Positive solutions of the system of operator equations $A_1X=C_1, XA_2=C_2, A_3XA^*_3=C_3, A_4XA^*_4=C_4$ in Hilbert $C^*$-modules

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POSITIVE SOLUTIONS OF THE SYSTEM OF OPERATOR EQUATIONS $A_1X = C_1$, $X A_2 = C_2$, $A_3 X A_3^* = C_3$, AND $A_4 X A_4^* = C_4$ IN HILBERT $C^*$-MODULES

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Abstract. Necessary and sufficient conditions are given for the operator system $A_1X = C_1$, $X A_2 = C_2$, $A_3 X A_3^* = C_3$, and $A_4 X A_4^* = C_4$ to have a common positive solution, where $A_i$’s and $C_i$’s are adjointable operators on Hilbert $C^*$-modules. This corrects a published result by removing some gaps in its proof. Finally, a technical example is given to show that the proposed investigation in the setting of Hilbert $C^*$-modules is different from that of Hilbert spaces.

Key words. Hilbert $C^*$-module, Operator equation, Orthogonally complemented submodule.

AMS subject classifications. 15A24, 46L08, 47A05, 47A62.

1. Introduction. Let $\mathfrak{A}$ be a $C^*$-algebra. A Hilbert $\mathfrak{A}$-module is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot , \cdot \rangle : H \times H \to \mathfrak{A}$ such that $H$ is complete with respect to the induced norm defined by $\| x \| = \| \langle x , x \rangle \|^{1/2}$ for $x \in H$. Suppose that $H$ and $K$ are Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(H, K)$ be the set of maps $A : H \to K$ for which there is a map $A^* : K \to H$, called the adjoint operator of $A$, such that

$$\langle Ax , y \rangle = \langle x , A^* y \rangle \quad \text{ for each } x \in H \text{ and } y \in K.$$  

It is known that each element $A$ of $\mathcal{L}(H, K)$ must be a bounded linear operator, which is also $\mathfrak{A}$-linear in the sense that $A(xa) = (Ax)a$ for each $x \in H$ and $a \in \mathfrak{A}$. We use the notations $\mathcal{L}(H)$ and $\mathcal{L}(H)_+$ to denote the $C^*$-algebra $\mathcal{L}(H, H)$ and the set of positive elements of $\mathcal{L}(H)$, respectively. Let $A \in \mathcal{L}(H)$. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we mean the range and the null space of $A$, respectively. By [3, Lemma 4.1], we know that $A$ is positive if and only if $\langle Ax , x \rangle \geq 0$ for all $x \in H$.

Let $H$ be a Hilbert $\mathfrak{A}$-module. A closed submodule $K$ of $H$ is said to be orthogonally complemented in $H$ if $H = K \oplus K^\perp$, where

$$K^\perp = \{ x \in H : \langle x , y \rangle = 0 \text{ for all } y \in K \}.$$

Evidently, $K$ is orthogonally complemented in $H$ if and only if there exists a projection $P$ on $H$, whose range is $K$ and $\mathcal{R}(P) \oplus \mathcal{N}(P) = H$.

Throughout the rest of this section, $H$ and $K$ are Hilbert $C^*$-modules, and $A$ is an element of $\mathcal{L}(H, K)$. Recall that an operator $A$ is regular if $\mathcal{R}(A)$ is closed in $K$.  

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LEMMA 1.1. (See [3, Theorem 3.2] and [9, Remark 1.1]) The closedness of any one of the following sets implies the closedness of the remaining three sets:

\[ \mathcal{R}(A), \mathcal{R}(A^*), \mathcal{R}(AA^*), \text{ and } \mathcal{R}(A^*A). \]

If \( \mathcal{R}(A) \) is closed, then \( \mathcal{R}(A) = \mathcal{R}(AA^*), \mathcal{R}(A^*) = \mathcal{R}(A^*A) \), and the following orthogonal decompositions hold:

\[ H = N(A) \oplus \mathcal{R}(A^*), \quad K = \mathcal{R}(A) \oplus N(A^*). \]

Recall that each element \( A^- \) of \( A\{1\} = \{X \in \mathcal{L}(K,H) : AXA = A\} \) is called an *inner inverse* of \( A \). Clearly, it can be deduced from [9, Theorem 2.2] that \( A \) has an inner inverse if and only if \( A \) is regular. In this case, we put

\[ L_A := I - A^-A, \]

where \( A^- \in A\{1\} \) is unspecified.

