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## RANKS OF QUANTUM STATES WITH PRESCRIBED REDUCED STATES\*

CHI-KWONG LI<sup>†</sup>, YIU-TUNG POON<sup>‡</sup>, AND XUEFENG WANG<sup>§</sup>

**Abstract.** Let  $\mathcal{M}_n$  be the set of  $n \times n$  complex matrices. In this note, all the possible ranks of a bipartite state in  $\mathcal{M}_m \otimes \mathcal{M}_n$  with prescribed reduced states in the two subsystems, are determined. The results are used to determine the Choi rank of quantum channels  $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$  sending  $I/m$  to a specific state  $\sigma_2 \in \mathcal{M}_n$ .

**Key words.** Quantum state, Reduced state, Partial trace, Quantum channel, Rank, Eigenvalue.

**AMS subject classifications.** 15A18, 15A60, 15A42, 15B48, 46N50.

**1. Introduction.** In quantum information science, quantum states are used to store, process, and transmit information. Mathematically, quantum states are represented by density matrices, i.e., positive semidefinite matrices of trace 1.

Let  $\mathcal{M}_n$  ( $\mathcal{H}_n$ ) be the set of  $n \times n$  complex (Hermitian) matrices. Let  $\mathcal{D}_n$  be the set of density matrices in  $\mathcal{H}_n$ . Suppose  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$  are two quantum states. Their product state is  $\sigma_1 \otimes \sigma_2 \in \mathcal{D}_{mn}$ . The combined system is known as the bipartite system, and a general quantum state is represented by a density matrix  $\rho \in \mathcal{D}_{mn}$ .

Two important quantum operations used to extract information of the subsystems from a quantum state of the bipartite system are the partial traces defined as

$$\mathrm{tr}_1(\sigma_1 \otimes \sigma_2) = \sigma_2 \quad \text{and} \quad \mathrm{tr}_2(\sigma_1 \otimes \sigma_2) = \sigma_1$$

on tensor states  $\sigma_1 \otimes \sigma_2 \in \mathcal{D}_{mn}$ , and extended by linearity for general states in  $\mathcal{D}_{mn}$ . In particular, suppose  $\rho = (\rho_{ij})_{1 \leq i, j \leq m} \in \mathcal{D}_{mn}$  such that  $\rho_{ij} \in \mathcal{M}_n$ . Then

$$\mathrm{tr}_1(\rho) = \rho_{11} + \cdots + \rho_{mm} \in \mathcal{M}_n \quad \text{and} \quad \mathrm{tr}_2(\rho) = (\mathrm{tr} \rho_{ij})_{1 \leq i, j \leq m} \in \mathcal{M}_m.$$

Let  $\mathrm{rank}(\sigma)$  be the rank of a matrix  $\sigma$ . The purpose of this note is to give a complete answer to the following.

**PROBLEM.** Determine all the possible values of  $\mathrm{rank}(\rho)$  for  $\rho \in \mathcal{D}_{mn}$  in the set

$$\mathcal{S}(\sigma_1, \sigma_2) = \{\rho \in \mathcal{D}_{mn} : \mathrm{tr}_1(\rho) = \sigma_2, \mathrm{tr}_2(\rho) = \sigma_1\}$$

for given  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$ .

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It is well known and easy to verify that the maximum rank of  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  equals  $R = \text{rank}(\sigma_1)\text{rank}(\sigma_2)$ . In Section 2, we will present a finite algorithm for determining the minimum rank value  $r$  of  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  in terms of the eigenvalues of  $\sigma_1$  and  $\sigma_2$ . Moreover, we will show that there is  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  with  $\text{rank}(\rho) = k$  for every  $k \in \{r, r + 1, \dots, R\}$ .

In Section 3, we will describe some implications of the results in Section 2 to the study of quantum channels. In particular, the results allow us to determine all the possible Choi ranks of a quantum channel  $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$  having a prescribed state  $\Phi(I_m/m) \in \mathcal{D}_n$ .

