Electronic Journal of Linear Algebra

Manuscript 3514

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

SURJECTIVE ADDITIVE RANK-1 PRESERVERS ON HESSENBERG MATRICES*

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Abstract. Let $H_n(\mathbb{F})$ be the space of all $n \times n$ upper Hessenberg matrices over a field $\mathbb{F}$, where n is a positive integer greater than two. In this paper, surjective additive maps preserving rank-1 on $H_n(\mathbb{F})$ are characterized.

Key words. Additive maps, Rank-1 preservers, Hessenberg matrices.

AMS subject classifications. 15A03, 15A86, 15B99.

1. Introduction. During the past twenty years, there are various research concerning additive preserver problems (APPs). APPs are problems similar to LPPs (linear preserver problems) except that these maps preserve the addition while preserving the scalar multiplication is not required. In general, additive rank-1 preservers are among the most studied subjects for examples in 2003 Cao and Zhang [1] gave the structure of additive rank-1 preserving surjections on symmetric matrix spaces over a field of characteristic not 2 or 3. Surjective additive rank-1 preservers on the full matrix algebra over any field were characterized by Cao and Zhang [2] in 2004. A year later, this work was extended by Zhang and Sze [7] to studying additive rank-1 preservers between spaces of full matrices of different dimensions.

This paper is motivated by the work of Cao and Zhang [2] and that of us [5] which provided the structure of linear rank-1 preservers on the space of all $n \times n$ upper Hessenberg matrices over an arbitrary field $\mathbb{F}$, where $n \geq 3$. Note that a square matrix (a_{ij}) is *upper Hessenberg* if $a_{ij} = 0$ whenever $j + 1 < i$. Likewise upper triangular matrices, upper Hessenberg matrices over a field form a vector space. Moreover, we can rewrite complex matrices into a Hessenberg decomposition [4], i.e., "for a complex matrix A, there exist a unitary matrix P and a Hessenberg matrix H such that $A = PH\overline{P}^t$". In another word, the similarity of such matrices A and H leads to some shared properties, for examples, their rank, determinant and eigenvalues. Particularly, Hessenberg matrices play an important role in the QR algorithm by reducing workload of each iteration: $\mathcal{O}(n^3)$ for a general matrix and $\mathcal{O}(n^2)$ for the Hessenberg form of the original matrix [6].

In this paper, some properties of Hessenberg matrices are given in Section 2 and surjective additive rank-1 preservers are characterized in Section 3 by using only basic concepts in the matrix theory.

2. Preliminaries. For convenience, we begin with the following definitions and notation used througout this paper. Let $M_{mn}(\mathbb{F})$, $T_n(\mathbb{F})$ and $H_n(\mathbb{F})$ be the set of all $m \times n$ matrices over a field $\mathbb{F}$, the set of all $n \times n$ upper triangular matrices over a filed $\mathbb{F}$ and the set of all $n \times n$ upper Hessenberg matrices over a field $\mathbb{F}$, respectively. Furthermore, $\rho(A)$ and A^t denote the rank of a matrix A and the transpose of a matrix A, respectively. A map φ on a space V is *additive* if $\varphi(a + b) = \varphi(a) + \varphi(b)$ for any elements a and b in V. A subspace V of a vector space is called a *rank*-1 *subspace* if each element in V is the zero matrix or has rank one. In addition, a map T on $H_n(\mathbb{F})$ is called a *rank*-1 *preserver* if $\rho(T(A)) = 1$ whenever $\rho(A) = 1$ for

*Received by the editors on March 29, 2017. Accepted for publication on December 12, 2018. Handling Editor: Joao Filipe Queiro. Corresponding Author: Sajee Pianskool.
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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

any $A \in H_n(\mathbb{F})$. Besides, a map T on $H_n(\mathbb{F})$ is called a *rank preserver* if T preserves all ranks. The symbol $x \otimes y$ denotes xy^t for any column vectors x and y. We also use the common notation $e_1, \ldots, e_m$ to denote the standard bases of $M_{m1}(\mathbb{F})$ and E_{ij} to denote the elementary matrix over $\mathbb{F}$ whose (i,j)-entry is one and others are zero. It is easy to verify that $E_{ij} = e_i \otimes e_j$ for all i and j.

Set

$$x \otimes M_{n1}(\mathbb{F}) = \{x \otimes y \,|\, y \in M_{n1}(\mathbb{F})\}, \quad \text{where } x \in M_{m1}(\mathbb{F}),$$

$$M_{m1}(\mathbb{F}) \otimes y = \{x \otimes y \,|\, x \in M_{m1}(\mathbb{F})\}, \quad \text{where } y \in M_{n1}(\mathbb{F}),$$

$$\Omega = \{A \in H_n(\mathbb{F}) \,|\, \rho(A) = 1\},$$

$$H_n^u(\mathbb{F}) = \{(a_{ij}) \in H_n(\mathbb{F}) \,|\, a_{j+1,j} = 0 \text{ for all } j \in \{2, \ldots, n-1\}\}$$

and

$$H_n^d(\mathbb{F}) = \{(a_{ij}) \in H_n(\mathbb{F}) \,|\, a_{j+1,j} = 0 \text{ for all } j \in \{1, \ldots, n-2\}\}.$$

The following notation is first used in [3]. For an integer s with $1 \leq s \leq n$, let

$$U_s = \left\{ \begin{pmatrix} x_1 & \cdots & x_s & 0 & \cdots & 0 \end{pmatrix}^t \,|\, x_i \in \mathbb{F} \text{ for all } i \in \{1, \ldots, s\} \right\},$$

$$V_s = \left\{ \begin{pmatrix} 0 & \cdots & 0 & x_s & \cdots & x_n \end{pmatrix} \,|\, x_i \in \mathbb{F} \text{ for all } i \in \{s, \ldots, n\} \right\},$$

$$xV_s = \{xv \,|\, v \in V_s\} \quad \text{for each } x \in M_{n1}(\mathbb{F})$$

and

$$U_s y = \{uy \,|\, u \in U_s\} \quad \text{for each } y \in M_{1n}(\mathbb{F}).$$

For a matrix $A = (a_{ij})$ in $M_n(\mathbb{F})$, Chooi and Lim denoted $A^\sim$ the matrix (b_{ij}) in $M_n(\mathbb{F})$ such that $b_{ij} = a_{n+1-j,n+1-i}$ for any i and j. Observably, the diagonal line acts as the reflection-axis for the remaining elements but the elements on this line are fixed. Furthermore, $(A+B)^\sim = A^\sim + B^\sim$, $(AB)^\sim = B^\sim A^\sim$, $(A^\sim)^\sim = A$ and $\rho(A) = \rho(A^\sim)$ for all $A, B \in M_n(\mathbb{F})$.

