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THE HAFNIAN AND A COMMUTATIVE ANALOGUE OF THE GRASSMANN ALGEBRA*

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Abstract. A close relationship between the determinant, the pfaffian, and the Grassmann algebra is well-known. In this paper, a similar relation between the permanent, the hafnian, and a commutative analogue of the Grassmann algebra is described. Using the latter, some new properties of the hafnian are proved.

Key words. Hafnian, Permanent, Pfaffian, Determinant, Grassmann algebra.

AMS subject classifications. 15A15, 15A75, 15A78.

1. Introduction.

The hafnian was introduced by Caianiello in the middle of 20th century in the work [1] connected with some questions of quantum field theory. The hafnian has a simple combinatorial interpretation and can be used to enumerate the 1-factors (perfect matchings) of undirected graphs [2], [3]. Its older and more famous "brother" is the pfaffian introduced by A. Cayley in the middle of 19th century. The relationship between these polynomial functions is the same as between the determinant and the permanent. The hafnian, unlike pfaffian, "ignores" the signs of permutations, and this difference significantly affects its properties.

It is well known (see e.g. [4] and [5]) that one can use Grassmann algebra to formulate the definition and properties of pfaffian. The aim of this paper is to show how one can give the definition of the hafnian and derive its properties in a similar way by replacing the Grassmann algebra with a commutative associative algebra with nilpotent (index 2) generators (a commutative analogue of the Grassmann algebra). Using this approach, we succeeded to prove several properties of hafnian analogous to some well known properties of pfaffian, which, to the best of our knowledge, have not been described in literature.

2. Definitions.

First we give a definition of a commutative analogue of Grassmann algebra. Let K be a field of characteristic zero, e.g., the field of real or complex numbers. Consider an n-dimensional vector space V over K with basis $\iota_1, \iota_2, \ldots, \iota_n$. Let $P_n(\iota)$ be a new vector space whose basis consists of all formal products

$$\iota_{k_1}\iota_{k_2}\cdots\iota_{k_t}, \quad k_1 < k_2 < \cdots < k_t, \quad 0 \le t \le n.$$

The empty product ($p = 0$) corresponds to 1. Thus, $P_n(\iota)$ is 2^n-dimensional and each of its elements is represented uniquely in the following form:

$$p = p_0 + \sum_{t=1}^{n} \sum_{k_1<\cdots<k_t} p_{k_1\cdots k_t}\iota_{k_1}\cdots\iota_{k_t}, \quad p_0, p_{k_1\cdots k_t} \in K. \tag{2.1}$$

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We endow $P_n(\iota)$ with the structure of a commutative associative unital algebra over K by the rule

$$\iota_{q_1}\cdots\iota_{q_s}\cdot\iota_{r_1}\cdots\iota_{r_t} = \begin{cases} 0 & \text{if } \{q_1,\ldots,q_s\}\cap\{r_1,\ldots,r_t\}\neq\emptyset, \\ \iota_{k_1}\cdots\iota_{k_{s+t}} & \text{otherwise,}\end{cases} \tag{2.2}$$

where $\{k_1,\ldots,k_{s+t}\}$ is the union of $\{q_1,\ldots,q_s\}$ and $\{r_1,\ldots,r_t\}$, taken in increasing order. In terms of generating elements and defining relations $P_n(\iota)$ is an algebra generated over K by 1 and the elements $\iota_1,\ldots,\iota_n$, subject to relations:

$$\iota_k^2=0,\ \ \iota_k\iota_l=\iota_l\iota_k,\ \ (\iota_k\iota_l)\iota_m=\iota_k(\iota_l\iota_m),\quad 1\le k,l,m\le n. \tag{2.3}$$

Note that if we replace the first two relations in (2.3) by $\iota_k\iota_l=-\iota_l\iota_k$, we obtain the definition of the Grassmann algebra.

Let $m=\lambda\iota_{k_1}\cdots\iota_{k_t}$, $k_1<k_2<\cdots<k_t$, $\lambda\in K$ be a monomial. We say that the number t of generating elements of $P_n(\iota)$ in m is the length of m. An element $p\in P_n(\iota)$ is called homogeneous of degree t, $t\le n$, if its decomposition in the basis vectors (2.1) involves only monomials of length t. Each vector of the space V can be considered as a homogeneous element of degree 1 of algebra $P_n(\iota)$.

Now we give two simple properties of the algebra $P_n(\iota)$, which will be used below.