In our discussion, let  $\mathcal{M}_{m,n}$  be the set of  $m \times n$  complex matrices so that  $\mathcal{M}_n = \mathcal{M}_{n,n}$ . Denote by  $\mathcal{U}_n$  and  $\mathcal{H}_n$  the set of unitary matrices and the set of Hermitian matrices in  $\mathcal{M}_n$ , respectively. Let  $\mathbb{R}_\downarrow^n$  denote the set of  $n$ -tuples in  $\mathbb{R}^n$  with decreasing coordinates. Given  $\sigma \in \mathcal{H}_n$ ,  $\lambda(\sigma) = (\lambda_1(\sigma), \dots, \lambda_n(\sigma)) \in \mathbb{R}_\downarrow^n$  will denote the eigenvalues of  $\sigma$  arranged in decreasing order.

Let  $\{e_1^{(m)}, \dots, e_m^{(m)}\}$  and  $\{e_1^{(n)}, \dots, e_n^{(n)}\}$  be the standard bases for  $\mathbb{C}^m$  and  $\mathbb{C}^n$ , respectively. Then, clearly,  $\{e_1^{(m)} \otimes e_1^{(n)}, e_1^{(m)} \otimes e_2^{(n)}, \dots, e_m^{(m)} \otimes e_n^{(n)}\}$  is the standard basis for  $\mathbb{C}^m \otimes \mathbb{C}^n \equiv \mathbb{C}^{mn}$ . For simplicity, we use the notation  $e_i$  for  $e_i^{(m)}$  or  $e_i^{(n)}$  if the dimension of the vector is clear in the context. Also, we use  $e_i \otimes e_j$  instead of  $e_i^{(m)} \otimes e_j^{(n)}$ , and  $e_i e_j^t = E_{ij}$  to denote the basic complex unit of appropriate size.

The following observation is useful in our discussion.

LEMMA 1.1. *Let  $\sigma_1 \in \mathcal{D}_m$ ,  $\sigma_2 \in \mathcal{D}_n$ ,  $U \in \mathcal{M}_m$  and  $V \in \mathcal{M}_n$ . Then*

$$\mathcal{S}(U\sigma_1U^*, V\sigma_2V^*) = (U \otimes V)\mathcal{S}(\sigma_1, \sigma_2)(U \otimes V)^* = \{(U \otimes V)\rho(U \otimes V)^* : \rho \in \mathcal{S}(\sigma_1, \sigma_2)\}.$$

**2. Ranks of matrices in  $\mathcal{S}(\sigma_1, \sigma_2)$ .** In this section, we present results concerning the ranks of states in  $\mathcal{S}(\sigma_1, \sigma_2)$ . We will use the fact that  $\mathcal{S}(\sigma_1, \sigma_2)$  contains a rank one matrix if and only if  $\sigma_1$  and  $\sigma_2$  have the same nonzero eigenvalues counting multiplicities; see [2, 5].

For  $w = (w_1, \dots, w_{mn})^t \in \mathbb{C}^{mn}$ , let  $W = [w]$  be the  $m \times n$  matrix such that the  $i$ th row equals  $(w_{(i-1)n+1}, \dots, w_{in})$  for  $i = 1, \dots, m$ . Suppose  $W = [w]$  has singular value decomposition  $XS Y^t$  such that  $X \in \mathcal{M}_m$  is unitary with columns  $x_1, \dots, x_m$ ,  $Y \in \mathcal{M}_n$  is unitary with columns  $y_1, \dots, y_n$  and  $S = s_1 e_1 e_1^t + \dots + s_k e_k e_k^t$ , where  $k$  is the rank of  $W$ . It follows that  $W = [w] = \sum_{j=1}^k s_j x_j y_j^t$ , and  $w = \sum_{j=1}^k s_j x_j \otimes y_j$ , which is known as the Schmidt decomposition of  $w$ .

THEOREM 2.1. *Let  $(\sigma_1, \sigma_2) \in \mathcal{D}_m \times \mathcal{D}_n$  such that  $\text{rank}(\sigma_1) = r_1 \geq r_2 = \text{rank}(\sigma_2)$ . There is an element in  $\mathcal{S}(\sigma_1, \sigma_2)$  attaining the lowest rank  $r$  with  $r \leq r_1$ . Moreover, there exists  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  of rank  $k$  if and only if  $r \leq k \leq r_1 r_2$ .*

*Proof.* By Lemma 1.1, we may assume that

$$\sigma_1 = \text{diag}(\mu_1, \dots, \mu_{r_1}, 0, \dots, 0) \quad \text{and} \quad \sigma_2 = \text{diag}(\hat{\mu}_1, \dots, \hat{\mu}_{r_2}, 0, \dots, 0).$$

If  $\rho = \sum_{j=1}^k z_j z_j^* \in \mathcal{S}(\sigma_1, \sigma_2)$ , then  $\text{tr}_1(\rho) = \sigma_2$  and  $\text{tr}_2(\rho) = \sigma_1$ .