The following proposition is a useful tool to prove some of our results; however, we state without proof because its proof is straightforward.

Proposition 1. *Let $x, y, u, v \in M_{n1}(\mathbb{F})$. The following statements hold.*

(i) $x \otimes y = 0$ if and only if $x = 0$ or $y = 0$.
(ii) If $x \otimes y \neq 0$, then $x \otimes y = u \otimes v$ if and only if there exists $\alpha \in \mathbb{F}\backslash\{0\}$ such that $u = \alpha x$ and $y = \alpha v$.
(iii) If $x \otimes y + u \otimes v \in \Omega$, then $\{x, u\}$ or $\{y, v\}$ is linearly dependent.
(iv) For $n \geq 2$, if $u \neq 0$ and $v \neq 0$, then there exists $w \in \Omega$ such that $w \notin u \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes v$.

By making use of Proposition 1 (ii), the property of the decomposition rank of a matrix and the fact that $V_n \subseteq \cdots \subseteq V_1$ and $U_1 \subseteq \cdots \subseteq U_n$, the following corollary is obtained.

Corollary 2. *Let $x \in M_{n1}(\mathbb{F})\backslash\{0\}$ and $y \in M_{1n}(\mathbb{F})\backslash\{0\}$. The matrix $xy \in \Omega$ if and only if there exists $s \in \{1, \ldots, n+1\}$ such that $x \in U_s$ and $y \in V_{s-1}$, where $V_0 = V_1$ and $U_{n+1} = U_n$.*

In general, a product of Hessenberg matrices need not be a Hessenberg matrix. However, this can be a Hessenberg matrix under some conditions as follows.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
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Proposition 3. *For $n \geq 3$, let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n(\mathbb{F})$. Then $AB \in H_n(\mathbb{F})$ if and only if $a_{i,i-1} = 0$ or $b_{i-1,i-2} = 0$ for all $i \in \{3, 4, \ldots, n\}$.*

Proof. The proof is simple. ∎

In matrix theory, for each $m \times n$ matrix A of rank r, there exist nonsingular matrices P and Q in $M_m(\mathbb{F})$ and $M_n(\mathbb{F})$, respectively, such that $PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. This property is shown in the sense of Hessenberg matrices.

Proposition 4. *If $A \in H_n(\mathbb{F})$ of rank $r \neq 0$, then there exist nonsingular matrices $P, Q \in T_n(\mathbb{F})$ such that $PAQ = \sum_{i=1}^{r} E_{s_i t_i}$, where $s_i, t_i \in \{1, \ldots, n\}$ with $s_i \leq t_i + 1$ for all i and $s_i \neq s_j$, $t_i \neq t_j$ for all $i \neq j$.*

Proof. Let $A = (a_{ij})$ be a Hessenberg matrix of rank $r \neq 0$. Given $R_1, \ldots, R_n$ and $C_1, \ldots, C_n$ are the row vectors and column vectors of A, respectively. Let R_s be the first nonzero row vector from the last row of A and let a_{sq} be the leading entry of R_s. Multiply R_s by a_{sq}^{-1} and then for each $1 \leq i < s$, apply the row operation $R_i - a_{iq} R_s \to R_i$.

Next, for each $q < j \leq n$, apply the column operation $C_j - \frac{a_{sj}}{a_{sq}} C_q \to C_j$. Let X and Y be the product of matrices obtained by these row operations and these column operations, respectively. Then X and Y are nonsingular triangular matrices such that $XAY = E_{sq} + B$ with $B = \begin{pmatrix} U & V \\ 0 & 0 \end{pmatrix}$, where $U \in H_{s-1}(\mathbb{F})$ and $V \in M_{s-1,n-s+1}(\mathbb{F})$. By using the same argument with B, we obtain $X_2 B Y_2 = E_{s_2 t_2} + B_2$, where $t_2 \neq q$ and $s_2 < s$. Furthermore, $X_2 E_{sq} Y_2 = X_2 \big(e_s \otimes e_q\big) Y_2$ which is the product of the s-column of X_2 and the q-row of Y_2, and hence, it is the product of e_s and e_q^t, which is E_{sq}. This shows that $X_2 E_{sq} Y_2 = E_{sq}$. Continue the same process and then let $P = X_r \cdots X_2 X$ and $Q = Y Y_2 \cdots Y_r$. It follows that P and Q are nonsingular triangular matrices; moreover,

$$X_i E_{sq} Y_i = E_{sq} \quad \text{for all } \; 3 \leq i \leq r,$$

and for each $j \in \{2, \ldots, r\}$, we get $X_l E_{s_j t_j} Y_l = E_{s_j t_j}$ for all $j+1 \leq l \leq r$. Then for all $s_i, t_i \in \{1, \ldots, n\}$ with $s_i \leq t_i + 1$ for all i and $s_i \neq s_j$, $t_i \neq t_j$ whenever $i \neq j$, it follows that $PAQ = \sum_{i=1}^{r} E_{s_i t_i}$, where $E_{s_1 t_1} = E_{sq}$. ∎

The following proposition acts as a supplement of matrices main proof.

Proposition 5. *Let $A, B \in H_n(\mathbb{F})$ be nonsingular and $\varphi : H_n(\mathbb{F}) \to M_n(\mathbb{F})$ the map defined by $\varphi(X) = AXB$. Then $\operatorname{Im} \varphi \subseteq H_n(\mathbb{F})$ if and only if $A \in H_n^u(\mathbb{F})$ and $B \in H_n^d(\mathbb{F})$.*

Proof. The sufficiency is clear so we prove the necessity. Assume that $\operatorname{Im} \varphi \subseteq H_n(\mathbb{F})$. To show that $B = (b_{ij}) \in H_n^d(\mathbb{F})$, we first claim that, for any $2 \leq s \leq n-1$, if $y \in V_s$, then $yB \in V_s$. As a result, for any $2 \leq s \leq n-1$, if $y = \begin{pmatrix} 0 & \cdots & 0 & y_s & \cdots & y_n \end{pmatrix} \in V_s$ with $y_s \neq 0$, then $yB \in V_s$ with $y_s b_{s,s-1} = 0$ so that $b_{s,s-1} = 0$. We can conclude that $B \in H_n^d(\mathbb{F})$ as desired.

