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ON THE DISTANCE AND DISTANCE SIGNLESS LAPLACIAN EIGENVALUES OF GRAPHS AND THE SMALLEST GERŜGORIN DISC[*]

FOUZUL ATIK[†] AND PRATIMA PANIGRAHI[†]

Abstract. The *distance matrix* of a simple connected graph G is $D(G) = (d_{ij})$, where d_{ij} is the distance between the ith and jth vertices of G. The *distance signless Laplacian matrix* of the graph G is $D_Q(G) = D(G) + Tr(G)$, where $Tr(G)$ is a diagonal matrix whose ith diagonal entry is the transmission of the vertex i in G. In this paper, first, upper and lower bounds for the spectral radius of a nonnegative matrix are constructed. Applying this result, upper and lower bounds for the distance and distance signless Laplacian spectral radius of graphs are given, and the extremal graphs for these bounds are obtained. Also, upper bounds for the modulus of all distance (respectively, distance signless Laplacian) eigenvalues other than the distance (respectively, distance signless Laplacian) spectral radius of graphs are given. These bounds are probably first of their kind as the authors do not find in the literature any bound for these eigenvalues. Finally, for some classes of graphs, it is shown that all distance (respectively, distance signless Laplacian) eigenvalues other than the distance (respectively, distance signless Laplacian) spectral radius lie in the smallest Gerŝgorin disc of the distance (respectively, distance signless Laplacian) matrix.

Key words. Distance matrix, Distance eigenvalue, Distance spectral radius, Distance signless Laplacian matrix, Gerŝgorin disc.

AMS subject classifications. 05C50.

1. Introduction.

We consider an n-vertex simple connected graph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$ is the vertex set and E is the edge set of the graph. The *transmission* $tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. The graph G is said to be s-*transmission* regular if $tr(v) = s$ for every vertex $v \in V$. We denote the *maximum* and *minimum transmission* of G by tr_{max} and tr_{min}, respectively, that is $tr_{max} = \max\limits_{v \in V} tr(v)$ and $tr_{min} = \min\limits_{v \in V} tr(v)$. A real square matrix is said to be *nonnegative* (respectively, *positive*) if all of its entries are nonnegative (respectively, positive). A nonnegative matrix A is said to be *reducible* if there exists a permutation matrix P such that $P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B and D are square matrices of order r and $n-r$, respectively, and $1 \le r \le n-1$. If the matrix is not reducible then it is called *irreducible*. The largest eigenvalue of a matrix A is called the *spectral radius* of the matrix and is denoted by $\rho(A)$. By the well known Perron-Frobenius theorem [18], for irreducible matrix A, the eigenvalue $\rho(A)$ is simple and having a positive eigenvector. The *distance matrix* $D(G)$ of an n-vertex graph G is a square matrix of order n, whose (i, j)th entry is equal to d_{ij}, $i, j = 1, 2, \dots, n$, where d_{ij} is the distance (length of a shortest path) between the ith and jth vertices in G. We take $Tr(G)$ as a diagonal matrix whose ith diagonal entry is the transmission of the vertex $i \in V$. The *distance signless Laplacian matrix* of the graph G, introduced in [2], is denoted by $D_Q(G)$ and is defined by $D_Q(G) = D(G) + Tr(G)$. The matrices $D(G)$ and $D_Q(G)$ both are symmetric, non-negative and irreducible. The largest eigenvalue of $D(G)$ (respectively, $D_Q(G)$), denoted by $\rho(D(G))$ (respectively, $\rho(D_Q(G))$), is called the *distance spectral radius* (respectively, *distance signless Laplacian spectral radius*) of G. The eigenvalues and spectrum of $D(G)$ (respectively,

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$D_Q(G)$) are said to be the *distance eigenvalues* (respectively, *distance signless Laplacian eigenvalues*), in short D-eigenvalues, and *distance spectrum* (respectively, *distance signless Laplacian spectrum*), in short D-spectrum, of G, respectively.

The distance eigenvalues of graphs have been studied by researchers for many years. In the literature, very few graphs are there whose full distance spectrum are known. For early work, see Graham and Lovász [14], where they have discussed about the characteristic polynomial of distance matrix of a tree. Ruzieh and Powers [29] found all the eigenvalues and eigenvectors of the distance matrix of path P_n on n vertices. In [13], Fowler et al. determined all the distance eigenvalues of cycle C_n with n vertices. Ilić [19] characterized the distance spectrum of integral circulant graphs and calculated the D-spectrum of unitary Cayley graphs. Lin et al. [22] characterized all connected graphs with least D-eigenvalue -2 and all connected graphs of diameter 2 with exactly three distinct distance eigenvalues when distance spectral radius is not an integer. In [4], Atik and Panigrahi found the distance spectrum of some distance regular graphs including the well known Johnson graphs. Balaban et al. [5] proposed the use of distance spectral radius as a molecular descriptor, while Gutman and Medeleanu [15] successfully used it to infer the extent of branching and model boiling points of alkanes. The distance spectral radius is a useful molecular descriptor in QSPR modeling as demonstrated by Consonni and Todeschini [8, 30]. In [36] and [37], Zhou and Trinajstic provided upper and lower bounds for $\rho(D(G))$ in terms of the number of vertices, Wiener index and Zagreb index of G. Das [11] determined upper and lower bounds for $\rho(D(G))$ of a connected bipartite graph and characterized graphs for which these bounds are exact. Indulal [20] found sharp bounds on the distance spectral radius of graphs. Some results on least distance eigenvalue of graphs can be found in [23, 35]). For more results related to D-spectrum of graphs, readers may see the survey [1]. Also, very few results, mostly bounds on spectral radius, are there for distance signless Laplacian matrix of graphs. For results on $D_Q(G)$, researchers may follow [10, 17, 21, 32, 33].

