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BOUNDS ON THE SUM OF MINIMUM SEMIDEFINITE RANK OF A GRAPH AND ITS COMPLEMENT*

SIVARAM K. NARAYAN[†] AND YOUSRA SHARAWI[‡]

Abstract. The minimum semi-definite rank (msr) of a graph is the minimum rank among all positive semi-definite matrices associated to the graph. The graph complement conjecture gives an upper bound for the sum of the msr of a graph and the msr of its complement. It is shown that when the msr of a graph is equal to its independence number, the graph complement conjecture holds with a better upper bound. Several sufficient conditions are provided for the msr of different classes of graphs to equal to its independence number.

Key words. Minimum semidefinite rank, Matrix of a graph, Independence number, Graph complement conjecture.

AMS subject classifications. 15A18, 15A57, 05C50.

1. Introduction. We will use V to denote the set of vertices of a graph G . The cardinality of V represents the *order* of G , denoted $|G|$. An edge is an unordered pair of vertices and we use E to denote the set of edges in a graph G . A graph is said to be *simple* if it has no loops or multiple edges. Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, we can associate a graph $G(A)$ to the matrix A in such a way that the set of vertices is $V = \{v_1, v_2, \dots, v_n\}$ and the set of edges is $E = \{\{v_i, v_j\} : a_{ij} \neq 0, i \neq j\}$. The diagonal entries of A do not affect the structure of $G(A)$. The graph $G(A)$ is an undirected simple graph. A Hermitian matrix $A \in M_n(\mathbb{C})$ is called *positive semidefinite* (psd) if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ [7].

Suppose G is a simple connected graph with vertex set $\{v_1, v_2, \dots, v_n\}$. We associate a set of vectors $\vec{V} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathbb{C}^m to the vertices such that, for $i \neq j$, $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$ if and only if v_i and v_j are adjacent vertices in G . The set \vec{V} is called a vector representation (or orthogonal representation) of G . If $X = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ is an $m \times n$ matrix then $A = X^*X$ is a psd matrix associated with G . Since every psd matrix A associated with G can be written as $A = B^*B$ for some matrix B , we can always find a vector representation \vec{V} that produces the matrix A with $rank(A) = dim(span \vec{V})$ [7].

If E is a block matrix given by $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is nonsingular and D is square, then the matrix $D - CA^{-1}B$ is called the Schur complement of A in E . If E is Hermitian (i.e, $C = B^*$) and A and $D - B^*A^{-1}B$ are both psd, then E is also psd, and $rank(E) = rank(A) + rank(D - B^*A^{-1}B)$ [7]. If the upper-block B consists of zeros, we call E a *block lower triangular* matrix.

A symmetric matrix A satisfies the Strong Arnold Hypothesis if the only real symmetric matrix X such that $AX = A \circ X = X \circ I = 0$ is the zero matrix, where \circ denotes the entrywise product of matrices, and I is the $n \times n$ identity matrix [5].

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Let $\mathcal{P}(G)$ denote the set of all complex Hermitian psd matrices A whose graph $G(A) = G$. Let $S_+(G)$ denote the set of all real symmetric psd matrices A whose graph $G(A) = G$.

The *complex minimum semidefinite rank* of G is denoted by $mr_+^{\mathbb{C}}(G)$ and is defined as $mr_+^{\mathbb{C}}(G) = \min\{\text{rank}(A) : A \in \mathcal{P}(G)\}$ and the *real minimum semidefinite rank* of G is denoted $mr_+^{\mathbb{R}}(G)$ and is defined as $mr_+^{\mathbb{R}}(G) = \min\{\text{rank}(A) : A \in S_+(G)\}$. Since $S_+(G) \subseteq \mathcal{P}(G)$ we conclude $mr_+^{\mathbb{C}}(G) \leq mr_+^{\mathbb{R}}(G)$. An example where strict inequality holds is given in [1]. When $mr_+^{\mathbb{C}}(G) = mr_+^{\mathbb{R}}(G)$ we write $mr_+(G)$.

The paper is organized as follows: In Section 2, we present graph theory preliminaries and some known results about the *msr* that will be used throughout the paper. In Section 3, we show that when the *msr* of a graph is equal to its independence number, we get an upper bound for the sum of the *msr* of a graph and its complement that is better than the one given in the graph complement conjecture. We present different classes of graphs for which the *msr* is equal to the independence number.

2. Preliminaries. In this section, we present some graph theory definitions and some known results related to the *msr*(G) which will be used throughout the paper.

