilde{X} AA^\dagger$ is a solution of the equation $XB = A^\dagger B$. Since $B^* = (B\tilde{X} A)^* = A^* \tilde{X} B^*$, we have $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$. Applying Lemma 2.1, there exists $Y \in \mathcal{B}(\mathcal{H})$ such that $B = YA$. Hence,

$$BP_{\mathcal{R}(A^*)} \tilde{X} AA^\dagger = B(A^\dagger A)\tilde{X} (AA^\dagger) = Y(A^\dagger A)\tilde{X} (AA^\dagger)$$

$$= (YA\tilde{X} A)A^\dagger = (B\tilde{X} A)A^\dagger = BA^\dagger.$$ 

Therefore, $P_{\mathcal{R}(A^*)} \tilde{X} AA^\dagger$ is a solution of the equation $B = BA^\dagger$. Thus $P_{\mathcal{R}(A^*)} \tilde{X} AA^\dagger$ is a solution of the system $XB = A^\dagger B, BX = BA^\dagger$.

($\Leftarrow$) : Suppose that $\tilde{X}$ is a solution of the system $XB = A^\dagger B, BX = BA^\dagger$. It follows from the assumption $B \preceq A$ that $B = AP_{\mathcal{R}(B^*)}$ and $B = P_{\mathcal{R}(B)} A$. Hence, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$. It follows from $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ to Lemma 3.2 that $AA^\dagger B = B$. Hence, $AX B = A(A^\dagger B) = AA^\dagger B = B$. It follows from $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$ and Lemma 2.1 that there exists $Z^* \in \mathcal{B}(\mathcal{H})$ such that $B = ZA$. Hence,

$$B\tilde{X} A = (BA^\dagger)A = BA^\dagger A = ZAA^\dagger A = ZA = B.$$ 

Therefore, $\tilde{X}$ is a solution of the system $BX A = B = AX B$. \hfill $\Box$
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**Lemma 3.10.** [4, Theorem 4.2]. Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$ and $A, B, M = B^*(I - A^1A)$ have closed ranges. Then, the system $AX = C$, $XB = D$ has a hermitian solution $X \in \mathbb{B}(\mathcal{H})$ if and only if

$$AA^1C = C, \quad DB^1B = D, \quad AD = CB$$

and $AC^*$ and $B^*D$ are hermitian. In this case, the general hermitian solution is


where $W \in \mathbb{B}(\mathcal{H})$ is hermitian and $s(T) = D^* - B^*A^1C$ is the so-called Schur complement of the block matrix $T = \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix}$.

**Theorem 3.11.** Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ have closed ranges. If $B \leq A$ and $B^*A^1B, BA^1B$ are hermitian, then the system $BXA = B = AXB$ has a hermitian solution.

**Proof.** Replace $A, B, C, D$ in Lemma 3.10 by $B, B, BA^1, A^1B$ to get

$$AA^1C = BB^1(BA^1) = BA^1 = C, \quad DB^1B = (A^1B)B^1B = A^1B = D$$

and

$$AD = B(A^1B) = (BA^1)B = CB, \quad AC^* = B(BA^1)^* = BA^1B^*, \quad B^*D = B^*A^1B.$$

Using Lemma 3.10, the system $XB = A^1B, BX = BA^1$ has a hermitian solution, say, $\tilde{X}$. It follows from the assumption $B \leq A$ that $B = AP_{\mathcal{R}(B)}$ and $B = P_{\mathcal{R}(B)}A$. Hence, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$.

It follows from $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and Lemma 3.2 that $AA^1B = B$. Hence, $A\tilde{X}B = A(A^1B) = AA^1B = B$. It follows from $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$ and Lemma 2.1 that there exists $Z \in \mathbb{B}(\mathcal{H})$ such that $B = ZA$. Hence,

$$B\tilde{X}A = (BA^1)A = BA^1A = ZAA^1A = ZA = B.$$

Therefore, $\tilde{X}$ is a hermitian solution of the system $BXA = B = AXB$. \qed

**4. $\ast$-Order via other operator equations.** Generally speaking, the inequality $PB \leq B$ dose not hold for any $P \in \mathcal{P}(\mathcal{H})$ even if $\mathcal{R}(P) \subseteq \mathcal{R}(B)$. In [2, Lemma 2.6], some conditions are mentioned which give a one-sided description of the relation $A \leq B$ regarding (1.2).

The next result is known.

**Proposition 4.1.** [2, Proposition 2.6]. Let $B \in \mathbb{B}(\mathcal{H})$.

(a) If $P \in \mathcal{O}(\mathcal{P}(\mathcal{H})$ and $\mathcal{R}(P) \subseteq \mathcal{R}(B)$, then $PB \leq B$ if and only if $PBB^* = BB^*P$.

(b) If $Q \in \mathcal{O}(\mathcal{P}(\mathcal{H})$ and $\mathcal{R}(Q) \subseteq \mathcal{R}(B^*)$, then $BQ \leq B$ if and only if $QB^*B = B^*BQ$.

