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POSITIVE SOLUTIONS OF THE SYSTEM OF OPERATOR EQUATIONS $A_1X = C_1$, $XA_2 = C_2$, $A_3XA_3^* = C_3$, AND $A_4XA_4^* = C_4$ IN HILBERT C^*-MODULES*

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Abstract. Necessary and sufficient conditions are given for the operator system $A_1X = C_1$, $XA_2 = C_2$, $A_3XA_3^* = C_3$, and $A_4XA_4^* = C_4$ to have a common positive solution, where A_i's and C_i's are adjointable operators on Hilbert C^*-modules. This corrects a published result by removing some gaps in its proof. Finally, a technical example is given to show that the proposed investigation in the setting of Hilbert C^*-modules is different from that of Hilbert spaces.

Key words. Hilbert C^*-module, Operator equation, Orthogonally complemented submodule.

AMS subject classifications. 15A24, 46L08, 47A05, 47A62.

1. Introduction. Let $\mathfrak{A}$ be a C^*-algebra. A Hilbert $\mathfrak{A}$-module is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle : H \times H \to \mathfrak{A}$ such that H is complete with respect to the induced norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ for $x \in H$. Suppose that H and K are Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(H,K)$ be the set of maps $A : H \to K$ for which there is a map $A^* : K \to H$, called the *adjoint operator* of A, such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \text{for each } x \in H \text{ and } y \in K.$$

It is known that each element A of $\mathcal{L}(H,K)$ must be a bounded linear operator, which is also $\mathfrak{A}$-linear in the sense that $A(xa) = (Ax)a$ for each $x \in H$ and $a \in \mathfrak{A}$. We use the notations $\mathcal{L}(H)$ and $\mathcal{L}(H)_+$ to denote the C^*-algebra $\mathcal{L}(H,H)$ and the set of positive elements of $\mathcal{L}(H)$, respectively. Let $A \in \mathcal{L}(H)$. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we mean the range and the null space of A, respectively. By [3, Lemma 4.1], we know that A is positive if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Let H be a Hilbert $\mathfrak{A}$-module. A closed submodule K of H is said to be *orthogonally complemented* in H if $H = K \oplus K^\perp$, where

$$K^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in K\}.$$

Evidently, K is orthogonally complemented in H if and only if there exists a projection P on H, whose range is K and $\mathcal{R}(P) \oplus \mathcal{N}(P) = H$.

Throughout the rest of this section, H and K are Hilbert C^*-modules, and A is an element of $\mathcal{L}(H,K)$. Recall that an operator A is *regular* if $\mathcal{R}(A)$ is closed in K.

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Lemma 1.1. (See [3, Theorem 3.2] and [9, Remark 1.1]) *The closedness of any one of the following sets implies the closedness of the remaining three sets:*

$$\mathcal{R}(A),\ \ \mathcal{R}(A^*),\ \ \mathcal{R}(AA^*),\ \ \text{and}\ \mathcal{R}(A^*A).$$

If $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) = \mathcal{R}(AA^*)$, $\mathcal{R}(A^*) = R(A^*A)$, and the following orthogonal decompositions hold:

$$H = \mathcal{N}(A) \oplus \mathcal{R}(A^*) \quad \text{and} \quad K = \mathcal{R}(A) \oplus \mathcal{N}(A^*). \tag{1.1}$$

Recall that each element A^- of $A\{1\} = \{X \in \mathcal{L}(K,H) : AXA = A\}$ is called an *inner inverse* of A. Clearly, it can be deduced from [9, Theorem 2.2] that A has an inner inverse if and only if A is regular. In this case, we put

$$L_A := I - A^-A, \tag{1.2}$$

where $A^- \in A\{1\}$ is unspecified.

The Moore–Penrose inverse $A^\dagger$ of A (if it exists) is the unique element X of $\mathcal{L}(K,H)$ which satisfies

$$AXA = A,\ \ XAX = X,\ \ (AX)^* = AX,\ \ \text{and}\ \ (XA)^* = XA. \tag{1.3}$$

We remark that as in the Hilbert space case, $A^\dagger$ exists if and only if A is regular [9, Theorem 2.2], in which case $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, and

$$(A^\dagger)^* = (A^*)^\dagger \quad \text{and} \quad (AA^*)^\dagger = (A^*)^\dagger A^\dagger = (A^\dagger)^* A^\dagger. \tag{1.4}$$

If $H = K$ and A is Hermitian, then $A^\dagger$ is also Hermitian and $AA^\dagger = A^\dagger A$.