We will use $N(v)$ to denote the set of vertices that are adjacent to a vertex v in a simple graph G . The *degree* of a vertex v in G , denoted $deg_G(v)$, is the cardinality of $N(v)$. Let $\delta(G)$ denote the minimum degree of the vertices in G , and $\Delta(G)$ denote the maximum degree of the vertices in G . A graph G is *regular* if $\delta(G) = \Delta(G)$, and it is *k-regular* if $\delta(G) = \Delta(G) = k$.

A graph is a *tree* if it is a connected graph with n vertices and $n - 1$ edges. A *star graph* on n vertices is a tree with one vertex having degree $n - 1$, called the center, and the other $n - 1$ vertices having degree 1. A *cycle* of length n is the graph C_n on n vertices $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ with n edges $\{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{n-1}, v_0\}\}$. A *complete graph* or *clique* is a connected graph in which the vertices are pairwise adjacent, we use K_n to denote a complete graph on n vertices. The size of the largest complete subgraph of G is called the *clique number*, denoted $\omega(G)$ ([12], p. 192).

An *independent set* in a graph G is a set of vertices where no two of the vertices are adjacent. The cardinality of a largest independent set in G is called the *independence number* of G and is denoted by $\alpha(G)$ ([12], p. 113). It was shown in [4] that, for a connected graph G , $msr(G) \geq \alpha(G)$.

We use $G = (R \cup L, E)$ to denote a bipartite graph, where R is the set of vertices on the right side and L is the set of vertices on the left side in a pictorial representation of G . In a bipartite graph G with $|R| = m \geq |L| = n$, the set of vertices in both R and L are pairwise disjoint, so they form two independent sets. When G is connected, the independence number of G is m . Hence, $msr(G) \geq m$. Equality is achieved for a complete bipartite graph $K_{m,n}$, $m \geq n$ [3].

An *induced subgraph* H of G is a subgraph with $V(H) \subseteq V(G)$ and $E(H) = \{\{i, j\} \in E(G) : i, j \in V(H)\}$. Since a principal submatrix of a psd matrix is psd, and the rank of a submatrix can never be greater than that of a matrix [7], minimum semi-definite rank of any induced subgraph of a given graph G gives a lower bound for the minimum semi-definite rank of G . Moreover, for a psd matrix $A = [a_{ij}]$, the diagonal entry a_{ii} is a principal submatrix, and hence, $a_{ii} \geq 0$. If there is a nonzero off-diagonal entry in the *i*th row of A then $a_{ii} > 0$.

The *complement* \overline{G} of a simple graph G on n vertices is the simple graph with vertex set $V(G)$ and edge set $E(\overline{G})$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Since cliques become independent sets (and vice versa) under complementation, we have $\alpha(G) = \omega(\overline{G})$.

A subset S of vertices of a connected graph G is a *cut-set* if the graph induced by the set of vertices $V(G) - S$ is disconnected. The *vertex connectivity* of a graph G , denoted $k(G)$, is the minimum cardinality of a vertex set S such that $G - S$ is disconnected or has only one vertex. Since deleting the neighbors of any vertex in a graph G disconnects G , $k(G) \leq \delta(G)$ ([12], p. 149). A graph G is *k-connected* if every cut set of G has at least k vertices.

A graph G is a *superposition* of two graphs, G_1 and G_2 if G is obtained by identifying G_1 and G_2 at a set of vertices, keeping all the edges that are present in either G_1 or G_2 [3].

An *isomorphism* from G to H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G is *isomorphic* to H if there is an isomorphism from G to H .

The *Laplacian* matrix of a graph G is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose i th diagonal entry is the degree of vertex v_i in G , and $A(G)$ is the adjacency matrix of G . Since the Laplacian matrix $L(G)$ is psd and has rank $|G| - 1$ [8] when G is a connected graph, it follows that $msr(G) \leq |G| - 1$ for every connected graph G [6]. Moreover, $msr(G) = n - 1$ if and only if G is a tree on n vertices [11]. It is well known that for a connected graph G with $|G| \geq 2$, $msr(G) = 1$ if and only if $G = K_n$. Thus, for any connected graph G with $|G| \geq 2$ where G is neither a tree nor a complete graph, we have $2 \leq msr(G) \leq |G| - 2$. If a graph G is disconnected, then the direct sum of psd matrices for the connected components G_i of G gives a psd matrix for the entire graph. In this case, $msr(G) = \sum_i msr(G_i)$. Hence, we assume in this paper all graphs are simple and connected.