It remains to show the claim. Fix $s \in \{2, \ldots, n-1\}$. Assume that $y \in V_s$. Then $x \otimes y^t \in H_n(\mathbb{F})$ for every $x \in U_{s+1}$ by Corollary 2. Since $(Ax)(yB) = A(x \otimes y^t)B \in H_n(\mathbb{F})$ for any $x \in U_{s+1}$ and A is nonsingular, we obtain that both spaces $\{Ax \,|\, x \in U_{s+1}\}$ and U_{s+1} have the same dimensions. Hence, $yB \in V_s$; otherwise, it forces the dimension of $\{Ax \,|\, x \in U_{s+1}\}$ to be less than or equal to s which is a contradiction. The claim is now complete.

Similarly, by using the same manner, proving that $A \in H_n^u(\mathbb{F})$ is enough to use the fact that if $x \in U_{s+1}$, then $Ax \in U_{s+1}$ for all $1 \leq s \leq n-2$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

3. Main results. This section is devoted to provide the structure of surjective additive rank-1 preservers on $H_n(\mathbb{F})$. For certain mappings on $H_n(\mathbb{F})$, relationships between the first row and the last column of each matrix in $H_n(\mathbb{F})$ are shown as follows. Note that for a space V of matrices, set $V^t = \{A^t \mid A \in V\}$.

LEMMA 6. *Let φ be an additive rank-1 preserver on $H_n(\mathbb{F})$. Then for any $i \in \{1, 2, \dots, n\}$, there exist $s_i, q_i \in \{1, \dots, n\}$ with $s_i \le q_i + 1$, nonzero elements $x_i \in U_{s_i}$ and $y_i \in V_{q_i}^t$ and injective additive maps $g_i : V_{i-1}^t \to V_{q_i}^t$ and $d_i : V_{i-1}^t \to U_{s_i}$, where $V_0 = V_1$ such that*

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \text{for all } z \in V_{i-1}^t \tag{3.1}$$

$$\text{or} \quad \varphi(e_i \otimes z) = d_i(z) \otimes y_i \quad \text{for all } z \in V_{i-1}^t. \tag{3.2}$$

Proof. We only show the case $i = 1$. The other cases can be obtained similarly. Since $e_1 \otimes M_{n1}(\mathbb{F})$ is a rank-1 subspace and φ preserves all rank-1 matrices, it follows that $\varphi(e_1 \otimes M_{n1}(\mathbb{F}))$ is a rank-1 additive group. Since $\varphi(e_1 \otimes M_{n1}(\mathbb{F})) \cap \Omega \neq \emptyset$, there exist $s_1, q_1 \in \{1, \dots, n\}$ with $s_1 \le q_1 + 1$ such that

$$0 \neq x_1 \otimes y_1 \in \varphi(e_1 \otimes M_{n1}(\mathbb{F}))$$

for some nonzero $x_1 \in U_{s_1}$ and $y_1 \in V_{q_1}^t$.

Case 1: Suppose that $\{u, x_1\}$ is linearly dependent for any nonzero $u \otimes v \in \varphi(e_1 \otimes M_{n1}(\mathbb{F}))$. Then $u = \alpha_u x_1$ for some $\alpha_u \in \mathbb{F}\backslash\{0\}$ and $u \in U_{s_1}$, and hence, $v \in V_{q_1}^t$. It follows that

$$u \otimes v = (\alpha_u x_1) \otimes v = x_1 \otimes (\alpha_u v) \in x_1 \otimes V_{q_1}^t.$$

Thus, $\varphi(e_1 \otimes M_{n1}(\mathbb{F})) \subseteq x_1 \otimes V_{q_1}^t$. This implies that there exists an 1-1 additive map $g_1 : M_{n1}(\mathbb{F}) \to V_{q_1}^t$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}).$$

Case 2: Suppose that $\{u_0, x_1\}$ is linearly independent for some nonzero $u_0 \otimes v_0 \in \varphi(e_1 \otimes M_{n1}(\mathbb{F}))$. Since $u_0 \otimes v_0 + x_1 \otimes y_1 \in \varphi(e_1 \otimes M_{n1}(\mathbb{F}))$ and φ is an additive rank-1 preserver, $u_0 \otimes v_0 + x_1 \otimes y_1 \in \Omega \cup \{0\}$. Thus, $\{v_0, y_1\}$ is linearly dependent by (ii) and (iii) of Proposition 1. Then $v_0 = \alpha_{v_0} y_1$ for some $\alpha_{v_0} \in \mathbb{F}\backslash\{0\}$. It follows that $u_0 \otimes v_0 = u_0 \otimes (\alpha_{v_0} y_1) = (\alpha_{v_0} u_0) \otimes y_1$. Hence, for each nonzero $u \otimes v \in \varphi(e_1 \otimes M_{n1}(\mathbb{F}))$, we obtain

$$u \otimes v + (\alpha_{v_0} u_0) \otimes y_1 \in \Omega \cup \{0\} \quad \text{and} \quad u \otimes v + x_1 \otimes y_1 \in \Omega \cup \{0\}.$$

As a result, $\{v, y_1\}$ is linearly dependent; otherwise, by (iii) of Proposition 1, it forces $\{u_0, x_1\}$ to be linearly dependent which is a contradiction. Then there exists $\alpha_v \in \mathbb{F}\backslash\{0\}$, $v = \alpha_v y_1$ and then $v \in V_{q_1}^t$. It follows that

$$u \otimes v = u \otimes (\alpha_v y_1) = (\alpha_v u) \otimes y_1 \in U_{s_1} \otimes y_1.$$

Thus, $\varphi(e_1 \otimes M_{n1}(\mathbb{F})) \subseteq U_{s_1} \otimes y_1$. This implies that there exists an 1-1 additive map $d_1 : M_{n1}(\mathbb{F}) \to U_{s_1}$ such that

$$\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \text{for all } z \in M_{n1}(\mathbb{F}). \qquad \square$$

In the similar way, we can conclude the following.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

Lemma 7. *Let φ be an additive rank-1 preserver on $H_n(\mathbb{F})$. Then for any $i \in \{1,2,\dots,n\}$, there exist $p_i, r_i \in \{1,\dots,n\}$ with $p_i \le r_i + 1$, nonzero elements $u_i \in U_{p_i}$ and $v_i \in V_{r_i}^t$ and injective additive maps $h_i : U_{i+1} \to V_{r_i}^t$ and $c_i : U_{i+1} \to U_{p_i}$, where $U_{n+1} = U_n$ such that*

$$\begin{aligned} &\varphi(z\otimes e_i) = u_i \otimes h_i(z) \quad \text{for all } z \in U_{i+1} \\ \text{or} \quad &\varphi(z\otimes e_i) = c_i(z) \otimes v_i \quad \text{for all } z \in U_{i+1}. \end{aligned}$$

The next lemma provides a relationship between Lemma 6 and Lemma 7 on the first row and the last column of each Hessenberg matrix.