Gerŝgorin disc theorem, which is stated in Section 5, is fundamental and widely used result on locating the eigenvalues of square matrices. Varga's nice book [31] surveys various applications and extensions of this important theorem. Recently, Marsli and Hall [25] found an interesting result, which states that if λ is an eigenvalue of an $n \times n$ matrix A with geometric multiplicity k, then λ is in at least k of the n Gerŝgorin discs of A. Fiedler et al. [12] proved that for a triple of positive integers $k,\ r,\ t$ with $k \le r \le t$, there is a $t \times t$ complex matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t, and λ is in precisely r Gerŝgorin discs of A. Marsli and Hall extended these results in subsequent papers [24, 26, 27]. Bárány and Solymosi [6] showed that if the matrix entries are non-negative and an eigenvalue has geometric multiplicity at least two, then this eigenvalue lies in a smaller disk.

In this paper, we find upper and lower bounds for the spectral radius of any nonnegative matrix. Applying this result we find upper and lower bounds for the distance and distance signless Laplacian spectral radius of graphs and characterize the graphs for which these bounds are extremal. Also, we give upper bounds for the modulus of all distance (respectively, distance signless Laplacian) eigenvalues other than the distance (respectively, distance signless Laplacian) spectral radius of graphs. These bounds are probably first of their kind as we do not find in the literature any bound for these eigenvalues. For graphs satisfying $tr_{max} - tr_{min} \le n-2$ (respectively, $2tr_{max} - 2tr_{min} \le n$) we get that all distance (respectively, distance signless Laplacian) eigenvalues other than the distance (respectively, distance signless Laplacian) spectral radius lie in the smallest Gerŝgorin disc of the distance (respectively, distance signless Laplacian) matrix. We also give an example of a class of graphs with $tr_{max} - tr_{min} > n-2$ and whose distance matrix satisfy the above property. In case of distance matrix or distance signless Laplacian matrix of a graph of order

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n, if it happens that all the eigenvalues other than the spectral radius lie in the smallest Gerŝgorin disc of that matrix, then it actually means that there are $n-1$ eigenvalues, whatever may be their geometric multiplicities, they lie in $n-1$ Gerŝgorin disc of that matrix (see Figure 2 and Figure 3).

Next we state few known results which will be used in the sequel.

THEOREM 1.1. [18] *Let A be a nonnegative square matrix. Suppose that there is a positive vector x and a nonnegative real number λ such that either $Ax = \lambda x$ or $x^T A = \lambda x^T$. Then $\lambda = \rho(A)$.*

THEOREM 1.2. [18] *Let $A = (a_{ij})$ and $B = (b_{ij})$ be nonnegative square matrices such that $A \geq B$ (that is $a_{ij} \geq b_{ij}$), then $\rho(A) \geq \rho(B)$.*

THEOREM 1.3. [16] *Let $A,\ B$ be nonnegative, irreducible square matrix such that $A \gneq B$, then $\rho(A) > \rho(B)$.*

THEOREM 1.4. [28] *If A is a nonnegative matrix with maximal eigenvalue λ and row sums $r_1, r_2, \ldots, r_n$, then*

$$r \leq \lambda \leq R,$$

where $r = \min_i r_i$ and $R = \max_i r_i$. If A is irreducible, then equality can hold on both sides if and only if all row sums of A are equal.

The following theorem is the well known Courant-Weyl inequalities.

THEOREM 1.5. [9] *Let $A,\ B$ be $n \times n$ Hermitian matrices and eigenvalues of them be in decreasing order. Then,*

$$\lambda_j(A+B) \leq \lambda_i(A) + \lambda_{j-i+1}(B) \quad \text{for } 1 \leq i \leq j \leq n,$$

and

$$\lambda_j(A+B) \geq \lambda_i(A) + \lambda_{j-i+n}(B) \quad \text{for } n \geq i \geq j \geq 1.$$

2. Spectral radius of nonnegative matrices. In this section, we develop some results on spectral radius of nonnegative matrices. These results are used in the next section for finding bounds on distance and distance signless Laplacian eigenvalues of graphs.

Consider a real square matrix A whose rows and columns are indexed by elements in $X = \{1, 2, \ldots, n\}$ and a partition $\pi = \{X_1, X_2, \ldots, X_m\}$ of X. The characteristic matrix $S = (s_{ij})$ of π is an $n \times m$ order matrix such that $s_{ij} = 1$ if $i \in X_j$ and 0 otherwise. We partition the matrix A according to π as $\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$, where each A_{ij} is a submatrix (block) of A whose rows and columns are indexed by elements of X_i and X_j, respectively. If q_{ij} denotes the average row sum of A_{ij} then the matrix $Q = (q_{ij})$ is called a *quotient matrix* of A. If the row sum of each block A_{ij} is a constant then the partition π is called *equitable*. The following is an well known result on equitable partition of a matrix.

THEOREM 2.1. [9] *Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.*

The following result appears as Corollary 3.9.11 in [9].

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Theorem 2.2. [9] *Any divisor of a graph G has the index of G as an eigenvalue.*

This result says that the adjacency spectral radius of a graph is same as the spectral radius of a quotient matrix corresponding to any equitable partition of the adjacency matrix of the graph.