3. The graph complement conjecture. It has been conjectured in [2] that $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) \leq |G| + 2$ for all graphs G , which is known as the “graph complement conjecture (GCC+)”. Proposition 3.2 below gives a better upper bound for the sum of the msr of a graph and its complement.

LEMMA 3.1. [9] *If a graph G contains a clique of size k , then there exists a matrix $A \in S_+(G)$ that satisfies the Strong Arnold Hypothesis with $\text{rank}(A) = |G| - k + 1$.*

PROPOSITION 3.2. *Let G be a connected graph. If $mr_+^{\mathbb{R}}(G) = \alpha(G)$, then $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) \leq |G| + 1$.*

Proof. Since $mr_+^{\mathbb{R}}(G) = \alpha(G)$ and $\alpha(G) = \omega(\overline{G})$, where $\omega(\overline{G})$ is the size of the largest clique in \overline{G} , using Lemma 3.1, we get $mr_+^{\mathbb{R}}(\overline{G}) \leq |G| - \omega(\overline{G}) + 1 = |G| - mr_+^{\mathbb{R}}(G) + 1$. Hence, $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) \leq |G| + 1$. \square

In this section, we investigate classes of graphs whose msr is equal to their independence number and hence satisfy Proposition 3.2. In some cases, we give a better upper bound.

PROPOSITION 3.3. *If G is a connected graph obtained from a complete graph K_m by deleting all the edges of the subgraph K_r with $r \leq m - 1$, then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r$ and $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) = r + 1$.*

Proof. We show that $mr_+^{\mathbb{R}}(G) = r$. Since edges of K_r are removed from K_m , there is a set of r vertices in G no two of which are adjacent. Since this is a maximum independent set, $\alpha(G) = r$ and so $mr_+^{\mathbb{R}}(G) \geq r$ by [3]. To show that $mr_+^{\mathbb{R}}(G) \leq r$, we will exhibit a vector representation of G in \mathbb{R}^r . Label the vertices of G as $\{v_1, v_2, \dots, v_r, \dots, v_m\}$. Assume that the set of the vertices $\{v_1, v_2, \dots, v_r\}$ is the maximal independent set. Let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_r\}$ be the standard orthonormal basis in \mathbb{R}^r . We assign to the vertices v_1, v_2, \dots, v_r the vectors $\vec{v}_1 = \vec{e}_1, \vec{v}_2 = \vec{e}_2, \dots, \vec{v}_r = \vec{e}_r$. For the remaining $m - r$ vertices, we assign vectors of the form $\vec{v}_i = \sum_{j=1}^r \vec{e}_j, i = r + 1, r + 2, \dots, m$. The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \dots, \vec{v}_m\}$ is a vector representation of the graph G in \mathbb{R}^r , so $mr_+^{\mathbb{R}}(G) \leq r$. Hence, $mr_+^{\mathbb{R}}(G) = r$. It is clear that \overline{G} consists of $m - r$ isolated vertices and the complete graph K_r , so $m_+^{\mathbb{R}}(\overline{G}) = 1$. Therefore, $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) = r + 1$. \square

PROPOSITION 3.4. *If G is a connected graph obtained from a complete graph K_m , $m \geq 5$ by deleting all but s edges of the subgraph K_r , where $1 \leq s \leq r - 1$, $3 \leq r \leq m - 1$, such that all the s edges share a common vertex in G , then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r - 1$.*

Proof. Label the vertices of G as $\{v_1, v_2, \dots, v_r, \dots, v_m\}$. Assume that the set of the s edges of K_r are $\{\{v_1, v_2\}, \{v_1, v_3\}, \dots, \{v_1, v_{s+1}\}\} \subset E(G)$. Then, the set of vertices $\{v_2, v_3, \dots, v_r\}$ is a maximal independent set and so, the independence number of G is $\alpha(G) = r - 1$ and $mr_+^{\mathbb{R}}(G) \geq r - 1$. To show that $mr_+^{\mathbb{R}}(G) \leq r - 1$, we will exhibit a vector representation of G in \mathbb{R}^{r-1} . Let $\{\vec{e}_2, \dots, \vec{e}_s, \vec{e}_{s+1}, \dots, \vec{e}_r\}$ be the standard orthonormal basis in \mathbb{R}^{r-1} . To the vertices v_2, \dots, v_r we assign the vectors $\vec{v}_2 = \vec{e}_2, \vec{v}_3 = \vec{e}_3, \dots, \vec{v}_{s+1} = \vec{e}_{s+1}, \dots, \vec{v}_r = \vec{e}_r$, and to the vertex v_1 , we assign the vector $\vec{v}_1 = \sum_{j=2}^{s+1} \vec{e}_j$. To the remaining $m - r$ vertices, we assign the vectors $\vec{v}_i = \sum_{j=2}^r \vec{e}_j$ where $i = r + 1, r + 2, \dots, m$. The set of the vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ represents G in \mathbb{R}^{r-1} and so, $mr_+^{\mathbb{R}}(G) \leq r - 1$. Therefore, $mr_+^{\mathbb{R}}(G) = r - 1$. \square