The study of operator equations has been developed from matrices to infinite dimensional spaces; for example, arbitrary Hilbert spaces and Hilbert $\mathfrak{A}$-modules, by several mathematicians; see [1, 4, 8, 11, 12] and references therein. In [8], some necessary and sufficient conditions for the existence of common Hermitian and positive solutions $X \in \mathcal{L}(H)$ for the equations $AX = C$ and $XB = D$ are proposed and some formulas for the general forms of their common solutions are given.

In this paper, we give some necessary and sufficient conditions for the operator system $A_1X = C_1$, $XA_2 = C_2$, $A_3XA_3^* = C_3$, and $A_4XA_4^* = C_4$ to have a common positive solution, where A_i's and C_i's are adjointable operators on Hilbert C^*-modules. This corrects the main result of Song and Wang [7] by removing some gaps in its proof. Finally, we give a technical example and show that our investigation in the setting of Hilbert C^*-modules differs from that in the framework of Hilbert spaces.

2. Main results.

Throughout this section, H, K, L, and $K_i (1 \le i \le 4)$ are Hilbert $\mathfrak{A}$-modules.

The proof of Lemma 2.1 below is straightforward.

Lemma 2.1. *Let $A \in \mathcal{L}(H,K), C \in \mathcal{L}(L,K)$ be such that A is regular. Then the operator equation $AX = C$ has a solution $X \in \mathcal{L}(L,H)$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. In this case, the general solution to $AX = C$ is of the form*

$$X = A^-C + (I - A^-A)T, \tag{2.5}$$

where $T \in \mathcal{L}(L,H)$ is arbitrary.

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LEMMA 2.2. (See [8, Theorem 2.1]) *Let $A, C \in \mathcal{L}(H,K)$ be such that both A and CA^* are regular. Then the operator equation $AX = C$ has a solution $X \in \mathcal{L}(H)_+$ if and only if $CA^* \geq 0$ and $\mathcal{R}(C) = \mathcal{R}(CA^*)$. In this case, the general positive solution to $AX = C$ is of the form*

$$X = C^*(CA^*)^- C + L_A S L_A^*,$$

where $S \in \mathcal{L}(H)_+$ is arbitrary and $C^*(CA^*)^-C$ is a positive element, which is independent of the choice of the inner inverse $(CA^*)^-$.

LEMMA 2.3. (See [8, Theorem 3.7]) *Let $A_1, C_1 \in \mathcal{L}(H,K), A_2, C_2 \in \mathcal{L}(L,H)$,*

$$D = \begin{pmatrix} A_1 \\ A_2^* \end{pmatrix}, \quad E = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} C_1 A_1^* & C_1 A_2 \\ (A_1 C_2)^* & C_2^* A_2 \end{pmatrix}$$

be such that D and F are regular. Then the system

$$A_1 X = C_1, \quad X A_2 = C_2, \quad X \in \mathcal{L}(H) \tag{2.6}$$

has a solution $X \in \mathcal{L}(H)_+$ if and only if $F \geq 0$ and $\mathcal{R}(E) \subseteq \mathcal{R}(F)$. In this case, the general positive solution to system (2.6) can be expressed as

$$X = E^* F^- E + L_D T L_D^*,$$

where $T \in \mathcal{L}(H)_+$ is arbitrary and E^*F^-E is a positive element, which is independent of the choice of the inner inverse F^-.