Proposition 2.1. *The square of a nonzero homogeneous element p of degree 1 in $P_n(\iota)$ is equal to zero if and only if p is proportional to one of the generators ι_k, $k=1,\ldots,n$ with some coefficient in K.*

Proof. The equality $(\alpha\iota_k)^2=0$ follows directly from the multiplication rule (2.2) or from the defining relations (2.3). Now let $p=\sum_{k=1}^{n}\alpha_k\iota_k$ be an arbitrary nonzero homogeneous element of degree one in $P_n(\iota)$. Then $p^2=\sum_{k<l}^{n}2\alpha_k\alpha_l\iota_k\iota_l$. Thus, if at least two different coefficients α_i and α_j are not equal to zero, then p^2 is not zero. □

Proposition 2.2. *Let p be a homogeneous element of degree m in $P_n(\iota)$. Then $p^d=0$ if $d>\frac{n}{m}$.*

Proof. Let $p=\sum\lambda_{k_1k_2\cdots k_m}\iota_{k_1}\iota_{k_2}\cdots\iota_{k_m}$, $\lambda_{k_1k_2\cdots k_m}\in K$. Then each summand in p^d is proportional to the product of dm generating elements of $P_n(\iota)$: $p^d=\sum\mu_{k_1k_2\cdots k_{dm}}\iota_{k_1}\iota_{k_2}\cdots\iota_{k_{dm}}$, $\mu_{k_1k_2\cdots k_{dm}}\in K$. Since the number of different generating elements is n, and $dm>n$, then each summand in p^d has at least two of the same generating elements. Taking into account (2.2) or (2.3), we get that each summand in p^d is zero. □

Let $A=(a_{ij})$ be an $n\times n$ matrix over K. Recall that the permanent of A is the number

$$\operatorname{per}(A)=\sum_{\sigma}a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}, \tag{2.4}$$

where the sum runs over all permutations of the set $\{1,2,\ldots,n\}$. We are going to show that there exists a relation between the permanent and algebra $P_n(\iota)$ analogous to relation between the determinant and Grassmann algebra. Consider m vectors in the space V $(m\le n)$:

$$\begin{aligned} p_1 &= a_{11}\iota_1+a_{12}\iota_2+\cdots+a_{1n}\iota_n,\\ p_2 &= a_{21}\iota_1+a_{22}\iota_2+\cdots+a_{2n}\iota_n,\\ &\vdots \qquad\qquad \vdots\\ p_m &= a_{m1}\iota_1+a_{m2}\iota_2+\cdots+a_{mn}\iota_n.\end{aligned}$$

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Let A be a rectangular matrix composed of the coordinates of these vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Now let $A_{k_1 k_2 \cdots k_m}$ denote the matrix formed by the columns of A with numbers $k_1 < k_2 < \cdots < k_m$. Then it is not hard to see that

$$p_1 p_2 \cdots p_m = \sum_{k_1 < k_2 < \cdots < k_m} \operatorname{per}(A_{k_1 k_2 \cdots k_m}) \iota_{k_1} \iota_{k_2} \cdots \iota_{k_m}.$$

In particular, for $m = n$, we get

$$p_1 p_2 \cdots p_n = \operatorname{per}(A) \iota_1 \iota_2 \cdots \iota_n. \tag{2.5}$$

The last relation is given in the monograph [3] of H. Mink, with reference to T. Muir.

We give now several equivalent definitions of the hafnian. Let $A = (a_{ij})$ be a symmetric matrix of order $n = 2m$. Its hafnian is defined as

$$\operatorname{Hf}(A) = \sum_{(i_1 i_2 | \cdots | i_{n-1} i_n)} a_{i_1 i_2} \cdots a_{i_{n-1} i_n},$$

where the sum runs over all decompositions of the set $\{1, 2, \ldots, n\}$ into disjoint pairs

$$(i_1, i_2), \ldots, (i_{n-1} i_n)$$

up to an order of pairs and an order of elements in each pair.

