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UPPER BOUND FOR THE NUMBER OF DISTINCT EIGENVALUES OF A PERTURBED MATRIX*

SUNYO MOON† AND SEUNGKOOK PARK†

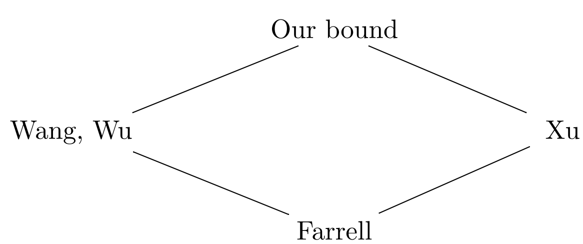
Abstract. In 2016, Farrell presented an upper bound for the number of distinct eigenvalues of a perturbed matrix. Xu (2017), and Wang and Wu (2016) introduced upper bounds which are sharper than Farrell's bound. In this paper, the upper bounds given by Xu, and Wang and Wu are improved.

Key words. Distinct eigenvalues, Perturbation, Geometric multiplicity, Algebraic multiplicity.

AMS subject classifications. 15A18, 65F15, 47A55.

1. Introduction.

A lot of research has been done on perturbation of matrices. However, most of the research is focused on rank-one perturbation of symmetric matrices [1, 3, 4, 7, 8, 9]. In 2016, Farrell established an interesting result which deals with arbitrary rank perturbation of arbitrary matrices. Farrell formulated an upper bound for the number of distinct eigenvalues of arbitrary matrices perturbed by arbitrary rank and proved that the number of Krylov iterations required to solve a linear system involving a diagonalizable matrix can at most double after a rank one perturbation [2]. Recently, Xu has improved the upper bound that was given by Farrell [11]. Also, Wang and Wu presented a new upper bound for the number of distinct eigenvalues of a matrix after perturbation [10]. In this paper, we give an upper bound for the number of distinct eigenvalues of a perturbed matrix which improves all of the upper bounds above. We also show that our upper bound includes the upper bounds in [2, 10, 11] as a special case. The comparison between the upper bounds are as follows. Both Xu's upper bound [11] and Wang and Wu's upper bound [10] improve Farrells' upper bound [2]. Xu's upper bound [11] and Wang and Wu's upper bound [10] cannot be compared in general. Our upper bound improves all the upper bounds in [2, 10, 11]. We depict the relations between the upper bounds in the following diagram.



The organization of the paper is as follows. In Section 2, we state the formulas for the upper bounds

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given in [2, 10, 11] and give comparison between the upper bounds. In Section 3, we give a new upper bound for the number of distinct eigenvalues of a perturbed matrix and show that it improves the upper bounds given in [2, 10, 11]. In Section 4, we give a concluding remark.

2. Preliminaries.

In this section, we introduce some notations and definitions that will be needed in the sequel sections and state the formulas for the upper bounds given in [2, 11, 10]. We also describe the relations between the upper bounds.

Let $\Lambda(M)$ be the set of distinct eigenvalues of a matrix M. Let $m_a(M,\lambda)$ and $m_g(M,\lambda)$ be the algebraic and geometric multiplicity of λ as an eigenvalue of M, respectively. Note that $m_a(M,\lambda) \geq m_g(M,\lambda) \geq 1$ for all $\lambda \in \Lambda(M)$.

Definition 2.1. The *defectivity* of an eigenvalue $\lambda \in \Lambda(M)$ is denoted by $d(M,\lambda)$, and it is the difference between its algebraic and geometric multiplicities, that is,

$$d(M,\lambda) := m_a(M,\lambda) - m_g(M,\lambda).$$

If $m_a(M,\lambda) > m_g(M,\lambda)$ for some $\lambda \in \Lambda(M)$, then M is called a *defective* matrix. If $m_a(M,\lambda) = m_g(M,\lambda)$ for all $\lambda \in \Lambda(M)$, then M is said to be *nondefective*.