The Moore–Penrose inverse \( A^\dagger \) of \( A \) (if it exists) is the unique element \( X \) of \( \mathcal{L}(K,H) \) which satisfies

\[ AXA = A, \quad XAX = X, \quad (AX)^* = AX, \text{ and } (XA)^* =XA. \]

We remark that as in the Hilbert space case, \( A^\dagger \) exists if and only if \( A \) is regular [9, Theorem 2.2], in which case \( \mathcal{R}(A^\dagger) = \mathcal{R}(A^*), N(A^\dagger) = N(A^*), \) and

\[ (A^\dagger)^* = (A^*)^\dagger \quad \text{and} \quad (AA^*)^\dagger = (A^*)^\dagger A^\dagger = (A^\dagger)^* A^\dagger. \]

If \( H = K \) and \( A \) is Hermitian, then \( A^\dagger \) is also Hermitian and \( AA^\dagger = A^\dagger A \).

The study of operator equations has been developed from matrices to infinite dimensional spaces; for example, arbitrary Hilbert spaces and Hilbert \( \mathfrak{A} \)-modules, by several mathematicians; see [1, 4, 8, 11, 12] and references therein. In [8], some necessary and sufficient conditions for the existence of common Hermitian and positive solutions \( X \in \mathcal{L}(H) \) for the equations \( AX = C \) and \( XB = D \) are proposed and some formulas for the general forms of their common solutions are given.

In this paper, we give some necessary and sufficient conditions for the operator system \( A_1X = C_1, \quad XA_2 = C_2, \quad A_3XA_3^* = C_3, \text{ and } A_4XA_4^* = C_4 \) to have a common positive solution, where \( A_i \)'s and \( C_i \)'s are adjointable operators on Hilbert \( C^* \)-modules. This corrects the main result of Song and Wang [7] by removing some gaps in its proof. Finally, we give a technical example and show that our investigation in the setting of Hilbert \( C^* \)-modules differs from that in the framework of Hilbert spaces.

2. Main results. Throughout this section, \( H, K, L, \) and \( K_i(1 \leq i \leq 4) \) are Hilbert \( \mathfrak{A} \)-modules.

The proof of Lemma 2.1 below is straightforward.

LEMMA 2.1. Let \( A \in \mathcal{L}(H,K), C \in \mathcal{L}(L,K) \) be such that \( A \) is regular. Then the operator equation \( AX = C \) has a solution \( X \in \mathcal{L}(L,H) \) if and only if \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \). In this case, the general solution to \( AX = C \) is of the form

\[ X = A^-C + (I - A^-A)T, \]

where \( T \in \mathcal{L}(L,H) \) is arbitrary.
LEMMA 2.2. (See [8, Theorem 2.1]) Let $A, C \in \mathcal{L}(H, K)$ be such that both $A$ and $CA^*$ are regular. Then the operator equation $AX = C$ has a solution $X \in \mathcal{L}(H)_+$ if and only if $CA^* \geq 0$ and $\mathcal{R}(C) = \mathcal{R}(CA^*)$. In this case, the general positive solution to $AX = C$ is of the form
\[
X = C^*(CA^*)^{-}C + LA SL_A^*,
\]
where $S \in \mathcal{L}(H)_+$ is arbitrary and $C^*(CA^*)^{-}C$ is a positive element, which is independent of the choice of the inner inverse $(CA^*)^{-}$.

LEMMA 2.3. (See [8, Theorem 3.7]) Let $A_1, C_1 \in \mathcal{L}(H, K), A_2, C_2 \in \mathcal{L}(L, H),$
\[
D = \left( \begin{array}{c} A_1 \\ A_2^* \end{array} \right), \quad E = \left( \begin{array}{c} C_1 \\ C_2^* \end{array} \right), \quad \text{and} \quad F = \left( \begin{array}{cc} C_1A_1^* & C_1A_2 \\ (A_1C_2)^* & C_2^*A_2 \end{array} \right)
\]
be such that $D$ and $F$ are regular. Then the system
\[
(2.6) \quad A_1X = C_1, \quad XA_2 = C_2, \quad X \in \mathcal{L}(H)
\]
has a solution $X \in \mathcal{L}(H)_+$ if and only if $F \geq 0$ and $\mathcal{R}(E) \subseteq \mathcal{R}(F)$. In this case, the general positive solution to system (2.6) can be expressed as
\[
X = E^*F^{-}E + LDTL_D^*,
\]
where $T \in \mathcal{L}(H)_+$ is arbitrary and $E^*F^{-}E$ is a positive element, which is independent of the choice of the inner inverse $F^{-}$.