Lemma 8. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ and let $x_1, y_1, u_n, v_n, g_1, d_1, h_n, c_n$ be defined as appeared in Lemma 6 and Lemma 7. Then*

$$\begin{aligned} (i) \qquad & \varphi(e_1\otimes z) = x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}) \\ \text{and} \quad & \varphi(z\otimes e_n) = c_n(z) \otimes v_n \quad \text{for all } z \in M_{n1}(\mathbb{F}), \text{ or} \end{aligned}$$

$$\begin{aligned} (ii) \qquad & \varphi(e_1\otimes z) = d_1(z) \otimes y_1 \quad \text{for all } z \in M_{n1}(\mathbb{F}) \\ \text{and} \quad & \varphi(z\otimes e_n) = u_n \otimes h_n(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}). \end{aligned}$$

Proof. By Lemma 6 in case $i = 1$, we obtain that

$$\varphi(e_1\otimes z) = x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}) \tag{3.3}$$

$$\text{or} \quad \varphi(e_1\otimes z) = d_1(z) \otimes y_1 \quad \text{for all } z \in M_{n1}(\mathbb{F}). \tag{3.4}$$

By Lemma 7 in case $i = n$, we also obtain that

$$\varphi(z\otimes e_n) = u_n \otimes h_n(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}) \tag{3.5}$$

$$\text{or} \quad \varphi(z\otimes e_n) = c_n(z) \otimes v_n \quad \text{for all } z \in M_{n1}(\mathbb{F}). \tag{3.6}$$

Nevertheless,

(I) (3.3) and (3.5) cannot hold simultaneously, and
(II) (3.4) and (3.6) cannot hold simultaneously.

We prove only (I). Suppose that (3.3) and (3.5) hold simultaneously. Since $x_1 \otimes g_1(e_n) = \varphi(e_1 \otimes e_n) = u_n \otimes h_n(e_1)$, by (ii) of Proposition 1, there exists a nonzero $\alpha \in \mathbb{F}$ such that $x_1 = \alpha u_n$. Thus, $\varphi(e_1 \otimes z) = \alpha u_n \otimes g_1(z) = u_n \otimes \alpha g_1(z) \in u_n \otimes M_{n1}(\mathbb{F})$ for all $z \in M_{n1}(\mathbb{F})$.

Case 1: $\varphi(\Omega) \subseteq u_n \otimes M_{n1}(\mathbb{F})$. In general, each Hessenberg matrix is the sum of finitely many rank-1 matrices. Then $\varphi\big(H_n(\mathbb{F})\big) \subseteq u_n \otimes M_{n1}(\mathbb{F})$ which contradicts the surjectivity of φ.

Case 2: $\varphi(\Omega) \nsubseteq u_n \otimes M_{n1}(\mathbb{F})$. Then there exist nonzero $x, y, u, v \in M_{n1}(\mathbb{F})$ with $x \otimes y \in \Omega$ such that $\varphi(x \otimes y) = u \otimes v$ and $\{u, u_n\}$ is linearly independent. We know that

$$\begin{aligned} &\varphi(x\otimes y) = u\otimes v \in \Omega, \\ &\varphi\big((x+e_1)\otimes y\big) = u\otimes v + \alpha u_n \otimes g_1(y) \in \Omega, \\ &\varphi\big(x\otimes (y+e_n)\big) = u\otimes v + u_n \otimes h_n(y) \in \Omega, \text{ and} \\ &\varphi\big((x+e_1)\otimes (y+e_n)\big) = u\otimes v + \alpha u_n \otimes g_1(y) + u_n\otimes h_n(y) + u_n \otimes h_n(e_1) \in \Omega. \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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By using (iii) of Proposition 1 repeatedly, we obtain that $\{v, g_1(y)\}$, $\{v, h_n(y)\}$ and $\{v, h_n(e_1)\}$ are linearly dependent so that there exists a nonzero $\beta \in \mathbb{F}$ such that $v = \beta h_n(e_1)$, and hence, $\varphi(x \otimes y) = u \otimes \beta h_n(e_1) = \beta u \otimes h_n(e_1) \in M_{n1}(\mathbb{F}) \otimes h_n(e_1)$.

As a conclusion, $\varphi(\Omega) \subseteq u_n \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes h_n(e_1)$, it follows that $\varphi\big(H_n(\mathbb{F})\big) \subseteq u_n \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes h_n(e_1)$ which contradicts the surjectivity of φ by (iv) of Proposition 1. ∎

The proofs of Lemma 9 and Lemma 10 use the same method, thereby we prove only Lemma 10.

Lemma 9. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying the condition (3.3) in the proof of Lemma 8. Then, for $1 \le i \le n-1$,*

$$\varphi(z \otimes e_i) = c_i(z) \otimes v_i \quad \textit{for all } z \in U_{i+1},$$

where c_i and v_i are given in Lemma 7.

Lemma 10. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying the condition (3.6) in the proof of Lemma 8. Then, for $2 \le i \le n$,*

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \textit{for all } z \in V_{i-1}^t,$$

where g_i and x_i are given in Lemma 6.