In Theorem 2.3 below, we give a relation between the spectral radius of any nonnegative square matrix with that of its quotient matrices with respect to equitable partitions. A particular case of this theorem, that is for nonnegative irreducible symmetric matrices, appears in [3]. More interestingly, this was conjectured by You et al. [34].

Theorem 2.3. *The spectral radius of a nonnegative square matrix A is same as the spectral radius of a quotient matrix of it corresponding to an equitable partition.*

Proof. We consider two cases depending on A is irreducible or not. We take Q as a quotient matrix of A corresponding to an equitable partition π.

Case 1. Suppose the matrix A is irreducible. Then Q is also nonnegative and irreducible. By Perron-Frobenius theorem the spectral radius $\rho(Q)$ of Q is positive and having a positive eigenvector, say v, associated with it. Then by Theorem 2.1 $\rho(Q)$ is an eigenvalue of A. Also we get that Sv is an eigenvector of A corresponding to the eigenvalue $\rho(Q)$, where S is the characteristic matrix corresponding to the equitable partition π. Since S is non-negative and every row of S has exactly one positive entry, the vector Sv is positive. As $ASv = \rho(Q)Sv$, from Theorem 1.1 we get $\rho(A) = \rho(Q)$.

Case 2. Now we consider that the matrix A is reducible. The quotient matrix Q is nonnegative but may not be irreducible. If Q is irreducible then the proof is same as Case 1. Otherwise we we define a matrix $A_\epsilon = A + \epsilon J$, where $\epsilon > 0$ and J is the all one square matrix of the order same as A. Then clearly A_ϵ is nonnegative and irreducible. Let Q_ϵ be the quotient matrix of A_ϵ corresponding to the same equitable partition π. Then $Q_\epsilon = Q + \epsilon J'$, where J' is the quotient matrix of J corresponding to the same equitable partition π. By Case 1 we have $\rho(A_\epsilon) = \rho(Q_\epsilon)$. Since eigenvalues and eigenvectors are continuous functions of entries of the matrix, we get $\rho(A_\epsilon) \to \rho(A)$ and $\rho(Q_\epsilon) \to \rho(Q)$ as $\epsilon \to 0$. Again $\rho(Q_\epsilon) = \rho(A_\epsilon) \to \rho(A)$ when $\epsilon \to 0$. So we have $\rho(A) = \rho(Q)$. ∎

Notation 2.1. For any nonnegative matrix of order n, r_i denote the i-th row sum of the matrix, $i = 1, 2, \ldots, n$. We also denote the maximum and minimum row sum of the matrix by r_{max} and r_{min}, respectively.

In the below, we give upper and lower bounds for the spectral radius of a nonnegative matrix and find out extremal matrices.

Theorem 2.4. *Let $A = (a_{ij})_{n\times n}$ be a nonnegative square matrix with $r_{min} = r_p$ and $r_{max} = r_q$ for some p and q, $1 \le p, q \le n$. Let $S = \{1, 2, \ldots, n\}$, $l = \max_{i\in S\setminus\{p\}} \{r_i - a_{ip}\}$, $m = \min_{i\in S\setminus\{q\}} \{r_i - a_{iq}\}$, $s = \max_{i\in S\setminus\{p\}} a_{ip}$ and $t = \min_{i\in S\setminus\{q\}} a_{iq}$. Then the spectral radius of A satisfies*

$$\frac{a_{qq} + m + \sqrt{(m - a_{qq})^2 + 4t(r_{max} - a_{qq})}}{2} \le \rho(A) \le \frac{a_{pp} + l + \sqrt{(l - a_{pp})^2 + 4s(r_{min} - a_{pp})}}{2}.$$

Moreover, equality holds in the left side if $r_i - a_{iq} = m$ and $a_{iq} = t$ for all $i \in S \setminus \{q\}$, and in the right side if $r_i - a_{ip} = l$ and $a_{ip} = s$ for all $i \in S \setminus \{p\}$.

Proof. Without loss of generality, we assume that $p = 1$ (because otherwise we can permute the rows

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and columns of the matrix A). Then A can be partitioned in block form as

$$A = \begin{bmatrix} a_{11} & E \\ B & C \end{bmatrix}, \tag{2.1}$$

where $E = (e_{ij})$ is an $1 \times (n-1)$ matrix, $B = (b_{ij})$ is an $(n-1) \times 1$ matrix and $C = (c_{ij})$ is an $(n-1) \times (n-1)$ square matrix. The row sum of E is $r_{min} - a_{11}$. Consider an $(n-1) \times 1$ matrix $B_1 = (b'_{ij})_{(n-1)\times 1}$ whose all entries are s. If row sum of any row in C is less than l then by adding positive numbers to entries of that row we get a matrix $C_1 = (c'_{ij})_{(n-1)\times(n-1)}$ such that $c'_{ij} \geq c_{ij}$ for all i, j and $\sum\limits_{j=1}^{n-1} c_{ij} = l$ for all i. Let A_1 be the matrix as given below.