REMARK 2.4. Suppose that $A \in \mathcal{L}(H, K)$ and $C \in \mathcal{L}(K)$ are both regular. It is indicated in [10, Lemma 3.2] that the equation
\[
(2.7) \quad AXA^* = C, \quad X \in \mathcal{L}(H),
\]
has a solution $X \in \mathcal{L}(H)_+$ if and only if
\[
(2.8) \quad C \geq 0 \quad \text{and} \quad \mathcal{R}(C) \subseteq \mathcal{R}(A).
\]
In this case, the general positive solution for equation (2.7) can be expressed as
\[
(2.9) \quad X = A^1C(A^1)^* + A^1C(A^1)^*VF_A + FA^*A^1C(A^1)^* + FA^*A^1C(A^1)^*VF_A + FAWF_A,
\]
where $F_A = I - A^1A$, $V \in \mathcal{L}(H)$ is arbitrary, and $W \in \mathcal{L}(H)_+$ is arbitrary.

The point is, as shown in [2] by Groß for matrices, we can replace $A^1$ in (2.9) by a general inner inverse $A^-$, and meanwhile give a simplified formula for $X$. For the sake of completeness, we give a detailed proof of Lemma 2.5 below, using a method somewhat different from that in [2].

LEMMA 2.5. (See [2, Theorem 1]) Suppose that $A \in \mathcal{L}(H, K)$ and $C \in \mathcal{L}(K)$ are both regular such that condition (2.8) is satisfied. Then the general positive solution to equation (2.7) can be expressed as
\[
(2.10) \quad X = [A^{-}B + LA Y][A^{-}B + LA Y]^* + LA S(L_A)^*,
\]
where $L_A$ is defined by (1.2), $Y \in \mathcal{L}(K, H)$ is arbitrary, $S \in \mathcal{L}(H)_+$ is arbitrary, and $B \in \mathcal{L}(K)$ is an arbitrary operator satisfying $BB^* = C$. 
Proof. Let $B \in \mathcal{L}(K)$ be chosen such that $BB^* = C$. By Lemma 1.1, we have $\mathcal{R}(B) = \mathcal{R}(C)$; hence, $AA^*B = B$, which means that each operator $X$ of the form (2.10) is a positive solution to equation (2.7).

Conversely, suppose that $X \in \mathcal{L}(H)_+$ is a solution to equation (2.7). Let $U = XA^* - A^*C$. Then $AU = 0$; hence, $XA^* = A^*C + L_AU$. Taking the $*$-operation, we have

$$ AX = C(A^*)^* + U^*(L_A)^* \overset{\text{def}}{=} C'. $$

Note that $C'A^* = AXA^* = C$, which is regular. Note also that $X$ is a positive solution to the equation $AZ = C', Z \in \mathcal{L}(H)$; so by Lemma 2.2, there exists $S \in \mathcal{L}(H)_+$ such that

$$ X = (C')^*(C'A)^1C' + LAS(L_A)^* = (C')^*(BB^*)^1C' + LAS(L_A)^*. $$

Clearly, $C(B')^* = BB^*(B')^* = B$, and, by (1.4), we have $(BB^*)^\dagger = (B^\dagger)^\dagger B^\dagger$. In view of the observation above, formula (2.10) for $X$ follows immediately from (2.11) and (2.12) by putting $Y = U(B^\dagger)^*$. □

**Lemma 2.6.** (See [6, Proposition 1.4.5]) Let $x$ and $a$ be elements in a $C^*$-algebra $\mathfrak{A}$ such that $a \geq 0$ and $x^*x \leq a$. If $0 < \beta < \frac{1}{2}$, then there exists $u \in \mathfrak{A}$ with $\|u\| \leq \|a^{1/2} - \beta\|$ such that $x = ua^\beta$.