Proof. By the condition (3.6) in the proof of Lemma 8 and Lemma 6, we only show that (3.2) in Lemma 6 does not hold. If (3.2) holded, then $d_i(e_n) \otimes y_i = \varphi(e_i \otimes e_n) = c_n(e_i) \otimes v_n$, and hence, $\varphi(\Omega) \subseteq U_{s_i} \otimes v_n \cup c_n(e_i) \otimes M_{n1}(\mathbb{F})$ contradicting (iv) of Proposition 1. ∎

The following proposition is a result of the combination of Lemmas 8–10. Recall the results from these lemmas:

(a) There exist an 1-1 additive map $g_1 : M_{n1}(\mathbb{F}) \to V_{q_1}^t$ and $x_1 \in U_{s_1} \backslash \{0\}$, where $s_1, q_1 \in \{1, \ldots, n\}$ with $s_1 \le q_1 + 1$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}).$$

(b) There exist an 1-1 additive map $c_n : M_{n1}(\mathbb{F}) \to U_{p_n}$ and $v_n \in V_{r_n}^t \backslash \{0\}$, where $p_n, r_n \in \{1, \ldots, n\}$ with $p_n \le r_n + 1$ such that

$$\varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \text{for all } z \in M_{n1}(\mathbb{F}).$$

(c) For $1 \le i \le n-1$, there exist an 1-1 additive map $c_i : U_{i+1} \to U_{p_i}$ and $v_i \in V_{r_i}^t \backslash \{0\}$, where $p_i, r_i \in \{1, \ldots, n\}$ with $p_i \le r_i + 1$ such that

$$\varphi(z \otimes e_i) = c_i(z) \otimes v_i \quad \text{for all } z \in U_{i+1}.$$

(d) For $2 \le i \le n$, there exist an 1-1 additive map $g_i : V_{i-1}^t \to V_{q_i}^t$ and $x_i \in U_{s_i} \backslash \{0\}$, where $s_i, q_i \in \{1, \ldots, n\}$ with $s_i \le q_i + 1$ such that

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \text{for all } z \in V_{i-1}^t.$$

Consider (a) and (b). We obtain

$$x_1 \otimes g_1(e_n) = \varphi(e_1 \otimes e_n) = c_n(e_1) \otimes v_n$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
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and then there exists $\alpha \in \mathbb{F}\setminus\{0\}$ such that $c_n(e_1) = \alpha x_1$ and $g_1(e_n) = \alpha v_n$ by Proposition 1. However, $c_n(e_1) = \alpha x_1 \in U_{s_1}$ but $c_n(M_{n1}(\mathbb{F})) \subseteq U_{p_n}$. This result forces $s_1 \leq p_n$. Similarly, $g_1(e_n) = \alpha v_n \in V^t_{r_n}$ but $g_1(M_{n1}(\mathbb{F})) \subseteq V^t_{q_1}$, and hence, $q_1 \leq r_n$.

Consider (a) and (c). In the case $\varphi(e_1 \otimes e_1)$, we get $s_1 \leq p_1$ and $q_1 \leq r_1$. Moreover, $s_1 \leq p_k$ and $q_1 \leq r_k$ if we focus on $\varphi(e_1 \otimes e_k)$ for $2 \leq k \leq n-1$. As a result, for each $i \in \{1,\dots,n\}$, $s_1 \leq p_i$ and $q_1 \leq r_i$. Consequently, $q_1 = 1$ and then $s_1 \leq 2$. If not, we get $q_1 \geq 2$ which forces $r_i \geq 2$ for all i. Since any Hessenberg matrix can be expressed as the form $\sum_{i=1}^n z \otimes e_i$ from (c), and thus, φ maps Hessenberg matrices into Hessenberg matrices such that the first column is zero, which contradicts the surjectivity of φ.

Consider (a), (b) and (d). We can see that $p_n = n$ and $r_n \geq n-1$ follow by a similar argument.

Lemma 11. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ which satisfies the condition:*

There exist $s_1, r_n \in \{1,\dots,n\}$ with $s_1 \leq 2$ and $r_n \geq n-1$, nonzero elements $x_1 \in U_{s_1}$ and $v_n \in V^t_{r_n}$ and injective additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\begin{aligned} \varphi(e_1 \otimes z) &= x_1 \otimes g_1(z) \quad \textit{for all } z \in M_{n1}(\mathbb{F}) \\ \textit{and} \quad \varphi(z \otimes e_n) &= c_n(z) \otimes v_n \quad \textit{for all } z \in M_{n1}(\mathbb{F}). \end{aligned}$$

Then the following statements hold.

(i) *There exist bijective additive maps $g_1,\dots,g_n$ and $x_1,\dots,x_n \in M_{n1}(\mathbb{F})$ such that $g_i : V^t_{i-1} \to V^t_{i-1}$, where $V_0 = V_1$ and*

$$x_i \in U_i \text{ for all } i \qquad \text{or} \qquad x_i \in \begin{cases} U_2, & \text{if } i = 1 \\ U_1, & \text{if } i = 2 \\ U_i, & \text{if } i \neq 1,2 \end{cases} \qquad \text{or} \qquad x_i \in \begin{cases} U_2, & \text{if } i = 1 \\ U_i, & \text{if } i \neq 1 \end{cases}$$

such that $\varphi(e_i \otimes z) = x_i \otimes g_i(z)$ for all $z \in V^t_{i-1}$. Moreover, such $x_1,\dots,x_n$ are linearly independent.

(ii) *There exist bijective additive maps $c_1,\dots,c_n$ and $v_1,\dots,v_n \in M_{n1}(\mathbb{F})$ such that $c_i : U_{i+1} \to U_{i+1}$, where $U_{n+1} = U_n$ and*

$$v_i \in V^t_i \text{ for all } i \qquad \text{or} \qquad v_i \in \begin{cases} V^t_i, & \text{if } i \neq n, n-1 \\ V^t_n, & \text{if } i = n-1 \\ V^t_{n-1}, & \text{if } i = n \end{cases} \qquad \text{or} \qquad v_i \in \begin{cases} V^t_i, & \text{if } i \neq n \\ V^t_{n-1}, & \text{if } i = n \end{cases}$$

such that $\varphi(z \otimes e_i) = c_i(z) \otimes v_i$ for all $z \in U_{i+1}$. Moreover, such $v_1,\dots,v_n$ are linearly independent.