For  $1 \leq i \leq r_1$  and  $1 \leq j \leq r_2$ , let  $e_i \otimes e_j$  denote  $e_i^{(m)} \otimes e_j^{(n)}$ , and let  $v_{ij} = \sqrt{\mu_i \hat{\mu}_j} e_i \otimes e_j$ . Now, for  $\ell = 1, \dots, r_1$ , let

$$z_\ell = \begin{cases} \sum_{i=1}^{r_2} v_{i+\ell-1, i} & \text{if } 1 \leq \ell \leq r_1 - r_2 + 1, \\ \sum_{i=1}^{r_1+1-\ell} v_{i+\ell-1, i} + \sum_{i=r_1+2-\ell}^{r_2} v_{i+\ell-1-r_1, i} & \text{if } r_1 - r_2 + 1 < \ell \leq r_1. \end{cases}$$

For example,  $z_1 = v_{11} + \dots + v_{r_2, r_2}$ ,  $z_2 = v_{21} + v_{32} + \dots + v_{r_2+1, r_2}, \dots, z_{r_1} = v_{r_1, 1} + v_{12} + \dots + v_{r_2-1, r_2}$ . Then

$$(2.1) \quad \text{tr}_1(z_\ell z_\ell^*) = \begin{cases} \sum_{i=1}^{r_2} \mu_{i+\ell-1} \hat{\mu}_i E_{ii} & \text{if } 1 \leq \ell \leq r_1 - r_2 + 1, \\ \sum_{i=1}^{r_1+1-\ell} \mu_{i+\ell-1} \hat{\mu}_i E_{ii} \\ + \sum_{i=r_1+2-\ell}^{r_2} \mu_{i+\ell-1-r_1} \hat{\mu}_i E_{ii} & \text{if } r_1 - r_2 + 1 < \ell \leq r_1, \end{cases}$$

and

$$(2.2) \quad \text{tr}_2(z_\ell z_\ell^*) = \begin{cases} \sum_{i=1}^{r_2} \mu_{i+\ell-1} \hat{\mu}_i E_{i+\ell-1, i+\ell-1} & \text{if } 1 \leq \ell \leq r_1 - r_2 + 1, \\ \sum_{i=1}^{r_1+1-\ell} \mu_{i+\ell-1} \hat{\mu}_i E_{i+\ell-1, i+\ell-1} \\ + \sum_{i=r_1+2-\ell}^{r_2} \mu_{i+\ell-1-r_1} \hat{\mu}_i E_{i+\ell-1-r_1, i+\ell-1-r_1} & \text{if } r_1 - r_2 + 1 < \ell \leq r_1. \end{cases}$$

Let

$$(2.3) \quad \rho = \sum_{\ell=1}^{r_1} z_\ell z_\ell^* \in \mathcal{S}(\sigma_1, \sigma_2).$$

Then  $\rho$  has rank  $r_1$ . It follows from (2.1) and (2.2) that  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ . In (2.3) if we replace  $z_1 z_1^*$  by

$$v_{11} v_{11}^* + \dots + v_{pp} v_{pp}^* + \left( \sum_{j>p} v_{jj} \right) \left( \sum_{j>p} v_{jj} \right)^*, \quad p = 1, \dots, r_2 - 1,$$

the resulting state is in  $\mathcal{S}(\sigma_1, \sigma_2)$  with rank  $p + r_1$  for  $p = 1, \dots, r_2 - 1$ . Similarly, we can replace each  $z_j z_j^*$  by a rank  $p + 1$  matrix for  $p = 1, \dots, r_2 - 1$ , in such a way that the resulting state still lies in  $\mathcal{S}(\sigma_1, \sigma_2)$ . Hence,  $\mathcal{S}(\sigma_1, \sigma_2)$  contains matrices of rank  $k$  for every  $k \in \{r_1, \dots, r_1 r_2\}$ .