Definition 2.2. The *defectivity* of a matrix M is denoted by $d(M)$, and it is the sum of the defectivities of its eigenvalues, that is,

$$d(M) := \sum_{\lambda\in\Lambda(M)} (m_a(M,\lambda) - m_g(M,\lambda)) = \sum_{\lambda\in\Lambda(M)} d(M,\lambda).$$

Since $d(M,\lambda) \geq 0$ for all $\lambda \in \Lambda(M)$, we have $d(M) \geq 0$. It is known that a matrix M is diagonalizable if and only if $d(M) = 0$. For example, Hermitian matrices are diagonalizable, and hence, their defectivity is 0.

We state the formulas for the upper bound given by Farrell [2], Xu [11] and Wang et al. [10].

Theorem 2.3. (Farrell, [2, Theorem 1.3]) *Let $A, B \in \mathbb{C}^{n\times n}$. If $C = A + B$, then*

$$|\Lambda(C)| \leq (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A).$$

Theorem 2.4. (Xu, [11, Theorem 3.1]) *Let $A, B \in \mathbb{C}^{n\times n}$, and let $C = A + B$. Then*

$$|\Lambda(C)| \leq (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C).$$

Theorem 2.5. (Wang and Wu, [10, Theorem 3.3]) *Let $A, B \in \mathbb{C}^{n\times n}$ and $C = A + B$. Then*

$$|\Lambda(C)| \leq (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A) - \sum_{\lambda\in\Lambda(A)} \max\{\varphi(\lambda), \psi(\lambda)\},$$

where $\varphi(\lambda) = \dim(R(A - \lambda I) \cap R(B))$, $\psi(\lambda) = \dim(R((A - \lambda I)^H) \cap R(B^H))$, B^H is the conjugate transpose of B, and $R(B)$ is the row space of B.

Xu's bound improves Farrell's bound by a factor of $d(C)$. Wang and Wu's bound improves Farrell's bound by a factor of $\sum_{\lambda\in\Lambda(A)} \max\{\varphi(\lambda), \psi(\lambda)\}$. If C is a diagonalizable matrix then Wang and Wu's bound

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is sharper than or equal to Xu's bound. But, in general, both bounds cannot be compared. The following example demonstrates a case in which Xu's bound is sharper than Wang and Wu's bound.

Example 2.6. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the perturbed matrix C and its Jordan canonical form $J(C)$ are

$$C = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad J(C) = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

respectively. Note that $|\Lambda(A)| = 3$, $d(A) = 1$, $\mathrm{rank}(B) = 1$, $d(C) = 2$,

$$\begin{aligned} \varphi(1) = 0, \quad \varphi(2) = 1, \quad \varphi(3) = 0, \\ \psi(1) = 0, \quad \psi(2) = 1, \quad \psi(3) = 0. \end{aligned}$$

Thus,

$$|\Lambda(C)| \le (\mathrm{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C) = 5,$$

and

$$|\Lambda(C)| \le (\mathrm{rank}(B) + 1)|\Lambda(A)| + d(A) - \sum_{\lambda \in \Lambda(A)} \max\{\varphi(\lambda), \psi(\lambda)\} = 6.$$

Hence, Xu's bound is sharper than Wang and Wu's bound.

We note that while Farrell's bound is less sharp, it relies only on information that is more readily available: In the typical case when the spectrum of the matrix A is known but the defectivity $d(C)$ of the matrix C is not known.

3. Main result.

In this section, we give an upper bound for the number of distinct eigenvalues of a perturbed matrix which improves the upper bounds given in [2, 10, 11]. We begin with a general result on the bounds for the rank of the sum of two matrices. The following theorem gives an upper bound for the rank of a sum [5].