$$A_1 = \begin{bmatrix} a_{11} & E \\ B_1 & C_1 \end{bmatrix}. \tag{2.2}$$

If $A_1 = (a'_{ij})_{n\times n}$ then $a'_{ij} \geq a_{ij}$ for all $i, j \in S$, that is $A \leq A_1$. Then by Theorem 1.2 $\rho(A) \leq \rho(A_1)$. We note that each block of A_1 has constant row sum. Then the corresponding quotient matrix for A_1 is

$$Q_1 = \begin{bmatrix} a_{11} & r_{min} - a_{11} \\ s & l \end{bmatrix}.$$

Now A_1 is also nonnegative, so by Theorem 2.3 spectral radius of A_1 is same as the spectral radius of Q_1. Thus, $\rho(A_1) = \rho(Q_1) = \frac{l+a_{11}+\sqrt{(l-a_{11})^2+4s(r_{min}-a_{11})}}{2}$. So we have $\rho(A) \leq \frac{l+a_{11}+\sqrt{(l-a_{11})^2+4s(r_{min}-a_{11})}}{2}$. Equivalently, by considering the matrix A in its original form we have

$$\rho(A) \leq \frac{a_{pp} + l + \sqrt{(l-a_{pp})^2 + 4s(r_{min} - a_{pp})}}{2}.$$

If $r_i - a_{ip} = l$ and $a_{ip} = s$ for all $i \in S \setminus \{p\}$ then we must have $A = A_1$ and so equality holds in the above inequality.

Following the similar technique we get the lower bound for $\rho(A)$. We assume that $q = 1$ and partition the matrix A in block form as

$$A = \begin{bmatrix} a_{11} & E' \\ B' & C' \end{bmatrix}, \tag{2.3}$$

where row sum of the block matrix E' is $r_{max} - a_{11}$. Consider another matrix

$$A'_1 = \begin{bmatrix} a_{11} & E' \\ B'_1 & C'_1 \end{bmatrix}, \tag{2.4}$$

where all the entries of B'_1 are t, and C'_1 is obtained from C by reducing some positive entries of each row of C whose row sum is grater than m so that C'_1 is nonnegative and each row sum of C'_1 is m. Then we have $A \geq A'_1$. Also the quotient matrix for A'_1 is

$$Q'_1 = \begin{bmatrix} a_{11} & r_{max} - a_{11} \\ t & m \end{bmatrix}.$$

By applying Theorem 1.2 and Theorem 2.3, we get $\rho(A) \geq \rho(A'_1) = \rho(Q'_1) = \frac{m+a_{11}+\sqrt{(m-a_{11})^2+4t(r_{max}-a_{11})}}{2}$. Hence, if the matrix is in its original form then we have

$$\rho(A) \geq \frac{a_{qq} + m + \sqrt{(m-a_{qq})^2 + 4t(r_{max} - a_{qq})}}{2},$$

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and equality holds whenever $r_i - a_{iq} = m$ and $a_{iq} = t$ for all $i \in S \setminus \{q\}$. ∎

Remark 2.5. Consider the nonnegative matrix $A = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 7 & 1 \\ 2 & 3 & 5 \end{bmatrix}$. By applying Theorem 2.4 we get that $8.449 < \rho(A) < 9.464$. Whereas, Theorem 1.4 gives that $8 < \rho(A) < 10$. Hence, one may find several nonnegative matrices for which bounds given in Theorem 2.4 are sharper than those in Theorem 1.4.

3. Bounds on eigenvalues of the distance matrix.

In this section, we first define a class of graphs for which our upper bound for spectral radius of distance matrix of a graph will have extremal value.

Definition 3.1. A simple connected graph G on n vertices is said to be *distinguished vertex deleted regular graph (DVDR)* if there exist a vertex v in G such that $deg_G(v) = n - 1$ and $G - v$ is regular graph. The vertex v is said to be a *distinguished vertex* of the DVDR graph G.

We note that complete graphs, star graphs, and wheel graphs are some trivial examples of this class of graphs. Some non-trivial DVDR graphs are shown in Figure 1. In the following, we give a representation of DVDR graphs in terms of almost transmission regular graphs.

Definition 3.2. A simple connected graph G is called *almost transmission regular* if each vertex of G except one vertex v has the same transmission. We say this vertex v also a distinguished vertex of the graph.

Lemma 3.3. *Let G be a simple connected graph of order n different from a complete graph. Then G is a DVDR graph if and only if G is almost transmission regular graph with transmission of the distinguished vertex is $n - 1$.*

Proof. Suppose G is a DVDR graph with distinguished vertex v and degree of each vertex in $G - v$ is k $(\neq n - 2)$. Then $tr(v) = n - 1$ and $tr(u) = k + 2(n - k - 1)$ for every u in $V(G) \setminus \{v\}$. Thus, we get that G is an almost transmission regular graph. Conversely, suppose that in an n-vertex graph G there exists a vertex $v \in V(G)$ such that $tr(v) = n - 1$ and each vertex in $V(G) \setminus \{v\}$ has transmission s. Then for any $u \in V(G) \setminus \{v\}$ we have $deg_G(u) + 2(n - 1 - deg_G(u)) = s$. Thus, $deg_G(u) = 2(n - 1) - s$ for all $u \in V(G) \setminus \{v\}$. That is degree of every vertex in $G - v$ is $2(n - 1) - s - 1$. Hence, G is a DVDR graph with distinguished vertex v. ∎

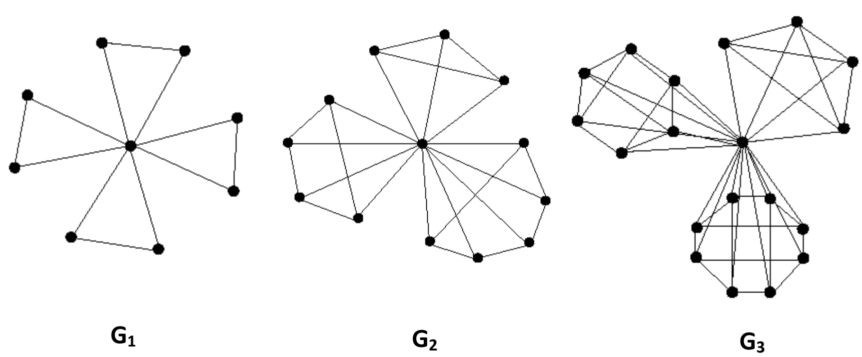

Figure 1. *Some examples of DVDR graphs.*

Next we present a theorem which gives upper and lower bounds for the distance spectral radius of a graph.