Now, suppose  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank  $r < r_1$ . We will show that there exists  $\hat{\rho} \in \mathcal{S}(\sigma_1, \sigma_2)$  with rank  $k$  for any  $r < k < r_1$ . We prove the result by induction on  $k$ . To this end, let

$$(2.4) \quad \rho = \sum_{j=1}^r z_j z_j^* \in \mathcal{S}(\sigma_1, \sigma_2).$$

By the Schmidt decomposition,  $z_j = \sum_{\ell=1}^{t_j} s_{j\ell} x_{j\ell} \otimes y_{j\ell}$  for each  $j$ , where  $t_j = \text{rank}([z_j])$ . Let  $w_{j\ell} = s_{j\ell} x_{j\ell} \otimes y_{j\ell}$ . Similar to the previous case, we can replace  $z_j z_j^*$  by

$$\sum_{\ell \leq p} w_{j\ell} w_{j\ell}^* + \left( \sum_{\ell > p} w_{j\ell} \right) \left( \sum_{\ell > p} w_{j\ell} \right)^*$$

so that the resulting state still lies in  $\mathcal{S}(\sigma_1, \sigma_2)$ . We may increase the number of summands in (2.4) by one at each time until we get  $\hat{\rho} = \sum_{j,\ell} w_{j\ell} w_{j\ell}^*$ . Note that in each step, the rank of the state will either stay the same or increase by 1, and  $\text{rank}(\hat{\rho}) \geq \text{rank}(\text{tr}_2(\hat{\rho})) = \text{rank}(\sigma_1) = r_1$ . As a result, the set  $\mathcal{S}(\sigma_1, \sigma_2)$  contains matrices of ranks  $r, \dots, r_1$ .  $\square$

We illustrate the above theorem by the following example.

EXAMPLE 2.2. Suppose  $\sigma_1 = \text{diag}(a_1, a_2, a_3)$  and  $\sigma_2 = \text{diag}(b_1, b_2)$  with  $a_1 \geq a_2 \geq a_3 > 0, b_1 \geq b_2 > 0$ . Let

$$z_1 = (\sqrt{a_1 b_1}, 0, 0, \sqrt{a_2 b_2}, 0, 0)^t, \quad z_2 = (0, 0, \sqrt{a_2 b_1}, 0, 0, \sqrt{a_3 b_2})^t, \\ z_3 = (0, \sqrt{a_1 b_2}, 0, 0, \sqrt{a_3 b_1}, 0)^t.$$

Then  $\rho = z_1 z_1^* + z_2 z_2^* + z_3 z_3^* \in \mathcal{S}(\sigma_1, \sigma_2)$  is of rank 3. One can replace  $z_1 z_1^*$  by  $v_1 v_1^* + v_2 v_2^*$  with  $v_1 = (\sqrt{a_1 b_1}, 0, 0, 0, 0, 0)^t$  and  $v_2 = (0, 0, 0, \sqrt{a_2 b_2}, 0, 0)^t$  to get a rank 4 matrix in  $\mathcal{S}(\sigma_1, \sigma_2)$ . Similarly, we can further replace  $z_2 z_2^*$  by  $v_3 v_3^* + v_4 v_4^*$ , and  $z_3 z_3^*$  by  $v_5 v_5^* + v_6 v_6^*$ , etc. to get matrices in  $\mathcal{S}(\sigma_1, \sigma_2)$  of rank 5 and 6.

COROLLARY 2.3. Let  $(\sigma_1, \sigma_2) \in \mathcal{D}_m \times \mathcal{D}_n$ . There is a rank one  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  if and only if  $\sigma_1$  and  $\sigma_2$  have the same (multi-)set of non-zero eigenvalues, say,  $\lambda_1 \geq \dots \geq \lambda_r$ .

In such a case, there exists  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  of rank  $k$  if and only if  $1 \leq k \leq r^2$ .

Proof. The first part follows from the fact that  $\rho = vv^* \in \mathcal{S}(\sigma_1, \sigma_2)$ , where  $v$  has Schmidt decomposition  $\sum_{j=1}^r s_j x_j \otimes y_j$ , if and only if  $\sigma_1 = \sum_{j=1}^r s_j^2 x_j x_j^*$  and  $\sigma_2 = \sum_{j=1}^r s_j^2 y_j y_j^*$ .

The second part follows from Theorem 2.1. □

We illustrate the above Corollary by the following example.