In the following, we state a generalization of Proposition 4.1.

**Proposition 4.2.** Let $B \in \mathbb{B}(\mathcal{H})$. If there exist $P, Q \in \mathcal{O}(\mathcal{P}(\mathcal{H})$ such that $\mathcal{R}(P) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(Q) \subseteq \mathcal{R}(B^*)$, then $PBQ \leq B$ if and only if $PBQB^* = BQB^*P$ and $QB^*PB = B^*PBQ$. 

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\textbf{Proof.} (\(\Rightarrow\)) : Let \(PBQ\leq B\). Applying (1.2), we get that
\[PBQB^* = (PBQ)B^* = B(PBQ)^* = BQB^*P\]
and
\[B^*PBQ = B^*(PBQ) = (PBQ)^*B = QB^*PB.\]

(\(\Leftarrow\) ) : Let \(PBQB^* = BQB^*P\) and \(QB^*PB = B^*PBQ\). Applying (1.2), we obtain that
\[(PBQ)(PBQ)^* = PBQB^*P = (BQB^*)P = QB^*P = B(PBQ)^*\]
and
\[(PBQ)^*(PBQ) = Q(B^*PB) = QB^*PB = (PBQ)^*B.\] \(\Box\)

The next known theorem gives a characterization of the order \(\leq^*\).

\textbf{Theorem 4.3.} [6, Theorem 2.3]. Let \(A \in \mathcal{B}(\mathcal{H})\) and \(C \in \mathcal{D}(\mathcal{H})\). Then, \(C\leq^* A\) if and only if there exists \(X \in \mathcal{B}(\mathcal{H})\) such that \(A = C + (I - C^*)X(I - C^*)\).

In the following, we establish an analogue of Theorem 4.3 for generalized projections on a Hilbert space. Recall that an operator \(A \in \mathcal{B}(\mathcal{H})\) is a generalized projection if \(A^2 = A^*\).

\textbf{Lemma 4.4.} [14, Theorem A.2]. Let \(A \in \mathcal{B}(\mathcal{H})\) be a generalized projection. Then, \(A\) is a closed range operator and \(A^3\) is an orthogonal projection on \(\mathcal{N}(A)\). Moreover, \(\mathcal{H}\) has decomposition
\[\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)\]
and \(A\) has the following matrix representation
\[A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},\]
where the restriction \(A_1 = A|\mathcal{R}(A)\) is unitary on \(\mathcal{R}(A)\).

\textbf{Theorem 4.5.} Let \(A \in \mathcal{B}(\mathcal{H})\) and \(B \in \mathcal{D}(\mathcal{H})\). Then, \(B\leq^* A\) if and only if there exists \(X \in \mathcal{B}(\mathcal{H})\) such that \(A = B + (I - BB^*)X(I - B^*B)\).

\textbf{Proof.} (\(\Rightarrow\)) : Let \(B \in \mathcal{D}(\mathcal{H})\) and \(B \leq A\). Employing Lemma 4.4, we infer that \(B\) has closed range and \(B^3 = P_{\mathcal{N}(B)}\). It follows from (1.1) that
\[\mathcal{R}(B^*) = \mathcal{R}(B^*B) = \mathcal{R}(B^3) = \mathcal{R}(BB^*) = \mathcal{R}(B)\]
Hence, \(P_{\mathcal{R}(B)} = P_{\mathcal{R}(B^*)} = BB^* = B^*B\). Therefore, \(P_{\mathcal{N}(B^*)} = I - BB^* = I - B^*B\). Applying Lemma 3.1 (c), we get \(A = B + P_{\mathcal{N}(B^*)}AP_{\mathcal{N}(B)}\). Hence, \(A = B + (I - BB^*)A(I - B^*B)\).

(\(\Leftarrow\) ) : Let \(X \in \mathcal{B}(\mathcal{H})\) be a solution of the equation \(A = B + (I - BB^*)X(I - B^*B)\). Since \(B\) is a generalized projection, so \(B^*BB^* = B^*\). Hence,
\[B^*A = B^*B + B^*(I - BB^*)X(I - B^*B) = B^*B\]
and
\[AB^* = BB^* + (I - BB^*)X(I - B^*B)B^* = BB^*\]
Therefore, \(B \leq A\) by (1.2). \(\Box\)
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In the next result, we show that if $A$ is a generalized projection and $B \leq A \wedge A^*$, then $AA^*$ can be written as the sum of two idempotents.

**Theorem 4.6.** Let $A \in \mathcal{G}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. If $B \leq A \wedge A^*$, then $B$ is an idempotent and there exists an idempotent $X$ such that $AA^* = B + X$ and $B^*X = XB^* = 0$.