COROLLARY 3.5. *If G is a connected graph obtained from a complete graph K_m , $m \geq 5$ by deleting all but one edge of the subgraph K_r , $r \leq m - 1$, then $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r + 1$.*

Proof. The graph \overline{G} consists of $m - r$ isolated vertices and the complete graph K_r minus one edge. Hence, $mr_+^{\mathbb{R}}(\overline{G}) = 2$ by Proposition 3.3. Since $mr_+^{\mathbb{R}}(G) = r - 1$ using Proposition 3.4, we get $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r + 1$. \square

LEMMA 3.6. [3] *If G is a superposition of two graphs G_1 and G_2 , then $msr(G) \leq msr(G_1) + msr(G_2)$.*

COROLLARY 3.7. *Let G be a connected graph obtained from a complete graph K_m , $m \geq 7$ by deleting all but s edges of the subgraph K_r , where $2 \leq s < r - 1$ and $4 \leq r \leq m - 1$. If all the s edges share a common vertex v in G , then $mr_+^{\mathbb{R}}(\overline{G}) \leq r - s$.*

Proof. In this case, the graph \overline{G} consists of $m - r$ isolated vertices and a connected component, call it \overline{G}' . Since $mr_+^{\mathbb{R}}$ of an isolated vertex is zero, we may only consider the connected component of \overline{G} . Since all the s edges share a common vertex in G , the connected component \overline{G}' can be viewed as the superposition of the complete graph K_{r-1} and the star graph on $r - s$ vertices whose center is the vertex v . The two graphs are identified at the set of vertices $V(\overline{G}') - S$, where S is the set of vertices on which the s edges are incident. Using Lemma 3.6, we get $mr_+^{\mathbb{R}}(\overline{G}) = mr_+^{\mathbb{R}}(\overline{G}') \leq 1 + r - s - 1 = r - s$. \square

PROPOSITION 3.8. *If G is a connected graph obtained from a complete graph K_m , $m \geq 5$ by deleting all but the s edges of the subgraph K_r , where $2 \leq s \leq \lfloor \frac{r}{2} \rfloor$ and $3 \leq r \leq m - 1$ such that no two of the s edges share a vertex, then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r - s$.*

Proof. Label the vertices of G as $\{v_1, v_2, \dots, v_r, \dots, v_m\}$. Since no two of the s edges share a vertex, they will be incident on $2s$ different vertices and the graph G will have $m - r$ vertices of degree $m - 1$. Let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{r-s}\}$ be the standard orthonormal basis in \mathbb{R}^{r-s} .

Case I. Suppose r is even and $s = \frac{r}{2}$. Assume that $\{\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{r-1}, v_r\}\} \subset E(G)$ is the set of s edges of K_r . By taking one vertex from each of the s existing edges in K_r we get a maximal set of $\frac{r}{2}$ independent vertices, so the independence number of G is $\alpha(G) = \frac{r}{2} = r - s$. Hence, $mr_+^{\mathbb{R}}(G) \geq r - s$. To show that $mr_+^{\mathbb{R}}(G) \leq r - s$, we will exhibit a vector representation of G in \mathbb{R}^{r-s} . To the vertices v_1, v_2, \dots, v_r we assign the vectors $\vec{v}_{2i} = \vec{v}_{2i-1} = \vec{e}_i$, where $1 \leq i \leq s$. To the remaining $m - r$ vertices, we assign the vectors $\vec{v}_t = \sum_{i=1}^s \vec{e}_i$, where $t = r + 1, r + 2, \dots, m$. The set of the vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ represents G in \mathbb{R}^{r-s} and so, $mr_+^{\mathbb{R}}(G) \leq r - s$. Therefore, $mr_+^{\mathbb{R}}(G) = r - s$.