REMARK 2.4. Suppose that $A \in \mathcal{L}(H,K)$ and $C \in \mathcal{L}(K)$ are both regular. It is indicated in [10, Lemma 3.2] that the equation

$$AXA^* = C, \quad X \in \mathcal{L}(H), \tag{2.7}$$

has a solution $X \in \mathcal{L}(H)_+$ if and only if

$$C \geq 0 \quad \text{and} \quad \mathcal{R}(C) \subseteq \mathcal{R}(A). \tag{2.8}$$

In this case, the general positive solution for equation (2.7) can be expressed as

$$X = A^\dagger C (A^\dagger)^* + A^\dagger C (A^\dagger)^* V F_A + F_A V^* A^\dagger C (A^\dagger)^* + F_A V^* A^\dagger C (A^\dagger)^* V F_A + F_A W F_A, \tag{2.9}$$

where $F_A = I - A^\dagger A$, $V \in \mathcal{L}(H)$ is arbitrary, and $W \in \mathcal{L}(H)_+$ is arbitrary.

The point is, as shown in [2] by Groß for matrices, we can replace $A^\dagger$ in (2.9) by a general inner inverse A^-, and meanwhile give a simplified formula for X. For the sake of completeness, we give a detailed proof of Lemma 2.5 below, using a method somewhat different from that in [2].

LEMMA 2.5. (See [2, Theorem 1]) *Suppose that $A \in \mathcal{L}(H,K)$ and $C \in \mathcal{L}(K)$ are both regular such that condition (2.8) is satisfied. Then the general positive solution to equation (2.7) can be expressed as*

$$X = [A^- B + L_A Y][A^- B + L_A Y]^* + L_A S (L_A)^*, \tag{2.10}$$

where L_A is defined by (1.2), $Y \in \mathcal{L}(K,H)$ is arbitrary, $S \in \mathcal{L}(H)_+$ is arbitrary, and $B \in \mathcal{L}(K)$ is an arbitrary operator satisfying $BB^* = C$.

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Proof. Let $B \in \mathcal{L}(K)$ be chosen such that $BB^* = C$. By Lemma 1.1, we have $\mathcal{R}(B) = \mathcal{R}(C)$; hence, $AA^-B = B$, which means that each operator X of the form (2.10) is a positive solution to equation (2.7).

Conversely, suppose that $X \in \mathcal{L}(H)_+$ is a solution to equation (2.7). Let $U = XA^* - A^-C$. Then $AU = 0$; hence, $XA^* = A^-C + L_AU$. Taking the $*$-operation, we have

$$AX = C(A^-)^* + U^*(L_A)^* \overset{def}{=} C'. \tag{2.11}$$

Note that $C'A^* = AXA^* = C$, which is regular. Note also that X is a positive solution to the equation $AZ = C', Z \in \mathcal{L}(H)$; so by Lemma 2.2, there exists $S \in \mathcal{L}(H)_+$ such that

$$X = (C')^*(C'A^*)^\dagger C' + L_AS(L_A)^* = (C')^*(BB^*)^\dagger C' + L_AS(L_A)^*. \tag{2.12}$$

Clearly, $C(B^\dagger)^* = BB^*(B^\dagger)^* = B$, and, by (1.4), we have $(BB^*)^\dagger = (B^\dagger)^*B^\dagger$. In view of the observation above, formula (2.10) for X follows immediately from (2.11) and (2.12) by putting $Y = U(B^\dagger)^*$. ◻

Lemma 2.6. (See [6, Proposition 1.4.5]) *Let x and a be elements in a C^*-algebra $\mathfrak{A}$ such that $a \geq 0$ and $x^*x \leq a$. If $0 < \beta < \frac{1}{2}$, then there exists $u \in \mathfrak{A}$ with $\|u\| \leq \left\|a^{\frac{1}{2}-\beta}\right\|$ such that $x = ua^\beta$.*

Lemma 2.7. *Let $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}(K)_+$ be such that $AA^* \leq B$. Then, for each $\beta \in (0, \frac{1}{2})$, there exists $C \in \mathcal{L}(H,K)$ such that $A = B^\beta C$.*

Proof. We consider the C^*-algebra $\mathcal{L}(H \oplus K)$, which contains $\widetilde{A}$ and $\widetilde{B}$, where

$$\widetilde{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

It is obvious that $\widetilde{A}\big(\widetilde{A}\big)^* \leq \widetilde{B}$; so, for each $\beta \in (0, \frac{1}{2})$, by Lemma 2.6, there exists $W = \begin{pmatrix} W_{11} & W_{12} \\ C & W_{22} \end{pmatrix} \in \mathcal{L}(H \oplus K)$ such that $\widetilde{A} = \widetilde{B}^\beta W$. Direct computation yields $A = B^\beta C$. ◻

Now we state the main result of this paper, which is a modification of [7, Theorem 3.5].