Proof. From the assumption, there exist $s_1, r_n \in \{1,\dots,n\}$ with $s_1 \leq 2$ and $r_n \geq n-1$, nonzero elements $x_1 \in U_{s_1}$ and $v_n \in V^t_{r_n}$ and 1-1 additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}) \tag{3.7}$$

$$\text{and} \quad \varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \text{for all } z \in M_{n1}(\mathbb{F}). \tag{3.8}$$

By Lemma 10 and Lemma 9, we obtain that for all $2 \leq i \leq n$, there exist $s_i, q_i \in \{1,\dots,n\}$ with $s_i \leq q_i + 1$, a nonzero element $x_i \in U_{s_i}$ and a 1-1 additive map $g_i : V^t_{i-1} \to V^t_{q_i}$ such that

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \text{for all } z \in V^t_{i-1} \tag{3.9}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

and for each $1 \leq i \leq n-1$, there exist $p_i, r_i \in \{1, \ldots, n\}$ with $p_i \leq r_i + 1$, a nonzero element $v_i \in V_{r_i}^t$ and a 1-1 additive map $c_i : U_{i+1} \to U_{p_i}$ such that

$$\varphi(z \otimes e_i) = c_i(z) \otimes v_i \quad \text{for all } z \in U_{i+1}. \tag{3.10}$$

It follows from (3.7) and (3.9) that for each nonzero $z \in V_{i-1}^t$ we get

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \in \Omega \quad \text{for all } 1 \leq i \leq n,$$

where $V_0^t = M_{n1}(\mathbb{F})$.

Now, first of all, since φ maps onto $H_n(\mathbb{F})$, for each $A \in H_n(\mathbb{F})$, there exists $B = \sum_{i=1}^n (e_i \otimes z_i^t) \in H_n(\mathbb{F})$ such that $\varphi(B) = A$, where each z_i is the i-row of B. It follows that

$$A = \varphi\Big(\sum_{i=1}^n (e_i \otimes z_i^t)\Big) = \sum_{i=1}^n \varphi\big((e_i \otimes z_i^t)\big) = \sum_{i=1}^n \big(x_i \otimes g_i(z_i^t)\big). \tag{3.11}$$

Consequently, every Hessenberg matrix A is represented by the sum of the form $x_i \otimes g_i(z_i^t)$, where each z_i is the i-row of B such that $\varphi(B) = A$.

Since $s_1 \leq 2$, we get $x_1 \in U_1$ or $x_1 \in U_2$ and then $\operatorname{Im} g_1 \in V_1^t$, where $V_0 = V_1$, besides, $x_i \in U_{s_i}$ and $\operatorname{Im} g_i \in V_{q_i}^t$ for all $2 \leq i \leq n$.

Case 1: $x_1 \in U_1$. In fact, E_{nn} is an element of $H_n(\mathbb{F})$ so that $E_{nn} = \sum_j \big(x_j \otimes g_j(z_j^t)\big)$ for some j. It follows that there exists an element in $\{x_2, x_3, \ldots, x_n\}$ such that its n-position must not be zero, say x_n. Since $x_n \in U_{s_n}$, we get $s_n = n$, and hence, $x_n \in U_n$ and $\operatorname{Im} g_n \in V_{n-1}^t$ by making use of Corollary 2. Furthermore, with the same argument, $E_{n-1,n-2} \in H_n(\mathbb{F})$ which forces that there exists an element in $\{x_2, x_3, \ldots, x_n\} \backslash \{x_n\}$ such that its $(n-1)$-position must not be zero by the structure of x_n and $\operatorname{Im} g_n$, say x_{n-1}. If the n-position of x_{n-1} is not equal to zero, we obtain that $x_{n-1} \in U_n$, and thus, $\operatorname{Im} g_{n-1} \in V_{n-1}^t$ which is impossible, therefore, x_{n-1} must be in U_{n-1} and $\operatorname{Im} g_{n-1} \in V_{n-2}^t$, and hence, $\{x_{n-1}, x_n\}$ is linearly independent. Similarly, the i-position of x_i must not be zero; moreover, $x_i \in U_i$ and $\operatorname{Im} g_i \in V_{i-1}^t$ for all $i \geq 2$, and hence, $\{x_2, \ldots, x_n\}$ is linearly independent. In addition, by the structure of x_1, we conclude that $\{x_1, \ldots, x_n\}$ is linearly independent.

Case 2: $x_1 \in U_2$. In a similar manner, we obtain that the i-position of x_i must not be zero; moreover, $x_i \in U_i$ and $\operatorname{Im} g_i \in V_{i-1}^t$ for all $i \geq 3$, and hence, $\{x_1, x_3, \ldots, x_n\}$ is linearly independent. Since E_{11} is an element of $H_n(\mathbb{F})$ and the structure of x_i for all $i \geq 3$, we get

$$E_{11} = x_1 \otimes g_1(z_1^t) + x_2 \otimes g_2(z_2^t) \tag{3.12}$$

for some $z_1, z_2 \in M_{1n}(\mathbb{F})$; moreover, $x_2 \in U_1$ or $x_2 \in U_2$. By (iii) of Proposition 1, we know that $\{x_1, x_2\}$ is linearly dependent or $\{g_1(z_1^t), g_2(z_2^t)\}$ is linearly dependent. In case $x_2 \in U_1$, $\{x_1, x_2\}$ is linearly dependent, which is impossible; for another, if $\{x_1, x_2\}$ is linearly dependent, then the 2-position of x_2 is not equal to zero and forces the second row of the right hand side of (3.12) is not zero, which is a contradiction. Therefore, $\{x_1, x_2\}$ is linearly independent and then $\{x_1, \ldots, x_n\}$ is linearly independent.

Furthermore, g_i is a bijective additive map on V_{i-1}^t for all $i \in \{1, \ldots, n\}$ by applying (3.7), (3.9), (3.11) and the fact that $\{x_1, \ldots, x_n\}$ is linearly independent. The proof of (i) is complete.

It implies from (3.8) and (3.10) that (ii) holds. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

Now, we are ready to present our main result.