EXAMPLE 2.4. Suppose  $\sigma_1 = \text{diag}(\lambda_1, \lambda_2, 0)$  and  $\sigma_2 = \text{diag}(\lambda_1, \lambda_2)$  such that  $\lambda_1 \geq \lambda_2 > 0$  and  $\lambda_1 + \lambda_2 = 1$ . Let  $f_1 = (\sqrt{\lambda_1}, 0, 0, 0, 0, 0)^t, f_2 = (0, 0, 0, \sqrt{\lambda_2}, 0, 0)^t$ . Then  $(f_1 + f_2)(f_1 + f_2)^* \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank 1, and  $f_1 f_1^* + f_2 f_2^* \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank 2. Let

$$v_{11} = (\lambda_1, 0, 0, 0, 0, 0)^t, \quad v_{12} = (0, 0, 0, \lambda_2, 0, 0)^t, \\ v_{21} = (0, 0, \sqrt{\lambda_2 \lambda_1}, 0, 0, 0)^t, \quad v_{31} = (0, \sqrt{\lambda_1 \lambda_2}, 0, 0, 0, 0)^t.$$

Then  $(v_{11} + v_{12})(v_{11} + v_{12})^* + v_{21} v_{21}^* + v_{31} v_{31}^* \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank 3 and  $v_{11} v_{11}^* + v_{12} v_{12}^* + v_{21} v_{21}^* + v_{31} v_{31}^* \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank 4.

Next, we determine the minimal rank of  $\rho$  in  $\mathcal{S}(\sigma_1, \sigma_2)$ . For  $w = (w_1, \dots, w_{mn})^t \in \mathbb{C}^{mn}$ , we continue to let  $W = [w]$  be the  $m \times n$  matrix such that the  $i$ th row equals  $(w_{(i-1)n+1}, \dots, w_{in})$  for  $i = 1, \dots, m$ . One can easily construct the inverse map which converts an  $m \times n$  matrix  $W$  to  $w = \text{vec}(W) \in \mathbb{C}^{mn}$  so that  $W = [w]$ . Note that  $\text{tr}_1(w w^*) = W^t (W^t)^*$  and  $\text{tr}_2(w w^*) = W W^*$ . Suppose  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  has rank  $\leq r$ . Then there exists an  $mn \times r$  matrix  $V$  such that  $\text{tr}_1(V V^*) = \sigma_2$  and  $\text{tr}_2(V V^*) = \sigma_1$ . For  $1 \leq j \leq r$ , let  $W_j = [v_j]$ , where  $v_j$  is the  $j$ th column of  $V$ . Then we have  $\sigma_1 = W_1 (W_1)^* + \dots + W_r (W_r)^*$  and  $\sigma_2 = W_1^t (W_1^t)^* + \dots + W_r^t (W_r^t)^*$ .

Given  $C \in H_m$ , let  $\lambda(C) = (c_1, \dots, c_m)$  denote the eigenvalues of  $C$  with  $c_1 \geq \dots \geq c_m$ . Let  $\mathbb{R}_\downarrow^m = \{(x_i) \in \mathbb{R}^m : x_1 \geq \dots \geq x_m\}$ . By a result of Klyachko [4, 2], the eigenvalues of a sum of Hermitian matrices can be characterized by a set  $LR(m, r)$  of  $r + 1$  tuples  $(J_0, J_1, \dots, J_r)$ , where  $J_0, \dots, J_r$  are subsets of  $\{1, \dots, m\}$ . More specifically, given  $\mathbf{a} = (a_1, \dots, a_m)$  and  $\mathbf{c}_j = (c_{1j}, \dots, c_{mj}) \in \mathbb{R}^m, 1 \leq j \leq r$ , there exist  $A, C_1, \dots, C_r \in H_n$  such that

$$A = \sum_{j=1}^r C_j \quad \text{with } \lambda(A) = \mathbf{a}, \quad \text{and } \lambda(C_j) = \mathbf{c}_j, \quad 1 \leq j \leq r$$

if and only if  $\text{tr} A = \sum_{j=1}^r \text{tr}(C_j)$  and for every  $(J_0, J_1, \dots, J_r) \in LR(m, r)$ , the inequality

$$(2.5) \quad \sum_{i \in J_0} a_i \leq \sum_{j=1}^r \sum_{i \in J_j} c_{ij}$$

holds. We have the following.