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Theorem 3.4. *Let G be a simple connected graph and u_1 and u_2 be vertices in G such that $tr(u_1) = tr_{min}$ and $tr(u_2) = tr_{max}$. Let $m_1 = \max\limits_{v\in V(G)\setminus u_1}\{tr(v) - d(u_1, v)\}$ and $m_2 = \min\limits_{v\in V(G)\setminus u_2}\{tr(v) - d(u_2, v)\}$. Then the distance spectral radius satisfies*

$$\frac{m_2 + \sqrt{m_2^2 + 4tr_{max}}}{2} \le \rho(D(G)) \le \frac{m_1 + \sqrt{m_1^2 + 4tr_{min}e(u_1)}}{2},$$

where $e(u_1)$ is the eccentricity of the vertex u_1. Moreover, equality holds on right side if and only if the graph is a DVDR graph and equality holds on left side if and only if the graph is a complete graph.

Proof. Recall that distance matrix of a graph is nonnegative and for this matrix the parameters in Theorem 2.4 are given by $a_{pp} = 0,\ l = m_1,\ r_{min} = tr_{min},\ s = e(u_1)$. Hence, from the upper bound of Theorem 2.4, we get

$$\rho(D(G)) \le \frac{m_1 + \sqrt{m_1^2 + 4tr_{min}e(u_1)}}{2}. \tag{3.5}$$

Suppose equality holds in (3.5). Since we are applying Theorem 2.4 taking $A = D(G)$, we get that $\rho(D(G))$ is same as the spectral radius of the matrix A_1. The matrix $D(G)$ as well as A_1 are nonnegative and irreducible. So by Theorem 1.3 we must have $D(G) = A_1$ and then $B = B_1$ and $C = C_1$. As all the entries of B_1 are equal to $e(u_1)$, $B = B_1$ is possible only when $e(u_1) = 1$. That is u_1 is adjacent to all other vertices of the graph. Again $C = C_1$ implies that $tr(v) = m_1 + 1$ for all $v \in V(G) \setminus u_1$. If $m_1 \ne n-2$, then by Lemma 3.3 G is a DVDR graph and if $m_1 = n-2$, then G is a complete graph, which is also a DVDR graph. Again if G is a DVDR graph then obviously equality holds in (3.5).

For the lower bound of $\rho(D(G))$ the parameters in Theorem 2.4 are given by $a_{qq} = 0,\ m = m_2,\ r_{max} = tr_{max},\ t = 1$. Hence, by the lower bound of Theorem 2.4, we get

$$\rho(D(G)) \ge \frac{m_2 + \sqrt{m_2^2 + 4tr_{max}}}{2}. \tag{3.6}$$

Suppose inequality (3.6) becomes equality. That is $\rho(D(G))$ is same as the spectral radius of the matrix A_1' in equation (2.4). Then by Theorem 1.3 we must have $D(G) = A_1'$. So we get $B' = B_1'$, and then u_2 is adjacent to all other vertices of the graph. Since $tr(u_2) = tr_{max}$, G must be a complete graph. Conversely, if G is a complete graph then obviously equality holds in (3.6). ∎

In the next two theorems, we give upper bounds for modulus of the distance eigenvalues of a graph other than the distance spectral radius. In the literature, we do not find any bound for these eigenvalues.

Theorem 3.5. *Let G be a simple connected graph and λ be any eigenvalue of $D(G)$ other than the distance spectral radius. Then the eigenvalue λ satisfies*

$$|\lambda| \le tr_{max} - n + 2.$$

Moreover, equality holds for all λ simultaneously if and only G is a complete graph.

Proof. We can write the distance matrix as $D(G) = A + B$, where A is the matrix whose all diagonal entries are zero and all other entries are one and B is a non-negative symmetric matrix.

For the matrix A we have $\lambda_1(A) = n-1$ and $\lambda_j(A) = -1$ for $j = 2, 3, \ldots, n$. The maximum row sum of the matrix B is $tr_{max} - n + 1$.