Theorem 12. *Let φ be a surjective additive map on $H_n(\mathbb{F})$. Then φ preserves rank-1 matrices if and only if there exist a field automorphism θ on $\mathbb{F}$ and nonsingular $P \in H_n^u(\mathbb{F})$ and $Q \in H_n^d(\mathbb{F})$ such that $\varphi(A) = PA^\theta Q$ for all $A \in H_n(\mathbb{F})$ or $\varphi(A) = P(A^\theta)^\sim Q$ for all $A = (a_{ij}) \in H_n(\mathbb{F})$, where $A^\theta = \big(\theta(a_{ij})\big)$.*

Proof. The sufficient part is clear. We prove only the necessary part. Assume that φ preserves rank-1 matrices. By Lemma 8, which can be written as follows:

(i) there exist $s_1, r_n \in \{1,\dots,n\}$ with $s_1 \le 2$ and $r_n \ge n-1$, nonzero elements $x_1 \in U_{s_1}$ and $v_n \in V_{r_n}^t$ and injective additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\begin{aligned} \varphi(e_1 \otimes z) &= x_1 \otimes g_1(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}) \\ \text{and} \quad \varphi(z \otimes e_n) &= c_n(z) \otimes v_n \quad \text{for all } z \in M_{n1}(\mathbb{F}), \text{ or} \end{aligned}$$

(ii) there exist $p_n, q_1 \in \{1,\dots,n\}$ with $p_n \le 2$ and $q_1 \ge n-1$, nonzero elements $u_n \in U_{p_n}$ and $y_1 \in V_{q_1}^t$ and injective additive maps d_1, h_n on $M_{n1}(\mathbb{F})$ such that

$$\begin{aligned} \varphi(e_1 \otimes z) &= d_1(z) \otimes y_1 \quad \text{for all } z \in M_{n1}(\mathbb{F}) \\ \text{and} \quad \varphi(z \otimes e_n) &= u_n \otimes h_n(z) \quad \text{for all } z \in M_{n1}(\mathbb{F}). \end{aligned}$$

Case 1: Assume that (i) holds. Lemma 11 yields that the sets $\{x_1,\dots,x_n\}$ and $\{v_1,\dots,v_n\}$ are linearly independent, where $x_1, x_2 \in U_1 \cup U_2$, $x_i \in U_i$ for all $i \in \{3,\dots,n\}$, $v_i \in V_i^t$ for all $i \in \{1,\dots,n-2\}$ and $v_{n-1}, v_n \in V_{n-1}^t \cup V_n^t$. Furthermore, g_i and c_i are also bijective additive maps on V_{i-1}^t and on U_{i+1}, respectively, for all $i \in \{1,\dots,n\}$.

Let $X = \begin{pmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{pmatrix}$ and $Y = \begin{pmatrix} — & v_1^t & — \\ & \vdots & \\ — & v_n^t & — \end{pmatrix}$. Then $X \in H_n^u(\mathbb{F})$ which is nonsingular and $Xe_i = x_i$ for all i. Put $P_1 = X^{-1}$. Then $e_i = P_1 x_i$ for all i and $P_1 \in H_n^u(\mathbb{F})$. Similarly, $Y \in H_n^d(\mathbb{F})$ which is nonsingular and $e_i Y = v_i^t$ for all i. Put $Q_1 = Y^{-1}$. Then $e_i = v_i^t Q_1$ for all i and $Q_1 \in H_n^d(\mathbb{F})$.

Let $\varphi_1 : H_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined by $\varphi_1(X) = P_1\varphi(X)Q_1$ for all $X \in H_n(\mathbb{F})$. Then $P_1\varphi(X)Q_1 \in H_n(\mathbb{F})$ for all $X \in H_n(\mathbb{F})$, i.e., $\varphi_1 : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ from Proposition 5. In fact, φ_1 is a surjective additive rank-1 preserver obtained from φ. Fix $i \in \{1,\dots,n\}$. For each $z \in M_{n1}(\mathbb{F})$ with $e_i \otimes z \in H_n(\mathbb{F})$, by applying (3.9) in the proof of Lemma 11, we obtain that

$$\varphi_1(e_i \otimes z) = P_1\varphi(e_i \otimes z)Q_1 = P_1\big(x_i \otimes g_i(z)\big)Q_1 = e_i \otimes Q_1^t g_i(z),$$

similarly, $\varphi_1(z \otimes e_i) = P_1\varphi(z \otimes e_i)Q_1 = P_1\big(c_i(z) \otimes v_i\big)Q_1 = P_1\big(c_i(z)\big) \otimes e_i$. Let $\psi_i(z) = Q_1^t g_i(z)$, where $z \in V_{i-1}^t$ when $V_0^t = M_{n1}(\mathbb{F})$, and $\phi_i(z) = P_1\big(c_i(z)\big)$, where $z \in V_{i+1}$ when $U_{n+1} = M_{1n}(\mathbb{F})$. Then ψ_i and ϕ_i are bijective additive maps on V_{i-1}^t and on U_{i+1}, respectively, for all i by the virtue of g_i and c_i, respectively.

Let $c \in \mathbb{F}$ and $i, j \in \{1,\dots,n\}$ with $i \le j+1$. Since

$$e_i \otimes \psi_i(ce_j) = \varphi_1(e_i \otimes ce_j) = \varphi_1(ce_i \otimes e_j) = \phi_j(ce_i) \otimes e_j,$$

it follows that there exists $\alpha_{ij}(c) \in \mathbb{F}\backslash\{0\}$ such that $\psi_i(ce_j) = \alpha_{ij}(c)e_j$ owing to (ii) of Proposition 1. Besides, $\alpha_{ij} : \mathbb{F} \to \mathbb{F}$ is a bijective additive map.

As a result, $\varphi_1(cE_{ij}) = \alpha_{ij}(c)E_{ij}$ for any $c \in \mathbb{F}$ and i, j such that $i \le j+1$, where $\alpha_{ij}(c) \in \mathbb{F}\backslash\{0\}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

Let $\varphi_2 : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ be defined by $\varphi_2(X) = P_2\varphi_1(X)Q_2$ for all $X \in H_n(\mathbb{F})$, where

$$Q_2 = diag\big(\alpha_{1n}(1)\alpha_{11}(1)^{-1}, \dots, \alpha_{1n}(1)\alpha_{1n}(1)^{-1}\big),\ P_2 = diag\big(\alpha_{1n}(1)^{-1}, \dots, \alpha_{nn}(1)^{-1}\big)$$

and $\alpha_{ij}(1)^{-1}$ is the inverse of $\alpha_{ij}(1)$ for all i, j. Then φ_2 is a surjective additive rank-1 preserver on $H_n(\mathbb{F})$. Furthermore,

$$\varphi_2(cE_{ij}) = \beta_{ij}(c)E_{ij}, \quad \text{where} \quad \beta_{ij}(c) = \alpha_{ij}(c)\alpha_{in}(1)^{-1}\alpha_{1n}(1)\alpha_{1j}(1)^{-1}.$$

For each $k \in \{1, \dots, n\}$, we know that $\varphi_2(E_{1k}) = E_{1k}$ and $\varphi_2(E_{kn}) = E_{kn}$. Hence,

$$\beta_{1k}(1) = 1 = \beta_{kn}(1) \quad \text{for all } k. \tag{3.13}$$

In addition, it can be shown similarly that β_{ij} is a bijective additive map on $\mathbb{F}$.