Equivalently, one can define the hafnian as:

$$\operatorname{Hf}(A) = \frac{1}{(m!) 2^m} \sum_{\sigma \in S_n} a_{\sigma(1), \sigma(2)} \cdots a_{\sigma(n-1), \sigma(n)}.$$

Finally, we give a third way to define the hafnian using the algebra $P_n(\iota)$ (similar to definition of the pfaffian of an antisymmetric matrix through Grassmann algebra). Consider the following element of degree 2 in $P_n(\iota)$:

$$a = \sum_{\substack{s,t=1 \\ s<t}}^{n} a_{st} \iota_s \iota_t = \frac{1}{2} \sum_{s,t=1}^{n} a_{st} \iota_s \iota_t.$$

Then, as one easily verifies,

$$\frac{a^m}{m!} = \operatorname{Hf}(A) \iota_1 \cdots \iota_n. \tag{2.6}$$

We assume that the hafnian of any matrix of odd order is zero.

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3. Properties of the hafnian. In this section, we will prove several properties of the hafnian using formula (2.6). These properties are analogues of the corresponding well known properties of the pfaffian, but, to the best of our knowledge, they have not yet been described in the mathematical literature.

The first property is intuitively clear, however we give a detailed proof.

Theorem 3.1. *Let A be a symmetric block-diagonal matrix:*

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}.$$

Then

$$\mathrm{Hf}(A) = \mathrm{Hf}(A_1)\mathrm{Hf}(A_2)\cdots\mathrm{Hf}(A_k). \tag{3.7}$$

Proof. Let the order of A be n. It is enough to consider the case $n = 2m$, since the case of odd n is trivial. Let the blocks $A_1, A_2, \ldots, A_k$ have orders $n_1, n_2, \ldots, n_k$, respectively. The fact that the matrix $A = (a_{ij})$ has the block-diagonal form means that one can divide the element $a = \sum_{\substack{s,t=1\\ s<t}}^{n} a_{st}\iota_s\iota_t$ into k summands as follows:

$$a = \left(\sum_{\substack{s,t=1\\ s<t}}^{n_1} a_{st}\iota_s\iota_t + \sum_{\substack{s,t=n_1+1\\ s<t}}^{n_1+n_2} a_{st}\iota_s\iota_t + \cdots + \sum_{\substack{s,t=n-n_k+1\\ s<t}}^{n} a_{st}\iota_s\iota_t \right).$$

Then, by (2.6), we get

$$\begin{aligned} \mathrm{Hf}(A)\iota_1\cdots\iota_n = \frac{a^m}{m!} &= \frac{1}{m!}\left(\sum_{\substack{s,t=1\\ s<t}}^{n_1} a_{st}\iota_s\iota_t + \cdots + \sum_{\substack{s,t=n-n_k+1\\ s<t}}^{n} a_{st}\iota_s\iota_t \right)^m \\ &= \frac{1}{m!} \sum_{d_1+\cdots+d_k=m} \frac{m!}{(d_1!)\cdots(d_k!)} \left(\sum_{\substack{s,t=1\\ s<t}}^{n_1} a_{st}\iota_s\iota_t \right)^{d_1} \cdots \left(\sum_{\substack{s,t=n-n_k+1\\ s<t}}^{n} a_{st}\iota_s\iota_t \right)^{d_k} \\ &= \sum_{d_1+\cdots+d_k=m} \frac{1}{d_1!} \left(\sum_{\substack{s,t=1\\ s<t}}^{n_1} a_{st}\iota_s\iota_t \right)^{d_1} \cdots \frac{1}{d_k!} \left(\sum_{\substack{s,t=n-n_k+1\\ s<t}}^{n} a_{st}\iota_s\iota_t \right)^{d_k}. \end{aligned}$$

Since $d_1 + \cdots + d_k = \frac{1}{2}(n_1 + \cdots + n_k)$, so if $d_i < \frac{n_i}{2}$, then there exists d_j such that $d_j > \frac{n_j}{2}$. Note now that if $d_i > \frac{n_i}{2}$, then by Proposition 2.2, we have $(\sum a_{st}\iota_s\iota_t)^{d_i} = 0$. Thus, the unique nonzero summand in the last sum corresponds to the case $d_i = \frac{n_i}{2}$, $i = 1, \ldots, k$. Now we have to consider the following two cases.

1. Some of n_i are odd. Then, for each set $\{d_1, \ldots, d_k\}$ of the exponents, there exists a k such that $d_k \neq \frac{n_k}{2}$, and hence, as it was explained above, $\mathrm{Hf}(A)\iota_1\cdots\iota_n = 0$. On the other hand, according to the definition of the hafnian of a symmetric matrix of odd order, the right part of the equality (3.7) is also equal to zero in this case.