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Now applying Theorem 1.5, we get for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D(G)) = \lambda_j(B + A) \leq \lambda_1(B) + \lambda_j(A). \tag{3.7}$$

As B is a nonnegative matrix so we have $\lambda_1(B) \leq tr_{max} - n + 1$. Hence, putting the value of $\lambda_j(A)$ in (3.7), we get for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D(G)) \leq tr_{max} - n. \tag{3.8}$$

Again by applying Theorem 1.5, we get for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D(G)) = \lambda_j(A + B) \geq \lambda_j(A) + \lambda_n(B) = -1 + \lambda_n(B). \tag{3.9}$$

For the matrix B we have

$$|\lambda_n(B)| \leq \lambda_1(B) \leq tr_{max} - n + 1 \tag{3.10}$$

$$\Rightarrow \ \lambda_n(B) \geq -(tr_{max} - n + 1). \tag{3.11}$$

From (3.9) and (3.11), we get for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D(G)) \geq -(tr_{max} - n + 2). \tag{3.12}$$

Hence, from (3.8) and (3.12), the distance eigenvalue λ satisfies

$$|\lambda| \leq tr_{max} - n + 2. \tag{3.13}$$

Suppose that equality holds in (3.13) for all λ, that is $|\lambda_j(D(G))| = tr_{max} - n + 2$ for all $j = 2, 3, \ldots, n$. If for some j, $2 \leq j \leq n$, λ_j is positive, then we get contradiction from (3.8) because $tr_{max} - n + 2$ can not be smaller or equal to $tr_{max} - n$. Thus, we get $\lambda_j(D(G)) = -(tr_{max} - n + 2)$ for all $j = 2, 3, \ldots, n$. Then the graph G has only two distinct distance eigenvalues, and hence, G is a complete graph.

Conversely, assume that G is a complete graph. Then for the graph G, $|\lambda_j(D(G))| = |-1| = 1$ for all $j = 2, 3, \ldots, n$. Also in this case, $tr_{max} = n - 1$. So $tr_{max} - n + 2 = 1$ and equality holds in (3.13) for all such distance eigenvalues. ∎

THEOREM 3.6. *Let G be a simple connected graph and λ be any eigenvalue of $D(G)$ other than the distance spectral radius. Then λ satisfies*

$$|\lambda| \leq \frac{m_1 - n + 2 + \sqrt{(m_1 - n + 2)^2 + 4(tr_{min} - n + 1)(e(u_1) - 1)}}{2} + 1,$$

where the parameters are the same as given in Theorem 3.4. Moreover, equality holds for all λ simultaneously if and only if G a is complete graph.

Proof. We follow the same steps up to inequality (3.10) of Theorem 3.5. Then, by the upper bound of Theorem 2.4, we get

$$\lambda_1(B) \leq \frac{m_1 - n + 2 + \sqrt{(m_1 - n + 2)^2 + 4(tr_{min} - n + 1)(e(u_1) - 1)}}{2}.$$

Hence, in equations (3.7) and (3.10), replacing the upper bound of $\lambda_1(B)$ by this new upper bound, we get the desired bound of $\lambda_j(D(G))$ for $j \in \{2, 3, \ldots, n\}$.

By the similar argument as in the previous theorem, we get that equality holds for all λ if and only if the graph G is a complete graph. ∎

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4. Bounds on eigenvalues of the distance signless Laplacian matrix. The result below is an application of Theorem 2.4 and its proof technique is analogous to that of Theorem 3.4.

THEOREM 4.1. *Let G be a simple connected graph with u_1 and u_2 be two vertices in G such that $tr(u_1) = tr_{min}$ and $tr(u_2) = tr_{max}$. Let $l_1 = \max_{v\in V(G)\setminus u_1}\{2tr(v) - d(u_1, v)\}$ and $l_2 = \min_{v\in V(G)\setminus u_2}\{2tr(v) - d(u_2, v)\}$. Then the distance signless Laplacian spectral radius satisfies*

$$\frac{l_2 + tr_{max} + \sqrt{(l_2 - tr_{max})^2 + 4tr_{max}}}{2} \leq \rho(D_Q(G)) \leq \frac{l_1 + tr_{min} + \sqrt{(l_1 - tr_{min})^2 + 4tr_{min}e(u_1)}}{2}.$$

Moreover, equality holds on right side if and only if the graph is DVDR graph and equality holds on left side if and only if the graph is a complete graph.

In the following, we give upper and lower bounds to distance signless Laplacian eigenvalues other than the distance signless Laplacian spectral radius of graphs. This result is analogous to Theorem 3.5, but proof technique is slightly different.

THEOREM 4.2. *Let G be a simple connected graph and λ be any eigenvalue of $D_Q(G)$ other than the distance signless Laplacian spectral radius. Then λ satisfies*

$$\max\{(2tr_{min} - 2tr_{max} + n - 2), 0\} \leq \lambda \leq 2tr_{max} - n.$$

Moreover, equality holds on both sides for all λ simultaneously if G is a complete graph.

Proof. We partition the distance signless Laplacian matrix as $D_Q(G) = A + B$, where A is the matrix whose all diagonal entries are tr_{min} and all other entries are equal to one, and B is a non-negative symmetric matrix.

For the matrix A we have $\lambda_1(A) = tr_{min} + n - 1$ and $\lambda_j(A) = tr_{min} - 1$ for $j = 2, 3, \ldots, n$. Then maximum row sum of the matrix B is $2tr_{max} - tr_{min} - n + 1$.

