Electronic Journal of Linear Algebra

Volume 34 *Volume 34 (2018)* Article 17

2018

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 231-239, May 2018.
http://repository.uwyo.edu/ela

NEW PERTURBATION BOUNDS IN UNITARILY INVARIANT NORMS FOR SUBUNITARY POLAR FACTORS*

LEI ZHU†, WEI-WEI XU‡, HAO LIU§, AND LI-JUAN MA¶

Abstract. Let $A \in \mathbb{C}^{m\times n}$ have generalized polar decomposition $A = QH$ with Q subunitary and H positive semidefinite. Absolute and relative perturbation bounds are derived for the subunitary polar factor Q in unitarily invariant norms and in Q-norms, that extend and improve existing bounds.

Key words. Perturbation upper bounds, Subunitary polar factor, Unitarily invariant norm, Q-Norm.

AMS subject classifications. 65F10.

1. Introduction. Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices and $\mathbb{C}_r^{m\times n}, \mathbb{C}_s^{m\times n}$ be the sets of $m \times n$ complex matrices with ranks r, s, respectively. The $n \times n$ identity matrix is denoted by I_n and the superscript $*$ denotes the conjugate transpose. We denote by $\|\cdot\|_2, \|\cdot\|_F$ and $\|\cdot\|$ the spectral norm, the Frobenius norm and any unitarily invariant norm, respectively. We denote by $A^\dagger$ the generalized inverse of matrix A. A unitarily invariant norm $\|\cdot\|$ is called a Q-norm if there exists another unitarily invariant norm $\|\cdot\|'$ such that $\|Y\| = (\|Y^*Y\|')^{\frac{1}{2}}$, which is denoted by $\|\cdot\|_Q$. Schatten norms are examples of Q-norms, see [2–5]. Let $A \in \mathbb{C}_r^{m\times n}$ and $\tilde{A} = A + E \in \mathbb{C}_s^{m\times n}$ have singular value decompositions

$$A = U \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad \tilde{A} = \tilde{U} \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{V}^* \tag{1.1}$$

and

$$Q = U_1 V_1^*, \quad H = V_1 \Sigma_1 V_1^*, \quad \tilde{Q} = \tilde{U}_1 \tilde{V}_1^*, \quad \tilde{H} = \tilde{V}_1 \tilde{\Sigma}_1 \tilde{V}_1^*,$$

where $U = (U_1, U_2)$, $\tilde{U} = (\tilde{U}_1, \tilde{U}_2) \in \mathbb{C}^{m\times m}$, $U_1 \in \mathbb{C}^{m\times r}$, $\tilde{U}_1 \in \mathbb{C}^{m\times s}$, $V = (V_1, V_2)$, $\tilde{V} = (\tilde{V}_1, \tilde{V}_2) \in \mathbb{C}^{n\times n}$, $V_1 \in \mathbb{C}^{n\times r}$, $\tilde{V}_1 \in \mathbb{C}^{n\times s}$, $\Sigma_1 = diag(\sigma_1, \dots, \sigma_r)$, $\tilde{\Sigma}_1 = diag(\tilde{\sigma}_1, \dots, \tilde{\sigma}_s)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, and $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_s > 0$.

The generalized polar decomposition of A and $\tilde{A}$ are defined by

$$A = QH, \quad \tilde{A} = \tilde{Q}\tilde{H}. \tag{1.2}$$

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*Received by the editors on April 17, 2017. Accepted for publication on April 14, 2018. Handling Editor: Francoise Tisseur. Corresponding Author: Lei Zhu.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 231-239, May 2018.
http://repository.uwyo.edu/ela

The matrices Q and $\tilde{Q}$ are called the (sub) unitary polar factors of A and $\tilde{A}$; H and $\tilde{H}$ are called the Hermitian positive (semi) definite factors of A and $\tilde{A}$. The decompositions (1.2) are unique if $R(Q^*) = R(H)$ and $R(\tilde{Q}^*) = R(\tilde{H})$, where $R(*)$ is the column space of a matrix (see [5]). The generalized polar decomposition is often called the canonical polar decomposition (see for example [Chapter 8, 9]) and Q is a partial isometry (another word for subunitary). Throughout the paper we always assume that the generalized polar decompositions satisfy the conditions $R(Q^*) = R(H)$ and $R(\tilde{Q}^*) = R(\tilde{H})$.