Theorem 3.1. ([5, Theorem 1]) *Let A and B be two matrices of the same size with entries in the field $\mathbb{C}$ of complex numbers. Let also their row spaces be $R(A)$ and $R(B)$, and their column spaces be $C(A)$ and $C(B)$. Then,*

$$\mathrm{rank}(A + B) \le \mathrm{rank}(A) + \mathrm{rank}(B) - \max\{\dim(R(A) \cap R(B)),\ \dim(C(A) \cap C(B))\}.$$

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Let $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\left(A \mid B\right)$ be the partitioned matrices. Then

$$R\begin{pmatrix} A \\ B \end{pmatrix} = R(A) + R(B),$$

and

$$C\left(A \mid B\right) = C(A) + C(B).$$

Thus, we have

$$\operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B) - \dim(R(A) \cap R(B)) \tag{3.1}$$

and

$$\operatorname{rank}\left(A \mid B\right) = \operatorname{rank}(A) + \operatorname{rank}(B) - \dim(C(A) \cap C(B)), \tag{3.2}$$

which provide formulas for $\dim\left(R(A) \cap R(B)\right)$ and $\dim\left(C(A) \cap C(B)\right)$ in terms of ranks. In the next corollary, we restate Theorem 3.1 in terms of ranks only.

Corollary 3.2. ([6, Corollary 8.1]) *Let A and B be two matrices of the same size with entries in the field $\mathbb{C}$ of complex numbers. Then*

$$\operatorname{rank}(A+B) \le \min\left\{\operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix},\ \operatorname{rank}\left(A \mid B\right)\right\}.$$

We can now formulate our main result.

Theorem 3.3. *Let $A,\ B \in \mathbb{C}^{n\times n}$. If $C = A + B$, then*

$$|\Lambda(C)| \le |\Lambda(A)| - n(|\Lambda(A)| - 1) + \sum_{\lambda\in\Lambda(A)} \min\left\{\operatorname{rank}\begin{pmatrix} A-\lambda I \\ B \end{pmatrix},\ \operatorname{rank}\left(A-\lambda I \mid B\right)\right\} - d(C).$$

Proof. Let $S_1 = \Lambda(C) \cap \Lambda(A)$ and $S_2 = \Lambda(C)\backslash\Lambda(A)$. Then

$$|\Lambda(C)| = |S_1| + |S_2|.$$

Let λ be an eigenvalue of A. By Corollary 3.2, we have

$$\operatorname{rank}(C-\lambda I) = \operatorname{rank}((A-\lambda I)+B) \le \min\left\{\operatorname{rank}\begin{pmatrix} A-\lambda I \\ B \end{pmatrix},\ \operatorname{rank}\left(A-\lambda I \mid B\right)\right\}.$$

For $\lambda \in S_1$, we get

$$\begin{aligned} m_g(C,\lambda) &= n - \operatorname{rank}(C-\lambda I) \\ &= n - \operatorname{rank}((A-\lambda I)+B) \\ &\ge n - \min\left\{\operatorname{rank}\begin{pmatrix} A-\lambda I \\ B \end{pmatrix},\ \operatorname{rank}\left(A-\lambda I \mid B\right)\right\}. \end{aligned}$$

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If $\lambda \in \Lambda(A)\backslash S_1$, then $C - \lambda I$ is nonsingular, and hence, $\operatorname{rank}(C - \lambda I) = n$. Thus,

$$m_g(C, \lambda) = n - \operatorname{rank}(C - \lambda I) = 0.$$

Since

$$0 = m_g(C, \lambda) \geq n - \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} \geq 0,$$

we have $n - \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} = 0$ for $\lambda \in \Lambda(A)\backslash S_1$. Now we compute an upper bound for $|S_2|$.

$$\begin{aligned}
|S_2| &\leq \sum_{\lambda \in S_2} m_g(C, \lambda) \\
&= \sum_{\lambda \in S_2} m_a(C, \lambda) - \sum_{\lambda \in S_2} d(C, \lambda) \\
&= n - \sum_{\lambda \in S_1} m_a(C, \lambda) - \sum_{\lambda \in S_2} d(C, \lambda) \\
&= n - \sum_{\lambda \in S_1} m_g(C, \lambda) - \sum_{\lambda \in S_1} d(C, \lambda) - \sum_{\lambda \in S_2} d(C, \lambda) \\
&\leq n - \sum_{\lambda \in S_1} \left(n - \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} \right) - d(C) \\
&\quad - \sum_{\lambda \in \Lambda(A)\backslash S_1} \left(n - \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} \right) \\
&= n - \sum_{\lambda \in \Lambda(A)} \left(n - \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} \right) - d(C) \\
&= n - n|\Lambda(A)| + \sum_{\lambda \in \Lambda(A)} \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} - d(C).
\end{aligned}$$