Now applying Theorem 1.5, we get for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D_Q(G)) = \lambda_j(B + A) \leq \lambda_1(B) + \lambda_j(A). \tag{4.14}$$

As B is a nonnegative matrix, we get $\lambda_1(B) \leq 2tr_{max} - tr_{min} - n + 1$. Hence, putting the value of $\lambda_j(A)$ in (4.14) we get, for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D_Q(G)) \leq 2tr_{max} - n. \tag{4.15}$$

Applying Theorem 1.5 we get, for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D_Q(G)) = \lambda_j(A + B) \geq \lambda_j(A) + \lambda_n(B) = tr_{min} - 1 + \lambda_n(B). \tag{4.16}$$

For the matrix B, we have

$$|\lambda_n(B)| \leq \lambda_1(B) \leq 2tr_{max} - tr_{min} - n + 1 \tag{4.17}$$

$$\Rightarrow \lambda_n(B) \geq -(2tr_{max} - tr_{min} - n + 1). \tag{4.18}$$

From (4.16) and (4.18), we get, for $j \in \{2, 3, \ldots, n\}$,

$$\lambda_j(D_Q(G)) \geq 2tr_{min} - 2tr_{max} + n - 2. \tag{4.19}$$

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Again distance signless Laplacian matrix is a nonnegative diagonally dominant symmetric matrix, so all of its eigenvalues are nonnegative. Hence, from (4.15) and (4.19), the distance signless Laplacian eigenvalue λ satisfies

$$\max\{(2tr_{min} - 2tr_{max} + n - 2), 0\} \leq \lambda \leq 2tr_{max} - n. \tag{4.20}$$

If G is a complete graph then obviously both the inequalities in (4.20) become equality. □

Our next result is analogous to Theorem 3.6 and proof technique is also similar to its proof.

THEOREM 4.3. *Let G be a simple connected graph and λ be any eigenvalue of $D_Q(G)$ other than the distance signless Laplacian spectral radius. Then λ satisfies*

$$\max\{(tr_{min} - 1 - S), 0\} \leq \lambda \leq tr_{min} - 1 + S,$$

where

$$S = \frac{l_1 - tr_{min} - n + 2 + \sqrt{(l_1 - tr_{min} - n + 2)^2 + 4(tr_{min} - n + 1)(e(u_1) - 1)}}{2},$$

and the parameters in S are the same as defined in Theorem 4.1. Moreover, equality holds on both sides for all λ simultaneously if G is a complete graph.

5. Distance and distance signless Laplacian eigenvalues and the Gerŝgorin discs. The following well known Gerŝgorin disc theorem guarantees that eigenvalues of a square matrix are contained in some easily computed discs.

THEOREM 5.1. (Gerŝgorin) *Let $A = [a_{ij}] \in M_n$, $R_i(A) = \sum_{j \neq i} |a_{ij}|$, $i = 1, 2, \ldots, n$, and consider the n Gerŝgorin discs*

$$\{z \in C : |z - a_{ii}| \leq R_i(A)\},\ i = 1, 2, \ldots, n.$$

Then the eigenvalues of A are in the union of Gerŝgorin discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in C : |z - a_{ii}| \leq R_i(A)\}.$$

Furthermore, if the union of k of the n discs that comprise $G(A)$ forms a set $G_k(A)$ that is disjoint from the remaining $n - k$ discs, then $G_k(A)$ contains exactly k eigenvalues of A, counted according to their algebraic multiplicities.

The distance matrix $D(G)$ of a simple connected graph G is real symmetric and having diagonal entries all zero. Thus, all the Gerŝgorin discs of $D(G)$ are concentric circles with center at zero and radius of the discs are the transmissions of the vertices of G (see Figure 2). So, according to Theorem 5.1 all the eigenvalues of $D(G)$ are contained in the circle having center at zero and radius tr_{max} of G. Again all the Gerŝgorin discs for distance signless Laplacian matrix of a graph are circles passing through the origin with centers on the real axis and radius of the discs are the transmissions of the vertices of the graphs (see Figure 3).

The following result is immediate from Theorem 3.5.

THEOREM 5.2. *If G is a simple connected graph satisfying $tr_{max} - tr_{min} \leq n - 2$, then all the eigenvalues of $D(G)$, other than the distance spectral radius, lie in the smallest Gerŝgorin disc.*

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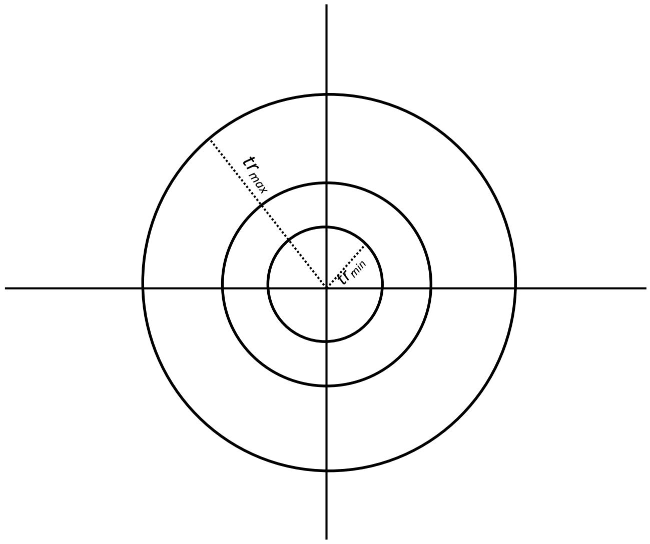

FIGURE 2. *Gerŝgorin discs for distance matrix of a graph.*

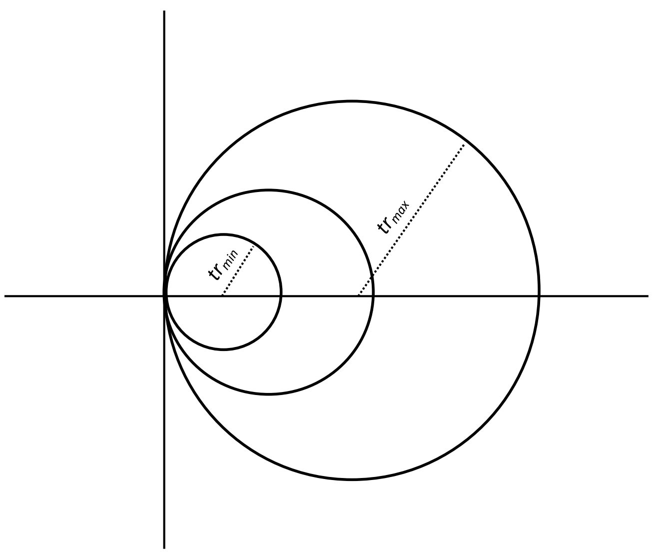

FIGURE 3. *Gerŝgorin discs for distance signless Laplacian matrix of a graph.*

For any nonnegative square matrix A we denote property $\mathcal{P}$ as below.