**THEOREM 2.5.** *Suppose  $m \leq n$ ,  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$ . The following conditions are equivalent:*

- (1) *There exists  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  with rank  $\leq r$ .*
- (2) *There exist  $C_1, \dots, C_r \in \mathcal{H}_m$  and  $\tilde{C}_1, \dots, \tilde{C}_r \in \mathcal{H}_n$  such that*

$$(i) \lambda(\tilde{C}_j) = \lambda(C_j \oplus O_{n-m}), \quad (ii) \sigma_1 = \sum_{j=1}^r C_j, \quad \text{and} \quad (iii) \sigma_2 = \sum_{j=1}^r \tilde{C}_j.$$

- (3) *There exists  $C \in \mathcal{H}_m$  such that (2) holds with  $\lambda(C_j) = \lambda(C)$  and  $\lambda(\tilde{C}_j) = \lambda(C \oplus O_{n-m})$  for all  $1 \leq j \leq r$ .*

*Proof.* “(1)  $\Rightarrow$  (2)”: Suppose there exists  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  with rank  $r$ . Then there exist  $W_1, \dots, W_r \in \mathcal{M}_{m,n}$  such that

$$\sigma_1 = W_1(W_1)^* + \dots + W_r(W_r)^* \quad \text{and} \quad \sigma_2 = W_1^t (W_1^t)^* + \dots + W_r^t (W_r^t)^*.$$

Then (2) holds with  $C_j = W_j W_j^*$  and  $\tilde{C}_j = W_j^t (W_j^t)^*$ .

“(2)  $\Rightarrow$  (1)” can be proven by reversing the above argument.

“(2)  $\Rightarrow$  (3)”: Suppose  $\lambda(\sigma_1) = (a_1, \dots, a_m)$ , and  $\lambda(C_j) = (c_{1j}, \dots, c_{mj})$  for  $j = 1, \dots, r$ . Let  $(J_0, J_1, \dots, J_r) \in LR(m, r)$ , and  $\pi = (1, 2, \dots, r)$  be the cyclic permutation of  $\{1, \dots, r\}$ , i.e.,  $\pi(j) = j + 1$  for  $1 \leq j < r$  and  $\pi(r) = 1$ . Then  $(J_0, J_{\pi^k(1)}, \dots, J_{\pi^k(r)}) \in LR(m, r)$ . Since  $\sigma_1 = C_1 + \dots + C_r$ , by (2.5) we have

$$\begin{aligned} \sum_{i \in J_0} a_i &\leq \sum_{j=1}^r \sum_{i \in J_j} c_{i\sigma^k(j)} \quad \text{for every } 0 \leq k \leq r-1 \\ \Rightarrow \sum_{i \in J_0} a_i &\leq \sum_{j=1}^r \sum_{i \in J_j} c_i, \quad \text{where } c_i = (c_{i1} + \dots + c_{ir})/r \\ \Rightarrow \sigma_1 &= \hat{C}_1 + \dots + \hat{C}_r \end{aligned}$$

for some  $\hat{C}_1, \dots, \hat{C}_r \in \mathcal{H}_m$ , where  $\lambda(\hat{C}_j) = (c_1, \dots, c_m)$  for  $1 \leq j \leq r$ . Similarly, we can choose  $\tilde{C}_j$  such that  $\lambda(\tilde{C}_j) = (c_1, \dots, c_m, 0, \dots, 0)$ .

Clearly, (3)  $\Rightarrow$  (2). □

**REMARK 2.6.** For every  $m, n \geq 1$ , there exists a permutation matrix  $P \in \mathcal{M}_{mn}$  such that  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  if and only if  $P\rho P^T \in \mathcal{S}(\sigma_2, \sigma_1)$ . For example, let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then for every  $\sigma_1 \in \mathcal{D}_2$  and  $\sigma_2 \in \mathcal{D}_3$ ,  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  if and only if  $P\rho P^T \in \mathcal{S}(\sigma_2, \sigma_1)$ . Thus, there is no loss of generality in the restriction of  $m \leq n$  in Theorem 2.5.