Perturbation bounds of subunitary polar factors of square and rectangular matrices have been discussed in [1–7]. When both A and its perturbation $\tilde{A}$ have rank r, the best bound in the Frobenius norm in [2] is given by

$$\|\tilde{Q} - Q\|_F \leq \frac{2}{\sigma_r + \tilde{\sigma}_r}\|E\|_F.$$

This bound however does not hold when A has rank r and $\tilde{A}$ has rank $s \neq r$. It is shown in [1] that in the latter case,

$$\|\tilde{Q} - Q\|_F \leq \frac{1}{\min\{\sigma_r, \tilde{\sigma}_s\}}\|E\|_F.$$

When both A and its perturbation $\tilde{A}$ have rank r, for any unitarily invariant norms in [5] the author showed the following bound

$$\|\tilde{Q} - Q\| \leq \frac{3}{\sigma_r + \tilde{\sigma}_r}\|E\|. \tag{1.3}$$

For Q-norm Li [5] presented the following result

$$\|\tilde{Q} - Q\|_Q \leq \frac{1+\sqrt{3}}{\sigma_r + \tilde{\sigma}_r}\|E\|_Q. \tag{1.4}$$

Many results have been given to improve the perturbation bounds of subunitary polar factors. However, the unitarily invariant norm and Q-norm bounds of perturbation with different ranks of subunitary polar factors have never been considered so far. By this motivation, in this paper, we present some new absolute and relative bounds in unitarily invariant norms and Q-norm, respectively.

The rest of this paper is organized as follows. In Section 2, we give some lemmas, which are useful to deduce our main results. In Section 3, we present perturbation upper bounds in unitarily invariant norms and Q-norm, respectively. In Section 4, the conclusion is drawn.

2. Some lemmas. In order to deduce our main results, some lemmas are needed.

LEMMA 2.1. [1] *Let $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$, $\tilde{\Sigma}_1 = diag(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_s)$, where $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_s > 0$, and let S be a complex matrix of suitable dimensions. Then there exists a unique solution X to the matrix equation $\Sigma_1 X + X\tilde{\Sigma}_1 = S$ and moreover, for unitarily invariant norm $\|\cdot\|$, we have*

$$\|X\| \leq \frac{1}{\sigma_r + \tilde{\sigma}_s}\|S\|.$$

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LEMMA 2.2. [5] *Let B have the block form*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Then

$$\|B\|_Q^2 \leq \|B_{11}\|_Q^2 + \|B_{12}\|_Q^2 + \|B_{21}\|_Q^2 + \|B_{22}\|_Q^2.$$

LEMMA 2.3. [8] *Let $A_{11} \in \mathbb{C}^{m_1\times n_1}, A_{12} \in \mathbb{C}^{m_1\times n_2}, A_{21} \in \mathbb{C}^{m_2\times n_1}, A_{22} \in \mathbb{C}^{m_2\times n_2}$. Then for unitarily invariant norm we have*

$$\left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \right\|.$$

LEMMA 2.4. *Let $A_{11} \in \mathbb{C}^{m_1\times n_1}, A_{12} \in \mathbb{C}^{m_1\times n_2}, A_{21} \in \mathbb{C}^{m_2\times n_1}$. Then for unitarily invariant norm $\|\cdot\|$ we have*

$$\left\| \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix} \right\|, \tag{2.1}$$

$$\left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix} \right\|. \tag{2.2}$$

Proof. It is easy to see that

$$\left\| \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix} \right\|. \tag{2.3}$$

Let the SVDs of A_{12} and A_{21} be

$$\begin{aligned} A_{12} &= W\Gamma Z^* = (W_1, W_2) \begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_1^* \\ Z_2^* \end{pmatrix} \\ &= W_1 \Gamma_1 Z_1^*, \end{aligned}$$

$$\begin{aligned} A_{21} &= \tilde{W}\tilde{\Gamma}\tilde{Z}^* = (\tilde{W}_1, \tilde{W}_2) \begin{pmatrix} \tilde{\Gamma}_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}_1^* \\ \tilde{Z}_2^* \end{pmatrix} \\ &= \tilde{W}_1 \tilde{\Gamma}_1 \tilde{Z}_1^*, \end{aligned}$$

where $W = (W_1, W_2),\ \tilde{W} = (\tilde{W}_1, \tilde{W}_2),\ Z = (Z_1, Z_2),\ \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)$ are unitary. Since

$$\begin{aligned} \begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix} &= \begin{pmatrix} W_1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \tilde{\Gamma}_1 \end{pmatrix} \begin{pmatrix} Z_1^* & 0 \\ 0 & \tilde{Z}_1^* \end{pmatrix} \\ &= \begin{pmatrix} I_{m_1} & 0 \\ 0 & \tilde{W}_1 \tilde{Z}_1^* \end{pmatrix} \begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix} \begin{pmatrix} I_{n_2} & 0 \\ 0 & \tilde{W}_1 \tilde{Z}_1^* \end{pmatrix}, \end{aligned} \tag{2.4}$$

taking a unitarily invariant norm in (2.4) yields

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 231-239, May 2018.
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$$\left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix}\right\| \le \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix}\right\|.$$