$\mathcal{P}$: All eigenvalues of the matrix A other than the spectral radius lie inside the smallest Gerŝgorin disc of A.

Next we give an example of a graph, namely the path $P_n, n \geq 5$, for which $tr_{max} - tr_{min} > n - 2$ and its distance matrix satisfies the property $\mathcal{P}$. The path P_n on n vertices is a graph in which $tr_{max} = \frac{n^2-n}{2}$

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and $tr_{min} = \frac{n^2}{4}$ or $\frac{n^2-1}{4}$ according as n is even or odd. Then for P_n with $n \geq 5$, $tr_{max} - tr_{min} > n-2$. In [29], Ruzieh and Powers have found all the distance eigenvalues of P_n, which are given below.

THEOREM 5.3. [29] *The distance eigenvalues* $d_i,\ i = 1, 2, \ldots, n$, *of the path* P_n *with* $n > 2$ *are as follows.*

1. $d_1 = 1/(\cosh\theta - 1)$, *where* θ *is the positive solution of* $\tanh(\theta/2)\tanh(n\theta/2) = 1/n$.
2. $d_i = 1/(\cos\theta - 1)$, *where (a)* θ *is the one of the* $[(n-1)/2]$ *solutions of* $\tanh(\theta/2)\tanh(n\theta/2) = -1/n$ *in the interval* $(0, \pi)$, *or (b)* $\theta = (2m-1)\pi/n$ *for* $m = 1, 2, \ldots, [n/2]$.

One sees that d_1 is the distance spectral radius of P_n and other distance eigenvalues are given by $1/(\cos\theta - 1)$, where θ lies in the interval $[\frac{\pi}{n}, \frac{(n-1)\pi}{n}]$. We note that $f(\theta) = 1/(\cos\theta - 1)$ is a strictly increasing function and $f(\theta) < 0$ in $[\frac{\pi}{n}, \frac{(n-1)\pi}{n}]$. So among all distance eigenvalues of P_n, other than the spectral radius, $f(\pi/n)$ is the smallest one. As all of them are negative so the eigenvalue $f(\pi/n)$ have maximum modulus. Now

$$\begin{aligned}
1 - \cos(\pi/n) &= 1 - \left[1 - \frac{\pi^2}{n^2 2!} + \frac{\pi^4}{n^4 4!} - \frac{\pi^6}{n^6 6!} + \frac{\pi^8}{n^8 8!} - \cdots\right] \\
&= \left(\frac{\pi^2}{n^2 2!} - \frac{\pi^4}{n^4 4!}\right) + \left(\frac{\pi^6}{n^6 6!} - \frac{\pi^8}{n^8 8!}\right) + \cdots \\
&> \frac{\pi^2}{2n^2} - \frac{\pi^4}{24n^4} \\
&= \frac{1}{n^2-1}\left[\frac{\pi^2(n^2-1)}{2n^2} - \frac{\pi^4(n^2-1)}{24n^4}\right] \\
&> \frac{4}{n^2-1} \quad \text{for } n \geq 5.
\end{aligned}$$

Thus, we get $\frac{1}{1-\cos(\pi/n)} < \frac{n^2-1}{4}$ for $n \geq 5$. Hence, $|\frac{1}{\cos(\pi/n)-1}| < \frac{n^2-1}{4} < \frac{n^2}{4}$ for $n \geq 5$. Therefore, distance matrix of the path $P_n,\ n \geq 5$, satisfies the property $\mathcal{P}$.

Next we get that property $\mathcal{P}$ also holds true for distance signless Laplacian matrix of some classes of graphs. The result below is immediate from Theorem 4.2.

THEOREM 5.4. *If* G *is a simple connected graph satisfying* $2tr_{max} - 2tr_{min} \leq n$, *then all the eigenvalues of* $D_Q(G)$, *other than the distance signless Laplacian spectral radius, lie in the smallest Gerŝgorin disc.*

Property $\mathcal{P}$ obviously holds true for distance and distance signless Laplacian matrix of transmission regular graphs. Based on the above discussions we leave the following questions for further research.

PROBLEM 5.1. Whether property $\mathcal{P}$ holds true for distance matrix of an arbitrary graph?

PROBLEM 5.2. Whether property $\mathcal{P}$ holds true for distance signless Laplacian matrix of an arbitrary graph?

We notice that distance matrix of an n-vertex connected graph is symmetric, and its diagonal entries are all zero and non-diagonal entries vary from 1 to $n-1$.

PROBLEM 5.3. There are many examples of symmetric $n \times n$ square matrices with diagonal entries all zero and non-diagonal entries vary from 1 to $n-1$, which do not satisfy property $\mathcal{P}$. What are all such matrices satisfying property $\mathcal{P}$?

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