Since $|S_1| \leq |\Lambda(A)|$, we obtain

$$\begin{aligned}
|\Lambda(C)| &= |S_1| + |S_2| \\
&\leq |\Lambda(A)| - n(|\Lambda(A)| - 1) + \sum_{\lambda \in \Lambda(A)} \min \left\{ \operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank} \left(A - \lambda I \mid B \right) \right\} - d(C). \qquad \square
\end{aligned}$$

From equations (3.1) and (3.2), we can deduce that if

$$\dim \left(R(A - \lambda I) \cap R(B) \right) = \dim \left(C(A - \lambda I) \cap C(B) \right) = 0,$$

then

$$\operatorname{rank} \begin{pmatrix} A - \lambda I \\ B \end{pmatrix} = \operatorname{rank}(A - \lambda I) + \operatorname{rank}(B) = \operatorname{rank} \left(A - \lambda I \mid B \right).$$

We show that the upper bound in Theorem 3.3 improves the upper bound in Theorem 2.4. In particular, it is shown that the upper bound in Theorem 2.4 is a special case of Theorem 3.3.

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COROLLARY 3.4. *The upper bound in Theorem 3.3 improves the one in Theorem 2.4.*

Proof. Showing the following inequality will complete the proof:

$$\text{(the upper bound in Theorem 2.4)} - \text{(the upper bound in Theorem 3.3)} \;\geq\; 0. \tag{3.3}$$

The subtraction in (3.3) yields

$$\begin{aligned}
&\Big[\ (\operatorname{rank}(B)+1)|\Lambda(A)| + d(A) - d(C)\ \Big] - \Big[\ |\Lambda(A)| - n(|\Lambda(A)|-1) \\
&\quad + \sum_{\lambda\in\Lambda(A)} \min\left\{\operatorname{rank}\begin{pmatrix} A-\lambda I \\ B\end{pmatrix},\ \operatorname{rank}\left(A-\lambda I \;\middle|\; B\right)\right\} - d(C)\ \Big] \\
&= \operatorname{rank}(B)|\Lambda(A)| + d(A) - \Bigg(\ n - \sum_{\lambda\in\Lambda(A)} n + \sum_{\lambda\in\Lambda(A)} (\operatorname{rank}(A-\lambda I)+\operatorname{rank}(B)) \\
&\quad - \sum_{\lambda\in\Lambda(A)} \max\left\{\dim\left(R(A-\lambda I)\cap R(B)\right),\ \dim\left(C(A-\lambda I)\cap C(B)\right)\right\}\ \Bigg).
\end{aligned} \tag{3.4}$$

The term "$n - \sum_{\lambda\in\Lambda(A)} n + \sum_{\lambda\in\Lambda(A)} (\operatorname{rank}(A-\lambda I)+\operatorname{rank}(B))$" in the above equality can be expanded as

$$\begin{aligned}
& n - \sum_{\lambda\in\Lambda(A)} n + \sum_{\lambda\in\Lambda(A)} (\operatorname{rank}(A-\lambda I)+\operatorname{rank}(B)) \\
&= n - \sum_{\lambda\in\Lambda(A)} (n - (\operatorname{rank}(A-\lambda I)) + \sum_{\lambda\in\Lambda(A)} \operatorname{rank}(B) \\
&= n - \sum_{\lambda\in\Lambda(A)} m_g(A,\lambda) + \sum_{\lambda\in\Lambda(A)} \operatorname{rank}(B) \\
&= \sum_{\lambda\in\Lambda(A)} (m_a(A,\lambda) - m_g(A,\lambda)) + \operatorname{rank}(B)|\Lambda(A)|,
\end{aligned} \tag{3.5}$$