**Proof.** Let $B \leq A \wedge A^*$. It follows from the assumption $A^2 = A^*$ and Lemma 3.1 (d) that $B^2 = (\mathcal{P}_{\mathcal{B}}(B^*)A^*)^2 = \mathcal{P}_{\mathcal{B}}(B^*)A^2 = \mathcal{P}_{\mathcal{B}}(B^*)AA^* = B$. Using Lemma 3.1, we get that

$$AB = A(AP_{\mathcal{B}}(B^*)) = A^2P_{\mathcal{B}}(B^*) = A^*P_{\mathcal{B}}(B^*) = B,$$

$$BA = (P_{\mathcal{B}}(A))A = P_{\mathcal{B}}(A^2) = P_{\mathcal{B}}(A^*) = B,$$

$$A^*B = A^*(A^*P_{\mathcal{B}}(B^*)) = A^*P_{\mathcal{B}}(B^*) = AP_{\mathcal{B}}(B^*) = B$$

and

$$BA^* = (P_{\mathcal{B}}(A^*)A^*) = P_{\mathcal{B}}(A^*)A^2 = P_{\mathcal{B}}(A^*)A = B.$$

Let $X = AA^* - B$. It follows from the assumption $B \leq A \wedge A^*$ that

$$X^2 = (AA^* - B)^2 = (AA^*)^2 + B^2 - AA^*B - BAA^*$$

$$= AA^* + B - AB - BA^*$$

$$= AA^* + B - B - B = AA^* - B = X.$$

Hence, $X$ is an idempotent. Applying (1.2), we have

$$B^*X = B^*(AA^* - B) = B^*AA^* - B^*B = B^*A - B^*B = B^*A - B^*B = 0$$

and

$$XB^* = (AA^* - B)B^* = AA^*B^* - BB^* = AB^* - BB^* = 0. \quad \Box$$

**Lemma 4.7.** Let $A \in \mathcal{G}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then, $B \leq A$ if and only if $B$ is an idempotent and there exists an idempotent $X$ such that $A = B + X$ and $B^*X = XB^* = 0$.

**Proof.** ($\Rightarrow$): Let $B \leq A$. It follows from the assumption $A^2 = A$ and Lemma 3.1 (d) that $B^2 = (\mathcal{P}_{\mathcal{B}}(A))A^2 = \mathcal{P}_{\mathcal{B}}(A^2) = \mathcal{P}_{\mathcal{B}}(A) = BP_{\mathcal{B}}(B^*) = B$. Utilizing Lemma 3.1 (d), we obtain that

$$AB = A(AP_{\mathcal{B}}(B^*)) = A^2P_{\mathcal{B}}(B^*) = AP_{\mathcal{B}}(B^*) = B$$

and

$$BA = (P_{\mathcal{B}}(A))A = P_{\mathcal{B}}(A^2) = P_{\mathcal{B}}(A^*) = B.$$

Hence, $X = A - B$ is an idempotent and $B^*X = B^*(A - B) = 0$ and $XB^* = (A - B)B^* = 0$.

($\Leftarrow$): Let $A = B + X$ and $B^*X = XB^* = 0$ for some idempotent $X$. Then, $B^*(A - B) = B^*X = 0$ and $(A - B)B^* = XB^* = 0$. Therefore, $B \leq A$ by (1.2). \quad \Box
Corollary 4.8. Let $A \in \mathcal{G}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then, $B \preceq AA^*$ if and only if $B$ is an idempotent and there exists an idempotent $X$ such that $AA^* = B + X$ and $B^*X = XB^* = 0$.

Proof. Let $A \in \mathcal{G}(\mathcal{H})$. Then, $(AA^*)^2 = AA^*AA^* = AA^*$. Hence, $AA^*$ is an idempotent. Now apply Lemma 4.7.

We end our work with the following result.

Proposition 4.9. Let $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{G}(\mathcal{H})$. Then, $B \in \mathcal{B}(\mathcal{H})$ is common $\preceq$ lower bound of $A$ and $CC^*$ if and only if $B$ is an idempotent and there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that

$$A = B + (I - B^*)X(I - B^*) \quad \text{and} \quad CC^* = B + Y,$$

where $B^*Y = YB^* = 0$.

Proof. ($\Rightarrow$): If $B$ be a common $\preceq$ lower bound of $A$ and $CC^*$, then $B \prec A$ and $B \prec CC^*$. It follows from the assumption $B \preceq CC^*$ and Lemma 4.7 that $B$ is an idempotent and there exists an idempotent $Y \in \mathcal{B}(\mathcal{H})$ such that $CC^* = B + R$, where $B^*R = RB^* = 0$. Since $B$ is an idempotent and $B \preceq A$, by Theorem 4.3, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $A = B + (I - B^*)S(I - B^*)$.

($\Leftarrow$): If there exists an idempotent $Y$ such that $CC^* = B + Y$ with $B^*Y = 0$ and $YB^* = 0$, then $B \preceq CC^*$. The assumption $A = B + (I - B^*)S(I - B^*)$ and the fact that $B$ is an idempotent yield $B^*(A - B) = 0$ and $(A - B)B^* = 0$. Hence, $B \preceq A$ and $B$ is a common $\preceq$ lower bound of $A$ and $CC^*$.

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