Case II. Suppose r is odd and $s = \frac{r-1}{2}$. In this case, we assume the vertex v_r is not joined to any of the vertices incident on the s edges. Assume that the s edges of K_r are $\{\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{r-2}, v_{r-1}\}\} \subset$

$E(G)$. By taking one vertex from each of the s existing edges in K_r and the vertex v_r we get a maximal set of $\frac{r-1}{2} + 1$ independent vertices, so the independence number of G is $\alpha(G) = \frac{r+1}{2} = r - s$. Hence, $mr_+^{\mathbb{R}}(G) \geq r - s$. To show that $mr_+^{\mathbb{R}}(G) \leq r - s$, we will exhibit a vector representation of G in \mathbb{R}^{r-s} . To the vertices v_1, v_2, \dots, v_{r-1} we assign the vectors $\vec{v}_{2i} = \vec{v}_{2i-1} = \vec{e}_i$, where $1 \leq i \leq s$. To the vertex v_r , we assign the vector $\vec{v}_r = \vec{e}_{s+1}$. To the remaining $m - r$ vertices, we assign the vectors $\vec{v}_t = \sum_{i=1}^{s+1} \vec{e}_i$, where $t = r + 1, r + 2, \dots, m$. The set of the vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ represents G in \mathbb{R}^{s+1} and so, $mr_+^{\mathbb{R}}(G) \leq s + 1 = r - s$. Therefore, $mr_+^{\mathbb{R}}(G) = r - s$.

Case III. Suppose $2 \leq s < \lfloor \frac{r}{2} \rfloor$. In this case, the graph G has $r - 2s$ pairwise disjoint vertices other than the vertices that the s edges are incident on. By taking one vertex from each of the s existing edges in K_r and the $r - 2s$ pairwise disjoint vertices we get a maximal set I of independent vertices. Since $|I| = s + r - 2s = r - s$, the independence number of G is $\alpha(G) = r - s$. Hence, $mr_+^{\mathbb{R}}(G) \geq r - s$. Assume that the s edges of K_r are $\{\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{2s-1}, v_{2s}\}\} \subset E(G)$. To show that $mr_+^{\mathbb{R}}(G) \leq r - s$, we will exhibit a vector representation of G in \mathbb{R}^{r-s} . To the vertices v_1, v_2, \dots, v_{2s} we assign the vectors $\vec{v}_{2i} = \vec{v}_{2i-1} = \vec{e}_i$, where $1 \leq i \leq s$. To the $r - 2s$ pairwise disjoint vertices, we assign the vectors $\vec{v}_j = \vec{e}_j$, where $2s + 1 \leq j \leq r$. To the remaining $m - r$ vertices, we assign the vectors $\vec{v}_t = \sum_{i=1}^s \vec{e}_i + \sum_{j=2s+1}^r \vec{e}_j$ where $t = r + 1, r + 2, \dots, m$. The set of the vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ represents G in \mathbb{R}^{r-s} , from this we get $mr_+^{\mathbb{R}}(G) \leq r - s$. Therefore, $mr_+^{\mathbb{R}}(G) = r - s$. \square

Since the msr of a complete bipartite graph is equal to its independence number, complete bipartite graphs satisfy Proposition 3.2. Next, we present upper bounds for the sum of the msr of a bipartite graph and its complement.

PROPOSITION 3.9. *Let G be a connected bipartite graph which is not a tree. Then $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) \leq \frac{3}{2}|G| - 1$*

Proof. Since the set of vertices in R forms an independent set and $|R| \geq \frac{|G|}{2}$, we get $\alpha(G) = \omega(\overline{G}) \geq \frac{|G|}{2}$, where $\omega(\overline{G})$ is the size of the largest clique in \overline{G} . Using Lemma 3.1, we get $mr_+^{\mathbb{R}}(\overline{G}) \leq |G| - \omega(\overline{G}) + 1 \leq \frac{|G|}{2} + 1$. Since $mr_+^{\mathbb{R}}(G) \leq |G| - 2$, it follows that $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) \leq |G| - 2 + \frac{|G|}{2} + 1 = \frac{3}{2}|G| - 1$. \square

In the next proposition, we give an upper bound for the msr of complements of two classes of bipartite graphs.