**Lemma 2.7.** Let $A \in \mathcal{L}(H, K)$ and $B \in \mathcal{L}(K)_+$ be such that $AA^* \leq B$. Then, for each $\beta \in (0, \frac{1}{2})$, there exists $C \in \mathcal{L}(H, K)$ such that $A = B^\beta C$.

Proof. We consider the $C^*$-algebra $\mathcal{L}(H \oplus K)$, which contains $\tilde{A}$ and $\tilde{B}$, where

$$ \tilde{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}. $$

It is obvious that $\tilde{A}(\tilde{A})^* \leq \tilde{B}$; so, for each $\beta \in (0, \frac{1}{2})$, by Lemma 2.6, there exists $W = \begin{pmatrix} W_{11} & W_{12} \\ C & W_{22} \end{pmatrix} \in \mathcal{L}(H \oplus K)$ such that $\tilde{A} = B^{\beta} W$. Direct computation yields $A = B^{\beta} C$. □

Now we state the main result of this paper, which is a modification of [7, Theorem 3.5].

**Theorem 2.8.** Let $A_1, C_1 \in \mathcal{L}(H, K_1), A_2, C_2 \in \mathcal{L}(K_2, H), A_3 \in \mathcal{L}(H, K_3), A_4 \in \mathcal{L}(H, K_4), C_3 \in \mathcal{L}(K_3)$, and $C_4 \in \mathcal{L}(K_4)$ be given such that $A_{11}, M, A_{33}, C_{33}, A_{44}, C_{44},$ and $A_{44}L_{A_{33}}$ are all regular, where

$$ A_{11} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad M = \begin{pmatrix} C_1A_1^* \\ C_2A_1^* \\ C_2A_2^* \end{pmatrix}, \quad N = (C_1^* C_2^*)M^\dagger \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad A_{33} = A_3L_{A_{11}}, \quad A_{44} = A_4L_{A_{11}}, \quad C_{33} = C_3 - A_3NA_3^*, \quad C_{44} = C_4 - A_4NA_4^*. $$

Then the system

$$ A_1X = C_1, \quad XA_2 = C_2, \quad A_3XA_3^* = C_3, \quad A_4XA_4^* = C_4, \quad X \in \mathcal{L}(H) $$

has a solution $X \in \mathcal{L}(H)_+$ if and only if the following three conditions hold:

(i) The operators $M, C_{33}$ and $C_{44}$ are all positive;

(ii) $\mathcal{R} \left( \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right) \subseteq \mathcal{R}(M), \mathcal{R}(C_{33}) \subseteq \mathcal{R}(A_{33}), \mathcal{R}(C_{44}) \subseteq \mathcal{R}(A_{44});$

(iii) There exist $S \in \mathcal{L}(H)_+$ and $T \in \mathcal{L}(K_3, K_4)$ such that

$$ C_S := C_{44} - A_{44}L_{A_{33}}SL_{A_{33}}^*A_{44}^\dagger \geq 0, $$

$$ \mathcal{R}(C_S^\dagger T - A_{44}A_{33}^*C_{33}^\dagger) \subseteq \mathcal{R}(A_{44}L_{A_{33}}). $$
If conditions (i)–(iii) are satisfied, then the general positive solution \( X \) to system (2.13) can be expressed as

\[
X = N + L_{A_{11}} \left( A_{33}^{-} C_{33}^{\frac{1}{2}} + L_{A_{33}} Y \right) \left( A_{33}^{-} C_{33}^{\frac{1}{2}} + L_{A_{33}} Y \right)^* (L_{A_{11}})^* + L_{A_{11}} L_{A_{33}} S L_{A_{33}}^* L_{A_{11}}^*,
\]

where \( Y \in \mathcal{L}(K_3, H) \) is defined by

\[
Y = (A_{44} L_{A_{33}})^{-} \left( C_{33}^{\frac{1}{2}} T - A_{44} A_{33}^{-} C_{33}^{\frac{1}{2}} \right) + W - (A_{44} L_{A_{33}})^{-} (A_{44} L_{A_{33}}) W,
\]

in which \( W \in \mathcal{L}(K_3, H) \) is arbitrary.