where $\sum_{\lambda\in\Lambda(A)} (m_a(A,\lambda) - m_g(A,\lambda)) = d(A)$. By combining (3.4) and (3.5), we have

$$\begin{aligned}
&\text{(the upper bound in Theorem 2.4)} - \text{(the upper bound in Theorem 3.3)} \\
&= \sum_{\lambda\in\Lambda(A)} \max\left\{\dim\left(R(A-\lambda I)\cap R(B)\right),\ \dim\left(C(A-\lambda I)\cap C(B)\right)\right\} \\
&\geq\; 0,
\end{aligned}$$

which completes the proof. Here, the equality holds if

$$\dim\left(R(A-\lambda I)\cap R(B)\right) = \dim\left(C(A-\lambda I)\cap C(B)\right) = 0 \ \text{ for all } \ \lambda\in\Lambda(A). \qquad \square$$

Two arbitrary matrices are chosen to illustrate Corollary 3.4 in the next example.

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EXAMPLE 3.5. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the perturbed matrix C and its Jordan canonical form $J(C)$ are

$$C = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad J(C) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

respectively. Note that $|\Lambda(A)| = 3$, $d(A) = 2$, $\operatorname{rank}(B) = 1$, $d(C) = 1$,

$$\operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix} = 4, \quad \operatorname{rank}\begin{pmatrix} A - I \\ B \end{pmatrix} = 5, \quad \operatorname{rank}\begin{pmatrix} A - 3I \\ B \end{pmatrix} = 4,$$
$$\operatorname{rank}\begin{pmatrix} A \mid B \end{pmatrix} = 5, \ \operatorname{rank}\begin{pmatrix} A - I \mid B \end{pmatrix} = 4, \ \operatorname{rank}\begin{pmatrix} A - 3I \mid B \end{pmatrix} = 4.$$

Thus, we have

$$|\Lambda(C)| \le (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A) - d(C) = 7$$

and

$$|\Lambda(C)| \le |\Lambda(A)| - n(|\Lambda(A)| - 1) + \sum_{\lambda \in \Lambda(A)} \min\left\{ \operatorname{rank}\begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \ \operatorname{rank}\begin{pmatrix} A - \lambda I \mid B \end{pmatrix} \right\} - d(C) = 4$$

as expected from Corollary 3.4.

Analogously, we also show that the upper bound in Theorem 3.3 improves the one in Theorem 2.5. In this case, two upper bounds in Theorem 3.3 and Theorem 2.5 are equal to each other if C is diagonalizable.

COROLLARY 3.6. *The upper bound in Theorem 3.3 improves the one in Theorem 2.5.*

Proof. In a similar manner to the proof of Corollary 3.4, the inequality

$$\text{(the upper bound in Theorem 2.5)} - \text{(the upper bound in Theorem 3.3)} \ \ge \ 0 \tag{3.6}$$

is shown here. First, recall the upper bounds of Theorem 3.3 and Theorem 2.5, and then, plug them into