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2. All of n_i are even. Then, from the previous reasoning, we get:

$$\begin{aligned}\mathrm{Hf}(A)\iota_1\cdots\iota_n &= \frac{1}{(n_1/2)!}\left(\sum_{\substack{s,t=1\\ s<t}}^{n_1} a_{st}\iota_s\iota_t\right)^{n_1/2}\cdots\frac{1}{(n_k/2)!}\left(\sum_{\substack{s,t=n-n_k+1\\ s<t}}^{n} a_{st}\iota_s\iota_t\right)^{n_k/2}\\ &= (\mathrm{Hf}(A_1)\iota_1\cdots\iota_{n_1})\cdots(\mathrm{Hf}(A_k)\iota_{n-n_k+1}\cdots\iota_n) = \mathrm{Hf}(A_1)\cdots\mathrm{Hf}(A_k)\iota_1\cdots\iota_n. \qquad \square\end{aligned}$$

To formulate the second property, we now introduce some notations. Let $Q_{k,n}$ denote the set of all (unordered) k-element subsets of the set $\{1, 2, \dots, n\}$. Let A be a matrix of order n and $\alpha = \{p_1, \dots, p_k\} \in Q_{k,n}$. We let $A[\alpha]$ denote the submatrix of A formed by the rows and columns of A with numbers in α and $A(\alpha)$ denote the submatrix of A formed from A by removing the rows and columns with numbers in α. Let ι_α denote the element $\iota_{p_1}\cdots\iota_{p_k}$ of $P_n(\iota)$. We assume that if $\alpha \in Q_{0,n}$, then $\iota_\alpha = 1$.

Theorem 3.2. *Let A, B be symmetric matrices of even order $n = 2m$. Then*

$$\mathrm{Hf}(A+B) = \sum_{k=0}^{m}\sum_{\alpha\in Q_{2k,n}} \mathrm{Hf}(A[\alpha])\mathrm{Hf}(B(\alpha)), \tag{3.8}$$

where $\mathrm{Hf}(A[\alpha]) = 1$, *if* $\alpha \in Q_{0,n}$, *and* $\mathrm{Hf}(B(\alpha)) = 1$, *if* $\alpha \in Q_{n,n}$.

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$. Then

$$\begin{aligned}\mathrm{Hf}(A+B)\iota_1\cdots\iota_n &= \frac{1}{m!}\left(\sum_{s<t}(a_{st}+b_{st})\iota_s\iota_t\right)^m\\ &= \frac{1}{m!}\left(\sum_{s<t}a_{st}\iota_s\iota_t + \sum_{s<t}b_{st}\iota_s\iota_t\right)^m\\ &= \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}\left(\sum_{s<t}a_{st}\iota_s\iota_t\right)^k\left(\sum_{s<t}b_{st}\iota_s\iota_t\right)^{m-k}\\ &= \sum_{k=0}^{m}\frac{1}{k!}\left(\sum_{s<t}a_{st}\iota_s\iota_t\right)^k\frac{1}{(m-k)!}\left(\sum_{s<t}b_{st}\iota_s\iota_t\right)^{m-k}\\ &= \sum_{k=0}^{m}\left(\sum_{\alpha\in Q_{2k,n}}\mathrm{Hf}(A[\alpha])\iota_\alpha\right)\left(\sum_{\beta\in Q_{n-2k,n}}\mathrm{Hf}(B[\beta])\iota_\beta\right)\\ &= \left(\sum_{k=0}^{m}\sum_{\substack{\alpha\in Q_{2k,n}\\ \beta\in Q_{n-2k,n}\\ \alpha\cap\beta=\emptyset}}\mathrm{Hf}(A[\alpha])\mathrm{Hf}(B[\beta])\right)\iota_1\cdots\iota_n\\ &= \left(\sum_{k=0}^{m}\sum_{\alpha\in Q_{2k,n}}\mathrm{Hf}(A[\alpha])\mathrm{Hf}(B(\alpha))\right)\iota_1\cdots\iota_n. \qquad \square\end{aligned}$$

Recall that the pfaffian has the following property. Let A be antisymmetric matrix of even order and B

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any matrix of the same order. Then

$$\mathrm{Pf}(BAB^t) = \det(B)\mathrm{Pf}(A). \tag{3.9}$$

We give below an analogue of this property for hafnian.