PROPOSITION 3.10. *Let $G = K_{m,n} \setminus p$ edges, $2 \leq p < n \leq m$. Then, $mr_+(\overline{G}) \leq p + 2$ if one of the following conditions is satisfied:*

- (i) *All the missing edges in G share a common vertex $v \in V(G)$.*
- (ii) *No two of the missing edges in G share a common vertex.*

Proof. To prove (i), assume that all the missing edges share a common vertex $v \in V(G)$. We consider two cases:

Case I. Suppose $v \in R$. In this case, the graph \overline{G} can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the cut vertex v , where $G_1 = K_m$ and G_2 is the superposition of K_n and $K_{1,p}$ identified at the p vertices of $K_{1,p}$. By Lemma 3.6, we get $mr_+(\overline{G}) = mr_+(G_1) + mr_+(G_2) \leq 1 + 1 + p = 2 + p$.

Case II. If $v \in L$, then the graph \overline{G} can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the cut vertex v , where $G_1 = K_n$ and G_2 is the superposition of K_m and $K_{1,p}$ that are identified at the p vertices of $K_{1,p}$. By Lemma 3.6, we get $mr_+(\overline{G}) = mr_+(G_1) + mr_+(G_2) \leq 1 + 1 + p = 2 + p$.

To prove (ii), assume that no two of the missing edges share a common vertex. Then, the graph \overline{G} can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the p vertices in L from which the edges are missing, where $G_1 = K_n$ and G_2 is the superposition of $\lfloor \frac{p}{2} \rfloor$ copies of C_4 and another graph G'_2 identified at the p vertices in R from which the edges are missing, where $G'_2 = K_m$ if p is even and $G'_2 = K_m + e$ if p is odd. We explain this as follows: since both sets of vertices L and R form independent sets in G , they form cliques in \overline{G} . Since no two of the missing edges share a common vertex, each two of those missing edges will be incident on four different vertices in \overline{G} , two of them, say y_1 and y_2 are in R and the other two, say x_1 and x_2 are in L where $y_1 \in N(y_2)$ and $x_1 \in N(x_2)$ in \overline{G} . So if p is even, then G_2 can be viewed as a superposition of K_m and $\frac{p}{2}$ copies of C_4 identified at the p vertices in R from which the edges are missing, and if p is odd then G_2 can be viewed as a superposition of $K_m + e$ and $\frac{p-1}{2}$ copies of C_4 identified at the p vertices in R from which the edges are missing. Since $mr_+(G_2) \leq 2\lfloor \frac{p}{2} \rfloor + mr_+(G'_2)$, $2\lfloor \frac{p}{2} \rfloor + mr_+(G'_2) = 1 + p$ if p is even, and $2\lfloor \frac{p}{2} \rfloor + mr_+(G'_2) = \frac{2(p-1)}{2} + 2 = p + 1$ if p is odd, we obtain $mr_+(\overline{G}) \leq mr_+(G_1) + mr_+(G_2) \leq 1 + 1 + p = 2 + p$. \square

COROLLARY 3.11. *Let $G = K_{m,n} \setminus p$ edges, where $m \geq n + 1$ and $2 \leq p < n$ such that all the missing edges share a common vertex $v \in R$, $R \subset V(G)$ with $|R| = m$. Then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$.*

Proof. In this case, the graph G can be viewed as the superposition of the two subgraphs $G - v = K_{m-1,n}$ and $K_{n-p,1}$ identified at the p vertices in L from which the edges are missing. So, by Lemma 3.6 we get $mr_+(G) \leq m - 1 + n - p = |G| - 1 - p$, and by Proposition 3.10 we get $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq |G| - 1 - p + 2 + p = |G| + 1$. \square

COROLLARY 3.12. *Let $G = K_{m,n} \setminus p$ edges, where $m \geq n$ and $2 \leq p < n$ such that all the missing edges share a common vertex $v \in L$, $L \subset V(G)$ with $|L| = n \leq m$. Then $mr_+(G) + mr_+(\overline{G}) \leq m + 2p + 2$.*

Proof. Since G is obtained from $K_{m,n}$ by deleting p edges, we get $mr_+(G) \leq mr_+(K_{m,n}) + p = m + p$ [3]. By Proposition 3.10, $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq m + p + 2 + p = m + 2p + 2$. \square

COROLLARY 3.13. *Let $G = K_{m,n} \setminus p$ edges, where $m \geq n$ and $p \geq 2$ such that no two edges share an end point. Then $mr_+(G) + mr_+(\overline{G}) \leq m + 2p + 2$. In addition, if $p \leq \frac{n}{2}$ then G satisfies GCC+.*

Proof. Since G is modified from $K_{m,n}$ by deleting p edges, we get $mr_+(G) \leq mr_+(K_{m,n}) + p = m + p$ [3]. By Proposition 3.10, $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq m + p + 2 + p = m + 2p + 2$. If $p \leq \frac{n}{2}$ then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$. \square