Conversely,

$$\left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix}\right\| \le \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix}\right\|.$$

Hence,

$$\left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix}\right\|. \tag{2.5}$$

From (2.3) and (2.5), we can deduce

$$\left\|\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21}^* \end{pmatrix}\right\|,$$

which yields (2.1). Since

$$\left\|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{12} & A_{11} \\ 0 & A_{21} \end{pmatrix}\right\|, \tag{2.6}$$

and by Lemma 2.3, we have

$$\left\|\begin{pmatrix} A_{12} & A_{11} \\ 0 & A_{21} \end{pmatrix}\right\| \ge \left\|\begin{pmatrix} A_{12} & 0 \\ 0 & A_{21} \end{pmatrix}\right\|. \tag{2.7}$$

Combining (2.5) and (2.6) with (2.7) we obtain (2.2). ∎

3. Perturbation bounds. In this section, we will present perturbation bounds of subunitary polar factors in unitarily invariant norms and in Q-norm, respectively.

Theorem 3.1. *Let $A \in \mathbb{C}_r^{m\times n}$, $\tilde{A} \in \mathbb{C}_s^{m\times n}$ have the singular value decompositions (1.1). Then, for unitarily invariant norm, we have*

$$\|\tilde{Q} - Q\| \le \left(\frac{2}{\sigma_r + \tilde{\sigma}_s} + \frac{1}{\min\{\sigma_r, \tilde{\sigma}_s\}}\right)\|E\|, \tag{3.1}$$

and for the Q-norm, we have

$$\|\tilde{Q} - Q\|_Q \le \left(\left(\frac{2}{\sigma_r + \tilde{\sigma}_s}\right)^2 + \frac{1}{\sigma_r^2} + \frac{1}{\tilde{\sigma}_s^2}\right)^{\frac{1}{2}}\|E\|_Q. \tag{3.2}$$

Proof. Let $E = \tilde{A} - A$. By the singular value decompositions (1.1) it follows that

$$E = \tilde{U}_1\tilde{\Sigma}_1\tilde{V}_1^* - U_1\Sigma_1V_1^*. \tag{3.3}$$

By (3.3), we can obtain the following equalities

$$\tilde{U}_1^*E\tilde{V}_1 = U_1^*\tilde{U}_1\tilde{\Sigma}_1 - \Sigma_1V_1^*\tilde{V}_1, \tag{3.4}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 231-239, May 2018.
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$$\tilde{U}_1^* E^* V_1 = \tilde{\Sigma}_1 \tilde{V}_1^* V_1 - \tilde{U}_1^* U_1 \Sigma_1, \tag{3.5}$$

$$\tilde{U}_1^* E^* V_2 = \tilde{\Sigma}_1 \tilde{V}_1^* V_2, \tag{3.6}$$

$$\tilde{V}_1^* E^* \tilde{U}_2 = -\Sigma_1 U_1^* \tilde{U}_2. \tag{3.7}$$

Combining (3.4) with (3.5), we have

$$\Sigma_1(U_1^* \tilde{U}_1 - V_1^* \tilde{V}_1) + (U_1^* \tilde{U}_1 - V_1^* \tilde{V}_1)\tilde{\Sigma}_1 = U_1^* E \tilde{V}_1 - V_1^* E^* \tilde{U}_1. \tag{3.8}$$

It follows from Lemma 2.1 with $X = U_1^* \tilde{U}_1 - V_1^* \tilde{V}_1$ and $S = U_1^* E \tilde{V}_1 - V_1^* E^* \tilde{U}_1$ that

$$\|U_1^* \tilde{U}_1 - V_1^* \tilde{V}_1\| \leq \frac{1}{\sigma_r + \tilde{\sigma}_s} \|U_1^* E \tilde{V}_1 - V_1^* E^* \tilde{U}_1\| \leq \frac{2}{\sigma_r + \tilde{\sigma}_s} \|E\|. \tag{3.9}$$