THEOREM 3.3. *Let A be a symmetric matrix of even order $n = 2m$. Let P be arbitrary permutation matrix of order n, D arbitrary diagonal matrix of order n, and let $C = PD$. Then*

$$\mathrm{Hf}(CAC^t) = \mathrm{per}(C)\mathrm{Hf}(A). \tag{3.10}$$

Proof. One can apply a scheme similar to that used in [5] for proving (3.9) with the only difference that instead of Grassmann algebra it is necessary to use $P_n(\iota)$.

Assume first that all diagonal elements of D are nonzero. Consider elements $\iota'_1, \iota'_2, \dots, \iota'_n$ in $P_n(\iota)$ such that

$$(\iota'_1, \iota'_2, \dots, \iota'_n) = (\iota_1, \iota_2, \dots, \iota_n)C,$$

i.e.,

$$\iota'_k = \sum_{p=1}^{n} c_{pk}\iota_p, \quad k = 1, \dots, n. \tag{3.11}$$

By the definition of C, the right part of (3.11) contains only one nonzero summand, and thus, ι'_k is proportional to one of ι_p:

$$\iota'_k = \alpha\iota_p, \quad \alpha \in K.$$

Therefore, the elements ι'_k have the nilpotency property:

$$(\iota'_k)^2 = 0, \quad k = 1, \dots, n,$$

so one can use them, just as ι_k, for the definition of the hafnian (see (2.6)). Let $A = (a_{ij})$, and consider the element

$$a = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\iota'_i\iota'_j. \tag{3.12}$$

Substituting (3.11) here, we obtain

$$a = \frac{1}{2}\sum_{i,j,p,q=1}^{n} a_{ij}c_{pi}c_{qj}\iota_p\iota_q = \frac{1}{2}\sum_{p,q=1}^{n} a'_{pq}\iota_p\iota_q, \tag{3.13}$$

where $a'_{pq} = \sum_{i,j=1}^{n} c_{pi}a_{ij}c_{qj}$. Let $A' = (a'_{pq})$, then $A' = CAC^t$. Raising (3.12) and (3.13) to power m, we get

$$a^m = (m!)\mathrm{Hf}(A)\iota'_1\iota'_2\cdots\iota'_n = (m!)\mathrm{Hf}(A')\iota_1\iota_2\cdots\iota_n. \tag{3.14}$$

On the other hand, from (2.5), (3.11), and the assertion that the permanent of a matrix is invariant under transpose, we have

$$\iota'_1\iota'_2\cdots\iota'_n = \mathrm{per}(C)\iota_1\iota_2\cdots\iota_n. \tag{3.15}$$

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Substituting (3.15) in (3.14), we get

$$\mathrm{Hf}(A)\mathrm{per}(C) = \mathrm{Hf}(A') = \mathrm{Hf}(CAC^t).$$

If some diagonal element of D is zero, then C will have a zero row, and the matrix CAC^t will have a zero row and a zero column with the same numbers. Hence, from the definition of the permanent and the hafnian, we see that both sides of (3.10) are zeros. □

Corollary 3.4. *Let A be a symmetric matrix of even order, and P a permutation matrix of the same order. Then*

$$\mathrm{Hf}(PAP^t) = \mathrm{Hf}(A), \tag{3.16}$$

i.e., the hafnian of a matrix is not changed if its rows and columns are simultaneously subjected to the same permutation.

Remark 3.5. The scheme used in the above proof can not be applied to arbitrary matrix C, as in this case, by Proposition 2.1, the elements ι'_k will not retain the nilpotency property. However, Theorem 3.3 does not completely describe the entire class of those C for which the property (3.10) holds. For example, this property holds trivially for an arbitrary matrix C of order n having a zero row. In this connection, a question arises as to whether there are any other types of matrices, in addition to the one above, for which (3.10) holds?

4. Conclusion. In his first work [1], connected with hafnians, Caianiello already said, "It would be quite interesting, and probably useful, to have a complete and systematic treatment of the relations among pfaffians, hafnians, determinants and permanents.". And we can say that one of the effective tools for performing this task can be the Grassmann algebra and its commutative analogue - the algebra $P_n(\iota)$. In general, the triple (the permanent, the hafnian, the algebra $P_n(\iota)$) has more "obstinate" character in comparison with the triple (the determinant, the pfaffian, the Grassmann algebra). Thus, the hafnian is less harmonious and more difficult for the study than the pfaffian. But many properties of the pfaffian also have analogues for the hafnian. One can often obtain them, drawing parallels between the triples mentioned above.

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