**Proof.** The proof is carried out along the same line initiated in [7]. We take two steps: firstly, we consider the necessity and secondly, we consider the sufficiency.

(1) Suppose that \( X_0 \in \mathcal{L}(H)_+ \) is a solution to system (2.13). Then from the first two equations in (2.13), we know that \( X_0 \) is a positive solution to the equation

\[
A_{11} X = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad X \in \mathcal{L}(H).
\]

As both \( A_{11} \) and \( M \) are regular, by Lemma 2.3, we conclude that

\[
M \geq 0 \quad \text{and} \quad \mathcal{R} \left( \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right) \subseteq \mathcal{R}(M),
\]

and there exists \( V \in \mathcal{L}(H)_+ \) such that

\[
X_0 = \left( \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right)^* M^{-} \left( \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right) + L_{A_{11}} V L_{A_{11}}^* = N + L_{A_{11}} V L_{A_{11}}^*.
\]

Substituting the expression of \( X_0 \) above into the third equation in (2.13) yields

\[
A_{33} V A_{33}^* = C_{33}.
\]

Therefore, \( V \) is a positive solution to the following equation:

\[
A_{33} X A_{33}^* = C_{33}, \quad X \in \mathcal{L}(H).
\]

As both \( A_{33} \) and \( C_{33} \) are regular, by (2.8), we conclude that

\[
C_{33} \geq 0 \quad \text{and} \quad \mathcal{R}(C_{33}) \subseteq \mathcal{R}(A_{33}),
\]

and by (2.10), there exist \( Y \in \mathcal{L}(K_3, H) \) and \( S \in \mathcal{L}(H)_+ \) such that

\[
V = \left[ A_{33} C_{33}^{\frac{1}{2}} + L_{A_{33}} Y \right] \left[ A_{33}^{-} C_{33}^{\frac{1}{2}} + L_{A_{33}} Y \right]^* + L_{A_{33}} S L_{A_{33}}^*.
\]

Since \( X_0 \) satisfies the last equation in (2.13), by (2.20), we can get

\[
A_{44} V A_{44}^* = C_{44}.
\]

As both \( A_{44} \) and \( C_{44} \) are regular, once again by (2.8), we have

\[
C_{44} \geq 0 \quad \text{and} \quad \mathcal{R}(C_{44}) \subseteq \mathcal{R}(A_{44}).
\]
We may combine (2.22) and (2.23) to get

\[(2.24) \quad \left[ A_{44} \left( A_{33} C_{33}^{\frac{1}{2}} + L A_{33} Y \right) \right] \left[ A_{44} \left( A_{33} C_{33}^{\frac{1}{2}} + L A_{33} Y \right) \right]^* = C_S, \]

which means that \( C_S \in \mathcal{L}(K_4)_+ \), and by Lemma 2.7, there exists \( T \in \mathcal{L}(K_3, K_4) \) such that

\[(2.25) \quad A_{44} \left( A_{33} C_{33}^{\frac{1}{2}} + L A_{33} Y \right) = C_{33}^{\frac{1}{2}} T. \]

Therefore, \( Y \) is a solution to the following equation

\[(2.26) \quad A_{44} L A_{33} X = C_{33}^{\frac{1}{2}} T - A_{44} A_{33} C_{33}^{\frac{1}{2}}, \quad X \in \mathcal{L}(K_3, H). \]

Since \( A_{44} L A_{33} \) is regular, by Lemma 2.1, there exists \( W \in \mathcal{L}(K_3, H) \) such that \( Y \) is given by (2.17). We may combine (2.20) with (2.22) to conclude that \( X_0 \) can be expressed as (2.16). This completes the proof of the necessity.

(2) Suppose that conditions (i)–(iii) are all satisfied. Let \( X \) be given by (2.16) with \( Y \) be formulated by (2.17). Then \( X \) is positive since its first term \( N \) in summation is positive by Lemma 2.3, and its other two terms are also positive. By (2.15), \( Y \) is a solution to (2.26); or equivalently, equation (2.25) is satisfied; hence, by the second equation in (2.14), we know that (2.24) is also valid.