Theorem 2.8. *Let $A_1, C_1 \in \mathcal{L}(H,K_1), A_2, C_2 \in \mathcal{L}(K_2,H)$, $A_3 \in \mathcal{L}(H,K_3), A_4 \in \mathcal{L}(H,K_4), C_3 \in \mathcal{L}(K_3)$, and $C_4 \in \mathcal{L}(K_4)$ be given such that $A_{11}, M, A_{33}, C_{33}, A_{44}, C_{44}$, and $A_{44}L_{A_{33}}$ are all regular, where*

$$A_{11} = \begin{pmatrix} A_1 \\ A_2^* \end{pmatrix}, \quad M = \begin{pmatrix} C_1A_1^* & C_1A_2 \\ C_2^*A_1^* & C_2^*A_2 \end{pmatrix}, \quad N = (C_1^* \;\; C_2)M^-\begin{pmatrix} C_1 \\ C_2^* \end{pmatrix},$$

$$A_{33} = A_3L_{A_{11}}, \quad A_{44} = A_4L_{A_{11}}, \quad C_{33} = C_3 - A_3NA_3^*, \quad C_{44} = C_4 - A_4NA_4^*.$$

Then the system

$$A_1X = C_1, \;\; XA_2 = C_2, \;\; A_3XA_3^* = C_3, \;\; A_4XA_4^* = C_4, \;\; X \in \mathcal{L}(H) \tag{2.13}$$

has a solution $X \in \mathcal{L}(H)_+$ if and only if the following three conditions hold:

(i) *The operators M, C_{33} and C_{44} are all positive;*

(ii) $\mathcal{R}\begin{pmatrix} C_1 \\ C_2^* \end{pmatrix} \subseteq \mathcal{R}(M)$, $\mathcal{R}(C_{33}) \subseteq \mathcal{R}(A_{33})$, $\mathcal{R}(C_{44}) \subseteq \mathcal{R}(A_{44})$;

(iii) *There exist $S \in \mathcal{L}(H)_+$ and $T \in \mathcal{L}(K_3, K_4)$ such that*

$$C_S := C_{44} - A_{44}L_{A_{33}}SL_{A_{33}}^*A_{44}^* \geq 0, \tag{2.14}$$

$$\mathcal{R}\left(C_S^{\frac{1}{3}}T - A_{44}A_{33}^-C_{33}^{\frac{1}{2}}\right) \subseteq \mathcal{R}(A_{44}L_{A_{33}}). \tag{2.15}$$

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If conditions (i)–(iii) *are satisfied, then the general positive solution* X *to system* (2.13) *can be expressed as*

$$X = N + L_{A_{11}} \left(A_{33}^- C_{33}^{\frac{1}{2}} + L_{A_{33}} Y\right) \left(A_{33}^- C_{33}^{\frac{1}{2}} + L_{A_{33}} Y\right)^* (L_{A_{11}})^* + L_{A_{11}} L_{A_{33}} S L_{A_{33}}^* L_{A_{11}}^*, \tag{2.16}$$

where $Y \in \mathcal{L}(K_3, H)$ *is defined by*

$$Y = (A_{44} L_{A_{33}})^- \left(C_S^{\frac{1}{3}} T - A_{44} A_{33}^- C_{33}^{\frac{1}{2}}\right) + W - (A_{44} L_{A_{33}})^- (A_{44} L_{A_{33}}) W, \tag{2.17}$$

in which $W \in \mathcal{L}(K_3, H)$ *is arbitrary.*

Proof. The proof is carried out along the same line initiated in [7]. We take two steps: firstly, we consider the necessity and secondly, we consider the sufficiency.