Next, let $c \in \mathbb{F}$. We claim that $\beta_{1j}(c) = \beta_{1n}(c) = \beta_{in}(c)$ for all $i, j \in \{1, \dots, n\}$. Without loss of generality, since $cE_{1j} + cE_{1n} + E_{ij} + E_{in} \in \Omega$ and φ_2 is an additive rank-1 preserver, $\beta_{1j}(c)E_{1j} + \beta_{1n}(c)E_{1n} + \beta_{ij}(1)E_{ij} + \beta_{in}(1)E_{in} \in \Omega$. We obtain $\beta_{1n}(c)\beta_{ij}(1) = \beta_{1j}(c)\beta_{in}(1) = \beta_{1j}(c)$ by (3.13). In particular, letting $c = 1$ implies $\beta_{ij}(1) = 1$. Hence, $\beta_{1n}(c) = \beta_{1j}(c)$ for all j. Similarly, $\beta_{in}(c) = \beta_{1n}(c)$ for all i.

In addition, by using the same argument as above, we obtain that $\beta_{pq}(c) = \beta_{1n}(c)$ for all $p, q \in \{1, \dots, n\}$. Put $\theta = \beta_{1n}$. Then θ is a bijective additive map on $\mathbb{F}$ such that for all $i, j \in \{1, \dots, n\}$, we get

$$\varphi_2(cE_{ij}) = \beta_{ij}(c)E_{ij} = \beta_{1n}(c)E_{ij} = \theta(c)E_{ij}.$$

Besides, $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in \mathbb{F}$, and hence, θ is a field automorphism on $\mathbb{F}$.

Now, for each $i, j \in \{1, \dots, n\}$, we know that $\varphi(cE_{ij}) = P\theta(c)E_{ij}Q$, where $P = XP_2^{-1}$ and $Q = Q_2^{-1}Y$. Hence, for each $A \in H_n(\mathbb{F})$, we obtain that $\varphi(A) = \varphi\Big(\sum c_{ij}E_{ij}\Big) = \sum P\theta(c_{ij})E_{ij}Q = P\Big(\sum \theta(c_{ij})E_{ij}\Big)Q = PA^\theta Q$, where $A^\theta = \big(\theta(a_{ij})\big)$.

Case 2: Assume that (ii) holds. By properties of the $\sim$ of any matrices and making use of Case 1, there exist $P \in H_n^u(\mathbb{F})$ and $Q \in H_n^d(\mathbb{F})$ such that $\varphi(A) = \big(PA^\theta Q\big)^\sim = Q^\sim(A^\theta)^\sim P^\sim$, where $Q^\sim \in H_n^u(\mathbb{F})$ and $P^\sim \in H_n^d(\mathbb{F})$. ∎

Corollary 13. *Let φ be a surjective additive map on $H_n(\mathbb{F})$. Then φ is a rank preserver if and only if there exist a field automorphism θ on $\mathbb{F}$ and nonsingular $P \in H_n^u(\mathbb{F})$ and $Q \in H_n^d(\mathbb{F})$ such that $\varphi(A) = PA^\theta Q$ for all $A \in H_n(\mathbb{F})$ or $\varphi(A) = P(A^\theta)^\sim Q$ for all $A = (a_{ij}) \in H_n(\mathbb{F})$, where $A^\theta = \big(\theta(a_{ij})\big)$.*

Finally, we would like to compare our main result, Theorem 12, and Theorem 3.23 in [5] which provided the structure of linear rank-1 preservers on $H_n(\mathbb{F})$ as follows:

If T is a linear map on $H_n(\mathbb{F})$ preserving rank-1 matrices, then

(i) $\operatorname{Im} T$ is an n-dimensional rank-1 subspace, or

(ii) there exist nonsingular upper Hessenberg matrices X and Y such that $T(A) = XAY$ for all $A \in H_n(\mathbb{F})$, or $T(A) = XA^\sim Y$ for all $A \in H_n(\mathbb{F})$.

Suppose that T is a surjective linear rank-1 preservers on $H_n(\mathbb{F})$. Then T is also a surjective additive rank-1 preservers on $H_n(\mathbb{F})$. By making use of Theorem 12, there exists a field automorphism θ on $\mathbb{F}$ such that $T(A) = PA^\theta Q$ or $T(A) = P(A^\theta)^\sim Q$. On the other hand, $T(A) = XAY$ or $T(A) = XA^\sim Y$. In fact, this θ is the identity map on $\mathbb{F}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 24-34, January 2019.
http://repository.uwyo.edu/ela

Acknowledgments. The first author was partially supported by *The* 90^{th} *Anniversary of Chulalongkorn University Fund* (Ratchadaphiseksomphot Endowment Fund), Thailand. Moreover, the authors would like to thank the referees for helpful comments.

REFERENCES

[1] C.G. Cao and X. Zhang. Additive rank-one preserving surjections on symmetric matrix spaces. *Linear Algebra and its Applications*, 362:145–151, 2003.

[2] C.G. Cao and X. Zhang. Additive surjections preserving rank one and applications. *Georgian Mathematical Journal*, 11:209–217, 2004.

[3] W.L. Chooi and M.H. Lim. Linear preservers on triangular matrices. *Linear and Multilinear Algebra*, 269:241–255, 1998.

[4] G.H. Golub and C.F. Van Loan. *Matrix Computations*, third edition. Johns Hopkins University Press, Baltimore, 1996.

[5] P. Khachorncharoenkul and S. Pianskool. Rank-1 preservers on Hessenberg matrices. *Linear and Multilinear Algebra*, 62:96–113, 2014.

[6] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery. *Numerical Recipes in C: The Art of Scientific Computing*, second edition. Cambridge University Press, Cambridge, 1992.

[7] X. Zhang and N.S. Sze. Additive rank-one preservers between spaces of rectangular matrices. *Linear and Multilinear Algebra*, 53:417–425, 2005.