COROLLARY 3.14. *Let $G = K_{m,n} \setminus p$ edges, where $2 \leq p < n$ such that all the missing edges share a common vertex $v \in L$, $L \subset V(G)$ with $|L| = n = |R| = m$. Then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$.*

Proof. In this case, the graph G can be viewed as a superposition of the two graphs $K_{m,n-1}$ and $K_{m-p,1}$ identified at the $m - p$ vertices in R . Hence, $mr_+(G) \leq m + m - p = |G| - p$. Using Proposition 3.10, we get $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$. \square

LEMMA 3.15. *Let M be an $m \times n$ matrix. If N is the $m \times m$ lower triangular matrix defined by $N[j, j] = 1$, $N[j, 1] = -\sum_{s=1}^n M[1, s] \cdot M[j, s]$ for $j = 2, \dots, m$, and $N[j, k] = -\sum_{s=1}^n M[k, s] \cdot M[j, s] - \sum_{s=1}^{k-1} N[k, s] \cdot N[j, s]$, where $k = 2, \dots, m$ and $j = k + 1, \dots, m$, then the rows of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ are mutually orthogonal.*

Proof. We have $\langle \text{row } i, \text{row } j \rangle = \sum_{s=1}^n M[i, s] \cdot M[j, s] + \sum_{k=1}^i N[i, k] \cdot N[j, k]$. Note that $N[i, k] = 0$ for $k > i$, and $N[j, k] = 0$ for $k > j$. Now $\sum_{s=1}^n M[i, s] \cdot M[j, s] = -N[j, i] - \sum_{s=1}^{i-1} N[i, s] \cdot N[j, s] = -\sum_{s=1}^i N[i, s] \cdot N[j, s]$ since $N[i, i] = 1$. Hence, $\langle \text{row } i, \text{row } j \rangle = 0$ for all $i \neq j$ where $1 \leq i \leq m$ and $1 \leq j \leq m + n$. \square

THEOREM 3.16. *Consider the matrix $C = \begin{bmatrix} M & N \end{bmatrix}^T$, where $\begin{bmatrix} M & N \end{bmatrix}$ is the matrix constructed in*



Lemma 3.15. Let $A = \begin{bmatrix} I_{m+n} & D \\ D^* & I_m \end{bmatrix}$, $m \geq 3$, $n \geq 0$, where I_{m+n} and I_m are the $(m+n) \times (m+n)$ and the $m \times m$ identity matrices, respectively, and the matrix D is an $(m+n) \times m$ matrix whose column vectors are given by $\frac{\vec{c}_i}{\|\vec{c}_i\|}$, where \vec{c}_i are the column vectors of the matrix C for all i , $1 \leq i \leq m$. Then the graph G associated to A is a connected bipartite graph whose $mr_+^C(G)$ is equal to $\alpha(G) = m+n$.

Proof. Since the rows of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ are mutually orthogonal, the columns of C are mutually orthogonal. Taking Schur Complement with respect to the $(1,1)$ entry of A we get that the matrix A is psd with $rank(A) = m+n$. Since the columns of A are mutually orthogonal, we can associate a graph to the matrix A whose vertices associated to the column vectors of A can be partitioned into two sets of vertices, call them L and R with $|R| = m+n$ and $|L| = m$ such that the vertices in each set are pairwise disjoint. In this case, $\alpha(G) = m+n$ and since $rank(A) = m+n$, we get $mr_+^C(G) = m+n = \alpha(G)$. \square

In the next theorem, we show that $mr_+^C(G)$ of a class of bipartite graphs is equal to their independence number by associating a psd matrix to the graph and showing that the rank of the associated matrix is the independence number of the given graph.

THEOREM 3.17. Let G be a connected bipartite graph with $V(G) = L \cup R$, $|L| = m$ and $|R| = m+n$, $n \geq 0$. Label the vertices of L as v_1, v_2, \dots, v_m and the vertices of R as u_1, u_2, \dots, u_{m+n} . If v_1 is adjacent to u_i for all $1 \leq i \leq n+1$, $i \neq 2$ and for $2 \leq j \leq m$, the vertex v_j is adjacent to u_k for all $1 \leq k \leq n+j$, the vertex v_t is not adjacent to u_{n+s} for all $1 \leq t \leq m$ and all $t+1 \leq s \leq m+n$, then $mr_+^C(G) = m+n = \alpha(G)$.