It follows from (3.3)–(3.7) that

$$\begin{aligned} \tilde{U}^* E V &= \begin{pmatrix} \tilde{U}_1^* \\ \tilde{U}_2^* \end{pmatrix} E(V_1, V_2) \\ &= \begin{pmatrix} \tilde{\Sigma}_1 \tilde{V}_1^* V_1 - \tilde{U}_1^* U_1 \Sigma_1 & \tilde{\Sigma}_1 \tilde{V}_1^* V_2 \\ -\tilde{U}_2^* U_1 \Sigma_1 & 0 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Then, by Lemma 2.4 and (3.10), we have

$$\left\| \begin{pmatrix} \tilde{\Sigma}_1 \tilde{V}_1^* V_2 & 0 \\ 0 & -\Sigma_1 U_1^* \tilde{U}_2 \end{pmatrix} \right\| \leq \|E\|. \tag{3.11}$$

Since

$$\begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \Sigma_1 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^* V_2 & 0 \\ 0 & -U_1^* \tilde{U}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}_1 \tilde{V}_1^* V_2 & 0 \\ 0 & -\Sigma_1 U_1^* \tilde{U}_2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \tilde{V}_1^* V_2 & 0 \\ 0 & -U_1^* \tilde{U}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}_1^{-1} & 0 \\ 0 & \Sigma_1^{-1} \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_1 \tilde{V}_1^* V_2 & 0 \\ 0 & -\Sigma_1 U_1^* \tilde{U}_2 \end{pmatrix}$$

and together with (3.11), we have

$$\left\| \begin{pmatrix} \tilde{V}_1^* V_2 & 0 \\ 0 & -U_1^* \tilde{U}_2 \end{pmatrix} \right\| \leq \frac{1}{\min\{\sigma_r, \tilde{\sigma}_s\}} \|E\|. \tag{3.12}$$

Since

$$\begin{pmatrix} \tilde{U}_1^* \\ \tilde{U}_2^* \end{pmatrix} (\tilde{Q} - Q)(V_1, V_2) = \begin{pmatrix} \tilde{V}_1^* V_1 - \tilde{U}_1^* U_1 & \tilde{V}_1^* V_2 \\ -\tilde{U}_2^* U_1 & 0 \end{pmatrix}, \tag{3.13}$$

then

$$\|\tilde{Q} - Q\| \leq \left\| \begin{pmatrix} \tilde{V}_1^* V_1 - \tilde{U}_1^* U_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & \tilde{V}_1^* V_2 \\ -\tilde{U}_2^* U_1 & 0 \end{pmatrix} \right\|,$$

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A publication of the International Linear Algebra Society
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together with (3.9), (3.12) and Lemma 2.4, we have (3.1). For Q-norm, it follows from (3.13) and Lemma 2.2 that

$$\|\tilde{Q}-Q\|_Q^2 \leq \|\tilde{V}_1^*V_1-\tilde{U}_1^*U_1\|_Q^2+\|\tilde{V}_1^*V_2\|_Q^2+\|\tilde{U}_2^*U_1\|_Q^2. \tag{3.14}$$

From (3.8) and Lemma 2.1, we have

$$\|\tilde{V}_1^*V_1-\tilde{U}_1^*U_1\|_Q^2 \leq \left(\frac{2}{\sigma_r+\tilde{\sigma}_s}\right)^2\|E\|_Q^2. \tag{3.15}$$

It follows from (3.6) and (3.7) that

$$\tilde{V}_1^*V_2=\tilde{\Sigma}_1^{-1}\tilde{U}_1^*E^*V_2 \quad \text{and} \quad U_1^*\tilde{U}_2=-\Sigma_1^{-1}\tilde{V}_1^*E^*\tilde{U}_2.$$

Then

$$\|\tilde{V}_1^*V_2\|_Q^2 \leq \frac{1}{\tilde{\sigma}_s^2}\|E\|_Q^2 \quad \text{and} \quad \|\tilde{U}_2^*U_1\|_Q^2 \leq \frac{1}{\sigma_r^2}\|E\|_Q^2. \tag{3.16}$$

Hence, we can obtain (3.2) from (3.14)–(3.16). This completes the proof. ∎

Remark 3.1. When $r=s$, (3.1) reduces to

$$\|\tilde{Q}-Q\| \leq \left(\frac{2}{\sigma_r+\tilde{\sigma}_r}+\frac{1}{\min\{\sigma_r,\tilde{\sigma}_r\}}\right)\|E\|,$$

which is the existing bound in [4]. This means that the bounds in Theorem 3.1 extend the ones in [4]. The following example illustrates that (1.3) and (1.4) can not hold when $A\in\mathbb{C}_r^{m\times n},\ \tilde{A}\in\mathbb{C}_s^{m\times n}, r\neq s$.