EXAMPLE 2.7. Suppose  $m = 2$ ,  $n = 3$ ,  $\rho = \frac{1}{12} \begin{bmatrix} 3 & -2 & 1 & 1 & 2 & -1 \\ -2 & 2 & 0 & 0 & -2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & -2 & 0 & 0 & 2 & -2 \\ -1 & 2 & 1 & 1 & -2 & 3 \end{bmatrix}$ . Then  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  with

$$\sigma_1 = \frac{1}{12} \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}, \quad \sigma_2 = \frac{1}{12} \begin{bmatrix} 5 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 5 \end{bmatrix}.$$
 Let

$$C_1 = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_2 = \frac{1}{12} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix},$$

$$\tilde{C}_1 = \frac{1}{12} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \tilde{C}_2 = \frac{1}{12} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}.$$

We have:

- (i)  $\lambda(C_1) = \frac{1}{12}(3, 1)$ ,  $\lambda(\tilde{C}_1) = \frac{1}{12}(3, 1, 0)$ ,  $\lambda(C_2) = \frac{1}{12}(8, 0)$ ,  $\lambda(\tilde{C}_2) = \frac{1}{12}(8, 0, 0)$ .
- (ii)  $\sigma_1 = C_1 + C_2$ .
- (iii)  $\sigma_2 = \tilde{C}_1 + \tilde{C}_2$ .

Let  $c_1 = \frac{1}{2} \left( \frac{3}{12} + \frac{8}{12} \right) = \frac{11}{24}$  and  $c_2 = \frac{1}{2} \left( \frac{1}{12} + 0 \right) = \frac{1}{24}$ . Then we can choose

$$C'_1 = \frac{1}{24} \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}, \quad C'_2 = \frac{1}{24} \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix},$$

$$\hat{C}_1 = \frac{1}{24} \begin{bmatrix} 5 + 4\sqrt{\frac{2}{57}} & 1 - 14\sqrt{\frac{2}{57}} & 4 + 4\sqrt{\frac{2}{57}} \\ 1 - 14\sqrt{\frac{2}{57}} & 2 - 8\sqrt{\frac{2}{57}} & 1 - 14\sqrt{\frac{2}{57}} \\ 4 + 4\sqrt{\frac{2}{57}} & 1 - 14\sqrt{\frac{2}{57}} & 5 + 4\sqrt{\frac{2}{57}} \end{bmatrix},$$

and

$$\hat{C}_2 = \frac{1}{24} \begin{bmatrix} 5 - 4\sqrt{\frac{2}{57}} & 1 + 14\sqrt{\frac{2}{57}} & 4 - 4\sqrt{\frac{2}{57}} \\ 1 + 14\sqrt{\frac{2}{57}} & 2 + 8\sqrt{\frac{2}{57}} & 1 + 14\sqrt{\frac{2}{57}} \\ 4 - 4\sqrt{\frac{2}{57}} & 1 + 14\sqrt{\frac{2}{57}} & 5 - 4\sqrt{\frac{2}{57}} \end{bmatrix}.$$

Then we have  $\sigma_1 = C'_1 + C'_2$ ,  $\sigma_2 = \hat{C}_1 + \hat{C}_2$  with  $\lambda(C'_1) = \lambda(C'_2) = (c_1, c_2)$  and  $\lambda(\hat{C}_1) = \lambda(\hat{C}_2) = (c_1, c_2, 0)$ .

When  $n = rm$ , we have the following corollary of Theorem 2.5.

COROLLARY 2.8. Suppose  $n = rm$ ,  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$ . The following conditions are equivalent:

- (a) There exists  $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$  with  $\text{rank} \leq r$ .
- (b) There exists  $C = (C_{ij}) \in H_n$  with  $C_{ij} \in M_m$  such that  $\lambda(C) = \lambda(\sigma_2)$  and  $\lambda(C_{11} + \cdots + C_{rr}) = \lambda(\sigma_1)$ .
- (c) Condition (b) holds with  $\lambda(C_{11}) = \cdots = \lambda(C_{rr})$ .