Now, let \( V \) be defined by (2.22). Then (2.23) follows immediately from (2.22), (2.24), and (2.14). Since \( \mathcal{R}(C_{33}^{\frac{1}{2}}) = \mathcal{R}(C_{33}) \subseteq \mathcal{R}(A_{33}) \), equation (2.21) can be derived from (2.22). Furthermore, by (2.16) and (2.22), we can conclude that

\[(2.27) \quad X = N + L_{A_{33}} V L_{A_{33}}^*. \]

The equation above, together with (2.21) and (2.23), yields the last two equations in (2.13). In view of (2.19), we have

\[ A_{11} N = M^* M^{-} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M M^{-} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \]

and thus, \( X \) formulated by (2.27) is a solution to (2.18); that is, the first two equations in (2.13) are also true. This completes the proof of the sufficiency.

Remark 2.9. Due to Lemma 2.7, we choose the number \( \frac{1}{2} \) as the power of \( C_S \) in (2.25). Evidently, in the Hilbert space case this number can be changed more naturally to be \( \frac{1}{2} \), since each closed subspace of a Hilbert space is orthogonally complemented. In fact, based on the equation (2.24) a partial isometry \( T \) can be constructed which satisfies

\[ A_{44} \left( A_{33} C_{33}^{\frac{1}{2}} + L A_{33} Y \right) = C_{33}^{\frac{1}{2}} T \]

such that the equation of \( C_{33}^{\frac{1}{2}} T T^* C_{33}^{\frac{1}{2}} = C_S \) is satisfied automatically. It is remarkable that the same is not always true for general Hilbert \( C^* \)-modules. We construct a counterexample as follows.

Example 2.10. Let \( \Omega = \{ z \in \mathbb{C} : |z - 1| \leq 1 \} \) and \( \mathfrak{A} = C(\Omega) \) be the \( C^* \)-algebra consisting of all complex-valued continuous functions on \( \Omega \). With the inner product defined by \( (f, g) = f^* g \), for \( f, g \in \mathfrak{A} \), the
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$C^*$-algebra $\mathfrak{A}$ itself is also a Hilbert $\mathfrak{A}$-module. Define adjointable operators $A, B, C \in \mathcal{L}(\mathfrak{A})$ by

$$(Af)(z) = \begin{cases} |ze^{i4\arg z}f(z)|, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(Cf)(z) = \begin{cases} |ze^{i\arg z}f(z)|, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(Bf)(z) = |z|^2 f(z),$$

where $\arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $z \neq 0$ is the argument function and $\arg(0, 0) = 0$, which is discontinuous only at the origin $(0, 0)$. Then $B = B^*$ and

$$(A^*f)(z) = \begin{cases} |ze^{-i4\arg z}f(z)|, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(C^*f)(z) = \begin{cases} |ze^{-i\arg z}f(z)|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

It follows that $AA^* = A^*A = C^*C = CC^* = B$. We show that there does not exist an $X \in \mathcal{L}(\mathfrak{A})$ such that $AX = C$. Indeed, if such an $X$ exists, then, for each $z \neq 0$ and $f \in \mathfrak{A}$ with $f(0) \neq 0$, we have

$$|z|e^{i4\arg z}f(z) = (Cf)(z) = (Af)(z) = |z|e^{i4\arg z}(Xf)(z).$$

Hence, if $z \neq 0$, then

$$(2.28) \quad (Xf)(z) = e^{i3\arg z}f(z) \quad \text{for each } f \in \mathfrak{A} \text{ with } f(0) \neq 0.$$ 

Let $f$ satisfy the condition in (2.28). If $z \in \Omega$ and $z = re^{i\arg z} \to 0$ with $\arg z \to (\frac{\pi}{2})^-$, then $(Xf)(z) \to e^{i\frac{3\pi}{2}}f(0)$. On the other hand, $(Xf)(z) \to e^{-i\frac{3\pi}{2}}f(0)$ when $z \in \Omega$ and $z = re^{i\arg z} \to 0$ with $\arg z \to (-\frac{\pi}{2})^+$. Hence, $\lim_{z \to 0}(Xf)(z)$ does not exist; this shows that $Xf \notin \mathfrak{A}$.

**Remark 2.11.** The counterexample above shows that Lemma 3.4 stated in [7] is incorrect, which leads to the wrong expression of $Y$ given in (3.5) of [7] and the nonsufficiency of the conditions stated in [7, Theorem 3.5].

**References**