(1) Suppose that $X_0 \in \mathcal{L}(H)_+$ is a solution to system (2.13). Then from the first two equations in (2.13), we know that X_0 is a positive solution to the equation

$$A_{11} X = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix}, \quad X \in \mathcal{L}(H). \tag{2.18}$$

As both A_{11} and M are regular, by Lemma 2.3, we conclude that

$$M \geq 0 \quad \text{and} \quad \mathcal{R}\begin{pmatrix} C_1 \\ C_2^* \end{pmatrix} \subseteq \mathcal{R}(M), \tag{2.19}$$

and there exists $V \in \mathcal{L}(H)_+$ such that

$$X_0 = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix}^* M^- \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix} + L_{A_{11}} V L_{A_{11}}^* = N + L_{A_{11}} V L_{A_{11}}^*. \tag{2.20}$$

Substituting the expression of X_0 above into the third equation in (2.13) yields

$$A_{33} V A_{33}^* = C_{33}. \tag{2.21}$$

Therefore, V is a positive solution to the following equation:

$$A_{33} X A_{33}^* = C_{33}, \quad X \in \mathcal{L}(H).$$

As both A_{33} and C_{33} are regular, by (2.8), we conclude that

$$C_{33} \geq 0 \quad \text{and} \quad \mathcal{R}(C_{33}) \subseteq \mathcal{R}(A_{33}),$$

and by (2.10), there exist $Y \in \mathcal{L}(K_3, H)$ and $S \in \mathcal{L}(H)_+$ such that

$$V = \left[A_{33}^- C_{33}^{\frac{1}{2}} + L_{A_{33}} Y\right] \left[A_{33}^- C_{33}^{\frac{1}{2}} + L_{A_{33}} Y\right]^* + L_{A_{33}} S L_{A_{33}}^*. \tag{2.22}$$

Since X_0 satisfies the last equation in (2.13), by (2.20), we can get

$$A_{44} V A_{44}^* = C_{44}. \tag{2.23}$$

As both A_{44} and C_{44} are regular, once again by (2.8), we have

$$C_{44} \geq 0 \quad \text{and} \quad \mathcal{R}(C_{44}) \subseteq \mathcal{R}(A_{44}).$$

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We may combine (2.22) and (2.23) to get

$$\left[A_{44}\left(A_{33}^{-}C_{33}^{\frac{1}{2}}+L_{A_{33}}Y\right)\right]\left[A_{44}\left(A_{33}^{-}C_{33}^{\frac{1}{2}}+L_{A_{33}}Y\right)\right]^{*}=C_S, \tag{2.24}$$

which means that $C_S\in\mathcal{L}(K_4)_+$, and by Lemma 2.7, there exists $T\in\mathcal{L}(K_3,K_4)$ such that

$$A_{44}\left(A_{33}^{-}C_{33}^{\frac{1}{2}}+L_{A_{33}}Y\right)=C_S^{\frac{1}{3}}T. \tag{2.25}$$

Therefore, Y is a solution to the following equation

$$A_{44}L_{A_{33}}X=C_S^{\frac{1}{3}}T-A_{44}A_{33}^{-}C_{33}^{\frac{1}{2}},\quad X\in\mathcal{L}(K_3,H). \tag{2.26}$$

Since $A_{44}L_{A_{33}}$ is regular, by Lemma 2.1, there exists $W\in\mathcal{L}(K_3,H)$ such that Y is given by (2.17). We may combine (2.20) with (2.22) to conclude that X_0 can be expressed as (2.16). This completes the proof of the necessity.

(2) Suppose that conditions (i)–(iii) are all satisfied. Let X be given by (2.16) with Y be formulated by (2.17). Then X is positive since its first term N in summation is positive by Lemma 2.3, and its other two terms are also positive. By (2.15), Y is a solution to (2.26); or equivalently, equation (2.25) is satisfied; hence, by the second equation in (2.14), we know that (2.24) is also valid.