Example 3.1. Let $A=\begin{pmatrix}\sigma_1 & 0\\ 0 & 0\end{pmatrix}\in\mathbb{C}_1^{2\times 2}$ and $\tilde{A}=\begin{pmatrix}\sigma_1 & 0\\ 0 & \epsilon\end{pmatrix}\in\mathbb{C}_2^{2\times 2}$, where $0<\epsilon<\frac{\sigma_1}{4}$, setting $\|\cdot\|$ by $\|\cdot\|_F$, then

$$\frac{3}{\sigma_1+\epsilon}\|\tilde{A}-A\|_F=\frac{3\epsilon}{\sigma_1+\epsilon}<1=\|\tilde{Q}-Q\|_F$$

and

$$\left(\frac{2}{\sigma_1+\epsilon}+\frac{2}{\sigma_1}\right)\|\tilde{A}-A\|_F=\left(\frac{2}{\sigma_1+\epsilon}+\frac{2}{\sigma_1}\right)\epsilon<1=\|\tilde{Q}-Q\|_F,$$

which implies (1.3) and (1.4) can not hold. Also, for Q-norm, (1.6) can not hold when $A\in\mathbb{C}_r^{m\times n},\ \tilde{A}\in\mathbb{C}_s^{m\times n}, r\neq s$. Set $\|\cdot\|_Q$ by $\|\cdot\|_F$, then

$$\frac{1+\sqrt{3}}{\sigma_1+\epsilon}\|\tilde{A}-A\|_F=\frac{(1+\sqrt{3})\epsilon}{\sigma_1+\epsilon}<1=\|\tilde{Q}-Q\|_F,$$

which implies (1.6) can not hold.

In the following, we will present another kind of perturbation bounds which depend on the relative perturbations $\tilde{A}^\dagger E,\ A^\dagger E$ and $EA^\dagger$.

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A publication of the International Linear Algebra Society
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THEOREM 3.2. *Let $A \in \mathbb{C}_r^{m\times n}$, $\tilde{A} \in \mathbb{C}_s^{m\times n}$ have the singular value decompositions (1.1). Then, for unitarily invariant norm, we have*

$$\|\tilde{Q} - Q\| \le \left(1 + \frac{\tilde{\sigma}_1}{\sigma_r + \tilde{\sigma}_s}\right)\|\tilde{A}^\dagger E\| + \frac{\sigma_1}{\sigma_r + \tilde{\sigma}_s}\|A^\dagger E\| + \|EA^\dagger\|, \tag{3.17}$$

and for the Q-norm, we have

$$\|\tilde{Q} - Q\|_Q \le \left(\frac{(\tilde{\sigma}_1\|\tilde{A}^\dagger E\|_Q + \sigma_1\|A^\dagger E\|_Q)^2}{(\sigma_r + \tilde{\sigma}_s)^2} + \|EA^\dagger\|_Q^2 + \|\tilde{A}^\dagger E\|_Q^2\right)^{\frac{1}{2}}. \tag{3.18}$$

Proof. By the singular value decompositions (1.1) it follows that

$$A = U_1\Sigma_1V_1^*,\ A^\dagger = V_1\Sigma_1^{-1}U_1^*,\ \tilde{A} = \tilde{U}_1\tilde{\Sigma}_1\tilde{V}_1^*, \tilde{A}^\dagger = \tilde{V}_1\tilde{\Sigma}_1^{-1}\tilde{U}_1^*. \tag{3.19}$$

It follows from (3.19) that

$$V_1\Sigma_1^{-1}U_1^*\tilde{U}_1\tilde{\Sigma}_1\tilde{V}_1^* - V_1V_1^* = A^\dagger E. \tag{3.20}$$