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(3.6) to obtain

$$\begin{aligned}
&\Bigg[(\mathrm{rank}(B)+1)|\Lambda(A)| + d(A) - \sum_{\lambda\in\Lambda(A)} \max\{\varphi(\lambda),\, \psi(\lambda)\} \Bigg] - \Bigg[|\Lambda(A)| \\
&\quad - n(|\Lambda(A)|-1) + \sum_{\lambda\in\Lambda(A)} \min\left\{ \mathrm{rank}\begin{pmatrix} A-\lambda I \\ B \end{pmatrix},\, \mathrm{rank}\left(A-\lambda I \;\middle|\; B\right) \right\} - d(C) \Bigg] \\
&= \mathrm{rank}(B)|\Lambda(A)| + d(A) - \sum_{\lambda\in\Lambda(A)} \max\{\dim(R(A-\lambda I)\cap R(B)), \\
&\quad \dim(R((A-\lambda I)^H)\cap R(B^H))\} - \Bigg(n - \sum_{\lambda\in\Lambda(A)} n + \sum_{\lambda\in\Lambda(A)} (\mathrm{rank}(A-\lambda I) + \mathrm{rank}(B)) \\
&\quad - \sum_{\lambda\in\Lambda(A)} \max\{\dim(R(A-\lambda I)\cap R(B)),\, \dim(C(A-\lambda I)\cap C(B))\} - d(C) \Bigg) \\
&= \mathrm{rank}(B)|\Lambda(A)| + d(A) - \sum_{\lambda\in\Lambda(A)} \max\{\dim(R(A-\lambda I)\cap R(B)), \\
&\quad \dim(C((A-\lambda I))\cap C(B))\} - \Bigg(n - \sum_{\lambda\in\Lambda(A)} m_g(A,\lambda) + \sum_{\lambda\in\Lambda(A)} \mathrm{rank}(B) \\
&\quad - \sum_{\lambda\in\Lambda(A)} \max\{\dim(R(A-\lambda I)\cap R(B)),\, \dim(C(A-\lambda I)\cap C(B))\} - d(C) \Bigg) \\
&= \mathrm{rank}(B)|\Lambda(A)| + d(A) - \Bigg(\sum_{\lambda\in\Lambda(A)} (m_a(A,\lambda) - m_g(A,\lambda)) + \mathrm{rank}(B)|\Lambda(A)| - d(C) \Bigg) \\
&= d(C) \\
&\geq\; 0,
\end{aligned}$$

in which $\varphi(\lambda) = \dim(R(A-\lambda I)\cap R(B))$ and $\psi(\lambda) = \dim(R((A-\lambda I)^H)\cap R(B^H))$ are defined in Theorem 2.5. It is clear that the equality holds in (3.6) if $d(C) = 0$. ☐

Remark 3.7. If C is not diagonalizable then $d(C) \geq 1$. Therefore, Theorem 3.3 implies

$$|\Lambda(C)| \leq (\mathrm{rank}(B)+1)|\Lambda(A)| + d(A) - \sum_{\lambda\in\Lambda(A)} \max\{\varphi(\lambda),\, \psi(\lambda)\} - 1,$$

that is, the upper bound in Theorem 3.3 improves the upper bound in Theorem 2.5 by at least one.

We present an example supporting Corollary 3.6.

Example 3.8. Let

$$A = \begin{pmatrix} 1&0&0&0&0\\ 0&2&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&1\\ 0&0&0&0&3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0\\ 1&0&1&0&1\\ 0&0&0&0&0 \end{pmatrix}.$$

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Then the perturbed matrix C and its Jordan canonical form $J(C)$ are

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad J(C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

respectively. Note that $|\Lambda(A)| = 3$, $d(A) = 1$, $\operatorname{rank}(B) = 2$, $d(C) = 2$,

$$\begin{array}{ccc} \varphi(1) = 1, & \varphi(2) = 0, & \varphi(3) = 1, \\ \psi(1) = 2, & \psi(2) = 1, & \psi(3) = 2, \\ \operatorname{rank}\begin{pmatrix} A - I \\ B \end{pmatrix} = 5, & \operatorname{rank}\begin{pmatrix} A - 2I \\ B \end{pmatrix} = 5, & \operatorname{rank}\begin{pmatrix} A - 3I \\ B \end{pmatrix} = 5, \\ \operatorname{rank}\left(A - I \mid B\right) = 4, & \operatorname{rank}\left(A - 2I \mid B\right) = 4, & \operatorname{rank}\left(A - 3I \mid B\right) = 4. \end{array}$$

The above yields

$$|\Lambda(C)| \le (\operatorname{rank}(B) + 1)|\Lambda(A)| + d(A) - \sum_{\lambda \in \Lambda(A)} \max\{\varphi(\lambda), \psi(\lambda)\} = 5,$$

and

$$|\Lambda(C)| \le |\Lambda(A)| - n(|\Lambda(A)| - 1) + \sum_{\lambda \in \Lambda(A)} \min\left\{\operatorname{rank}\begin{pmatrix} A - \lambda I \\ B \end{pmatrix}, \operatorname{rank}\left(A - \lambda I \mid B\right)\right\} - d(C) = 3.$$

Hence, the upper bound in Theorem 3.3 improves the upper bound in Theorem 2.5 by 2.