Now, let V be defined by (2.22). Then (2.23) follows immediately from (2.22), (2.24), and (2.14). Since $\mathcal{R}\big(C_{33}^{\frac{1}{2}}\big)=\mathcal{R}(C_{33})\subseteq\mathcal{R}(A_{33})$, equation (2.21) can be derived from (2.22). Furthermore, by (2.16) and (2.22), we can conclude that

$$X=N+L_{A_{11}}VL_{A_{11}}^{*}. \tag{2.27}$$

The equation above, together with (2.21) and (2.23), yields the last two equations in (2.13). In view of (2.19), we have

$$A_{11}N=M^*M^-\begin{pmatrix}C_1\\C_2^*\end{pmatrix}=MM^-\begin{pmatrix}C_1\\C_2^*\end{pmatrix}=\begin{pmatrix}C_1\\C_2^*\end{pmatrix},$$

and thus, X formulated by (2.27) is a solution to (2.18); that is, the first two equations in (2.13) are also true. This completes the proof of the sufficiency. ∎

REMARK 2.9. Due to Lemma 2.7, we choose the number $\frac{1}{3}$ as the power of C_S in (2.25). Evidently, in the Hilbert space case this number can be changed more naturally to be $\frac{1}{2}$, since each closed subspace of a Hilbert space is orthogonally complemented. In fact, based on the equation (2.24) a partial isometry T can be constructed which satisfies

$$A_{44}\left(A_{33}^{-}C_{33}^{\frac{1}{2}}+L_{A_{33}}Y\right)=C_S^{\frac{1}{2}}T$$

such that the equation of $C_S^{\frac{1}{2}}TT^*C_S^{\frac{1}{2}}=C_S$ is satisfied automatically. It is remarkable that the same is not always true for general Hilbert C^*-modules. We construct a counterexample as follows.

EXAMPLE 2.10. Let $\Omega=\{z\in\mathbb{C}:|z-1|\le 1\}$ and $\mathfrak{A}=C(\Omega)$ be the C^*-algebra consisting of all complex-valued continuous functions on Ω. With the inner product defined by $\langle f,g\rangle=f^*g$, for $f,g\in\mathfrak{A}$, the

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C^*-algebra $\mathfrak{A}$ itself is also a Hilbert $\mathfrak{A}$-module. Define adjointable operators $A, B, C \in \mathcal{L}(\mathfrak{A})$ by

$$(Af)(z) = \begin{cases} |z|e^{i4\arg z} f(z), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(Cf)(z) = \begin{cases} |z|e^{i\arg z} f(z), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(Bf)(z) = |z|^2 f(z),$$

where $\arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $z \neq 0$ is the argument function and $\arg(0,0) = 0$, which is discontinuous only at the origin $(0,0)$. Then $B = B^*$ and

$$(A^* f)(z) = \begin{cases} |z|e^{-i4\arg z} f(z), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$(C^* f)(z) = \begin{cases} |z|e^{-i\arg z} f(z), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

It follows that $AA^* = A^*A = C^*C = CC^* = B$. We show that there does not exist an $X \in \mathcal{L}(\mathfrak{A})$ such that $AX = C$. Indeed, if such an X exists, then, for each $z \neq 0$ and $f \in \mathfrak{A}$ with $f(0) \neq 0$, we have

$$|z|e^{i\arg z} f(z) = (Cf)(z) = (AXf)(z) = |z|e^{i4\arg z}(Xf)(z)\,.$$

Hence, if $z \neq 0$, then

$$(Xf)(z) = e^{i3\arg z} f(z) \quad \text{for each } f \in \mathfrak{A} \text{ with } f(0) \neq 0. \tag{2.28}$$

Let f satisfy the condition in (2.28). If $z \in \Omega$ and $z = re^{i\arg z} \to 0$ with $\arg z \longrightarrow \left(\frac{\pi}{2}\right)^-$, then $(Xf)(z) \to e^{i\frac{3\pi}{2}} f(0)$. On the other hand, $(Xf)(z) \to e^{-i\frac{3\pi}{2}} f(0)$ when $z \in \Omega$ and $z = re^{i\arg z} \to 0$ with $\arg z \longrightarrow \left(-\frac{\pi}{2}\right)^+$. Hence, $\lim_{z\to 0}(Xf)(z)$ dose not exist; this shows that $Xf \notin \mathfrak{A}$.

Remark 2.11. The counterexample above shows that Lemma 3.4 stated in [7] is incorrect, which leads to the wrong expression of Y given in (3.5) of [7] and the nonsufficiency of the conditions stated in [7, Theorem 3.5].

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