From (3.20), we have

$$U_1^*\tilde{U}_1\tilde{\Sigma}_1 - \Sigma_1V_1^*\tilde{V}_1 = \Sigma_1V_1^*A^\dagger E\tilde{V}_1, \tag{3.21}$$

similarly, we have

$$\tilde{\Sigma}_1\tilde{V}_1^*V_1 - \tilde{U}_1^*U_1\Sigma_1 = \tilde{\Sigma}_1\tilde{V}_1^*\tilde{A}^\dagger EV_1, \tag{3.22}$$

$$\tilde{V}_1^*V_2 = \tilde{V}_1^*\tilde{A}^\dagger EV_2, \tag{3.23}$$

$$-\tilde{U}_2^*U_1 = \tilde{U}_2^*EA^\dagger U_1. \tag{3.24}$$

Combining (3.21) with (3.22) gives

$$(U_1^*\tilde{U}_1 - V_1^*\tilde{V}_1)\tilde{\Sigma}_1 + \Sigma_1(U_1^*\tilde{U}_1 - V_1^*\tilde{V}_1) = \Sigma_1V_1^*A^\dagger E\tilde{V}_1 - (\tilde{V}_1^*\tilde{A}^\dagger EV_1)^*\tilde{\Sigma}_1. \tag{3.25}$$

It follows from (3.23)–(3.25) and Lemma 2.1 that

$$\|U_1^*\tilde{U}_1 - V_1^*\tilde{V}_1\| \le \frac{\sigma_1\|A^\dagger E\| + \tilde{\sigma}_1\|\tilde{A}^\dagger E\|}{\sigma_r + \tilde{\sigma}_s}, \tag{3.26}$$

$$\|\tilde{V}_1^*V_2\| \le \|\tilde{A}^\dagger E\|, \tag{3.27}$$

$$\|\tilde{U}_2^*U_1\| \le \|EA^\dagger\|. \tag{3.28}$$

From (3.13) and (3.26)–(3.28), we have (3.17). For Q-norm, it follows that

$$\|\tilde{Q} - Q\|_Q^2 \le \|U_1^*\tilde{U}_1 - V_1^*\tilde{V}_1\|_Q^2 + \|\tilde{V}_1^*V_2\|_Q^2 + \|\tilde{U}_2^*U_1\|_Q^2. \tag{3.29}$$

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Combining (3.26)–(3.29) yields (3.18). □

Remark 3.2. The bounds (3.17) and (3.18) only depend on the relative perturbations $\tilde{A}^\dagger E,\ A^\dagger E$ and $EA^\dagger$. The following example shows that the bound (3.17) can be sharper than the bound (3.1).

Example 3.2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} 1 & 0.05 & 0.05 \\ 0 & 0 & 0 \\ 0 & 0 & 2.05 \end{pmatrix}$. A simple calculation yields that by Bound (3.1) we get

$$\|\tilde{Q} - Q\| \leq 3.6404;$$

by bound (3.17), we get

$$\|\tilde{Q} - Q\| \leq 3.1278.$$

Remark 3.3. It is easy to check that in Theorem 3.1

$$\left(\frac{2}{\sigma_r + \tilde{\sigma}_s} + \frac{1}{\min\{\sigma_r, \tilde{\sigma}_s\}}\right) \|E\|_Q \geq \left(\left(\frac{2}{\sigma_r + \tilde{\sigma}_s}\right)^2 + \frac{1}{\sigma_r^2} + \frac{1}{\tilde{\sigma}_s^2}\right)^{\frac{1}{2}} \|E\|_Q$$

and in Theorem 3.2

$$\begin{aligned} &\left(1 + \frac{\tilde{\sigma}_1}{\sigma_r + \tilde{\sigma}_s}\right) \|\tilde{A}^\dagger E\|_Q + \frac{\sigma_1}{\sigma_r + \tilde{\sigma}_s} \|A^\dagger E\|_Q + \|EA^\dagger\|_Q \\ &\geq \left[\frac{(\tilde{\sigma}_1 \|\tilde{A}^\dagger E\|_Q + \sigma_1 \|A^\dagger E\|_Q)^2}{(\sigma_r + \tilde{\sigma}_s)^2} + \|EA^\dagger\|_Q^2 + \|\tilde{A}^\dagger E\|_Q^2\right]^{\frac{1}{2}}. \end{aligned}$$

Hence, by using Q-norms, we can derive sharper bounds than the bounds by using general unitary invariant norms.

4. Conclusions. In this paper, we consider refined perturbation upper bounds of subunitary polar factors in unitarily invariant norm and in Q-norm, respectively, which extend some existing results. We present the absolute perturbation bounds in unitarily invariant norm and in Q-norm, respectively and the corresponding relative perturbation bounds are given.

Acknowledgment. The work was supported by Natural Science Foundation of China under grants no. 11501300 and no. U1533202, and Civil Aviation Science and Technology Project under grant no. 20150218.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 231-239, May 2018.
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