Finally, for comparison, we present an example with all the bounds. In the example, we fix a matrix A and perturb it with matrices having various ranks.

Example 3.9. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ 0 & -2 & 2 & -2 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_3 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that B_1, B_2, B_3 have rank 1, 2, 2, respectively. Let $C_i = A + B_i$ for $i = 1, 2, 3$. Then we have the following:

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	Our bound	Wang, Wu	Xu	Farrell
$\|\Lambda(C_1)\| = 4$	4	5	6	7
$\|\Lambda(C_2)\| = 4$	4	5	9	10
$\|\Lambda(C_3)\| = 3$	3	5	8	10

4. Conclusion.

The new upper bound in Theorem 3.3 improves both upper bounds in Theorem 2.4 and Theorem 2.5. By using the inequality

$$\operatorname{rank}(A+B) \leq \min\left\{\operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix},\ \operatorname{rank}\begin{pmatrix} A \mid B \end{pmatrix}\right\},$$

we improve the upper bound in Theorem 2.4 by a factor of

$$\sum_{\lambda\in\Lambda(A)} \max\left\{\dim\left(R(A-\lambda I)\cap R(B)\right),\, \dim\left(C(A-\lambda I)\cap C(B)\right)\right\}$$
$$= |\Lambda(A)|\operatorname{rank}(B) + \sum_{\lambda\in\Lambda(A)} \operatorname{rank}(A-\lambda I) - \sum_{\lambda\in\Lambda(A)} \min\left\{\operatorname{rank}\begin{pmatrix} A-\lambda I \\ B \end{pmatrix},\ \operatorname{rank}\begin{pmatrix} A-\lambda I \mid B \end{pmatrix}\right\}.$$

We utilize the facts that $m_g(C,\lambda) = m_a(C,\lambda) - d(C,\lambda)$ for all $\lambda \in \Lambda(C)$ and $m_g(C,\lambda) = 0$ for $\lambda \in \Lambda(A)\backslash S_1$ in order to improve the upper bound in Theorem 2.5 by a factor of $d(C)$.

REFERENCES

[1] J.R. Bunch, C.P. Nielsen, and D.C. Sorensen. Rank-one modification of the symmetric eigenproblem. *Numer. Math.*, 31:31–48, 1978.

[2] P.E. Farrell. The number of distinct eigenvalues of a matrix after perturbation. *SIAM J. Matrix Anal. Appl.*, 37:572–576, 2016.

[3] G.H. Golub. Some modified matrix eigenvalue problems. *SIAM Rev.*, 15:318–334, 1973.

[4] I.C.F. Ipsen and B. Nadler. Refined perturbation bounds for eigenvalues of Hermitian and non-Hermitian matrices. *SIAM J.Matrix Anal. Appl.*, 31:40–53, 2009.

[5] G. Marsaglia. Bounds for the rank of two matrices. Boeing Scientific Research Laboratories, Mathematical Note no. 344, 1964.

[6] G. Marsaglia and G.P.H. Styan. Equalities and inequalities for ranks of matrices. *Linear Multilinear Algebra*, 2:269–292, 1974.

[7] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Jordan forms of real and complex matrices under rank one perturbations. *Oper. Matrices*, 7:381–398, 2013.

[8] J. Moro and F.M. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[9] S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Math. Notes*, 74:557–568, 2003.

[10] Y. Wang and G. Wu. Refined bounds on the number of distinct eigenvalues of a matrix after perturbation. Preprint, arxiv161208369, 2016.

[11] X. Xu. An improved upper bound for the number of distinct eigenvalues of a matrix after perturbation. *Linear Algebra Appl.*, 523:109–117, 2017.