Proof. Since $|R| = m+n$, the independence number of G is $\alpha(G) = m+n$. Consider the matrix $C = \begin{bmatrix} M & N \end{bmatrix}^T$, where M is an $m \times n$ matrix that has the entry $M[2,1] = 0$ and all other entries are positive and the matrix N is constructed as in Lemma 3.15. To the graph G , associate the matrix $A = \begin{bmatrix} I_{m+n} & D \\ D^* & I_m \end{bmatrix}$, where I_{m+n} and I_m are the $(m+n) \times (m+n)$ and the $m \times m$ identity matrices respectively, the matrix D is an $(m+n) \times m$ matrix whose column vectors are given by $\frac{\vec{c}_i}{\|\vec{c}_i\|}$, where \vec{c}_i are the column vectors of the matrix C for all i , $1 \leq i \leq m$. Since $\alpha(G) = m+n$ and A is psd with $rank(A) = m+n$, we get $mr_+^C(G) = m+n$. \square

PROPOSITION 3.18. Let G be a k -regular graph on $2k$ vertices with $k \geq 2$, such that for each $v \in V(G)$ the set $N(v)$ forms an independent set. Then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$

Proof. We will show that G is isomorphic to the complete bipartite graph $K_{k,k}$. Let $v \in V(G)$. We claim that $V(G) - N(v) = N(w)$ for some $w \in N(v)$. If not, then $N(w) \cap N(v) \neq \emptyset$ for every $w \in N(v)$. This contradicts the assumption that $N(v)$ is an independent set. Therefore, G is isomorphic to $K_{k,k}$, and so $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$. \square

The k -connected Harary graph $H_{k,n}$, $k < n$, is constructed as follows [12]: Place n vertices in circular order. If $k = 2r$, form $H_{k,n}$ by making each vertex adjacent to the nearest r vertices in each direction around the circle. If $k = 2r + 1$ and n is even, then form $H_{k,n}$ by making each vertex adjacent to the nearest r vertices in each direction and to the vertex opposite it on the circle. In each case, the graph $H_{k,n}$ is regular. If $k = 2r + 1$ and n is odd, then index the vertices by the integers modulo n . Construct $H_{k,n}$ from $H_{2r,n}$ by adding the edges $\{i, i + \frac{(n+1)}{2}\}$ for $0 \leq i \leq \frac{(n-1)}{2}$. In this case, the graph $H_{k,n}$ cannot be regular since both k and n are odd numbers.

LEMMA 3.19. [10] Let G be a k -connected Harary graph $H_{k,n}$, where $k = 2r$, $r \geq 1$. Then, $mr_+^C(G) = |G| - \delta(G)$.

Let G be a connected graph and let $S = \{v_1, \dots, v_m\}$ be an ordered set of vertices of G . Denote by G_k the subgraph induced by v_1, v_2, \dots, v_k for each k , $1 \leq k \leq m$. Let H_k be the connected component of G_k such that $v_k \in V(H_k)$. If for each k , there exists $w_k \in V(G)$ such that $w_k \neq v_l$ for $l \leq k$, $w_k v_k \in E(G)$, and $w_k v_l \notin E(G)$, for all $v_l \in V(H_k)$ with $l \neq k$, then S is called a *vertex set of ordered subgraphs (or OS-vertex set)*. The *OS-number* of a graph G , denoted $OS(G)$, is the maximum cardinality among all *OS-vertex sets* of G [6].

The next theorem shows that the class of k -regular Harary graphs where k is even satisfies the GCC+. Its proof uses the notion of *OS-number*.

THEOREM 3.20. *Let G be a k -regular Harary graph, $H_{k,n}$ which is k -connected. If $k = 2r, r \geq 1$, then $mr_+^C(G) + mr_+^C(\overline{G}) \leq |G| + 2$.*

Proof. By Proposition 9 in [9], we have $OS(G) + mr_+^C(\overline{G}) \leq |G| + 2$. Using Proposition 4.11 in [10], we get $mr_+^C(\overline{G}) \leq |G| + 2 - OS(G) = \delta(G) + 2$. But since $\delta(G) = |G| - 1 - \delta(\overline{G})$, it follows that $mr_+^C(\overline{G}) \leq |G| - \delta(\overline{G}) + 1$. Since G is k -regular, \overline{G} is $(|G| - k - 1)$ -regular. So, $mr_+^C(\overline{G}) \leq k + 2$. Combining this and Lemma 3.19, we achieve the result. \square

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