*Proof.* “(a)  $\Rightarrow$  (b)”: Suppose (a) holds. Let  $C_j$  and  $\tilde{C}_j$ ,  $1 \leq j \leq r$ , be as given by condition (2) in Theorem 2.5. For each  $1 \leq j \leq r$ , there exists an  $n \times n$  unitary matrix  $U_j$  such that  $\tilde{C}_j = U_j (C_j \oplus O_{(r-1)m}) U_j^*$ . Let  $V_j \in \mathcal{M}_{n,m}$  be formed by the first  $m$  columns of  $U_j$ . Let  $R_j = V_j C_j^{1/2}$  and  $R = [R_1 | \cdots | R_r]$ . Set  $C = R^* R$ . Then

$$\lambda(C) = \lambda(R^* R) = \lambda(RR^*) = \lambda\left(\sum_{j=1}^r \tilde{C}_j\right) = \lambda(\sigma_2)$$

and

$$\lambda(C_{11} + \cdots + C_{rr}) = \lambda\left(\sum_{j=1}^r R_j^* R_j\right) = \lambda\left(\sum_{j=1}^r C_j\right) = \lambda(\sigma_1).$$

Therefore,  $C = R^* R$  will satisfy condition (b).

“(b)  $\Rightarrow$  (a)”: Suppose (b) holds. Let  $C = R^* R$  where  $R = [R_1 | \cdots | R_r]$ , with  $R_j \in \mathcal{M}_{n,m}$ . Then  $C_{ii} = R_i^* R_i$  and  $\lambda(C) = \lambda(RR^*)$ . Hence, we have

$$\lambda(\sigma_1) = \lambda(R_1^* R_1 + \cdots + R_r^* R_r) \quad \text{and} \quad \lambda(\sigma_2) = \lambda(R_1^* R_1 + \cdots + R_r^* R_r).$$

Since  $\lambda(R_i^* R_i) = \lambda(R_i R_i^* \oplus O_{(r-1)m})$ , the result follows from condition (2) in Theorem 2.5.

“(b)  $\Leftrightarrow$  (c)” follows from “(2)  $\Leftrightarrow$  (3)” in Theorem 2.5. □

Applying Theorem 2.5, we have the following algorithm for finding the minimal rank  $r$  of matrices in  $\mathcal{S}(\sigma_1, \sigma_2)$ .

ALGORITHM 2.9. Suppose  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$ , with  $n \geq m$ . Let

$$\lambda(\sigma_1) = (a_1, \dots, a_m) \quad \text{and} \quad \lambda(\sigma_2) = (b_1, \dots, b_n).$$

**Step 1.** If  $b_i = a_i$  for  $1 \leq i \leq m$ , then  $r = 1$ . If not,  $r > 1$ , then go to step 2.

**Step 2.** For  $r > 1$ , suppose the previous steps shows that the minimal rank is  $\geq r$ . Let

$$P_r(a_1, \dots, a_m) = \left\{ \mathbf{c} \in \mathbb{R}_{\downarrow}^m : c_1, \dots, c_m \geq 0, \sum_{j=1}^m c_j = 1/r, \right. \\ \left. \sum_{i \in J_0} a_i \leq \sum_{j=1}^r \sum_{i \in J_j} c_i \text{ for all } (J_0, J_1, \dots, J_r) \in LR(m, r) \right\},$$

$$Q_r(b_1, \dots, b_n) = \left\{ \mathbf{c} \in \mathbb{R}_{\downarrow}^m : c_1, \dots, c_m \geq 0, \sum_{j=1}^m c_j = 1/r, \right. \\ \left. \sum_{i \in J_0} b_i \leq \sum_{j=1}^r \sum_{i \in J_j} \hat{c}_i \text{ for all } (J_0, J_1, \dots, J_r) \in LR(n, r) \right\},$$

where in  $Q_r(b_1, \dots, b_n)$ ,  $\hat{c}_i = c_i$  for  $1 \leq i \leq m$  and  $\hat{c}_i = 0$  for  $m + 1 \leq i \leq n$ .



If  $S = P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n)$  is non-empty, then the minimal rank of  $\rho$  in  $\mathcal{S}(\sigma_1, \sigma_2)$  is  $r$ .

Otherwise, the minimal rank is larger than  $r$  and we have to repeat Step 2 with  $r$  increased by 1.

By Theorem 2.1, Algorithm 2.9 will terminate for some

$$r \leq \max\{\text{rank}(\sigma_1), \text{rank}(\sigma_2)\}.$$

In fact, we can focus on  $\sigma_1 \in \mathcal{D}_m$  and  $\sigma_2 \in \mathcal{D}_n$  with rank  $m$  and  $n$ , respectively.

The set  $P_r(a_1, \dots, a_m)$  and  $Q_r(b_1, \dots, b_n)$  are polyhedral. There are standard linear programming packages for checking whether  $S = P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n)$  is empty. Actually, we have the following.

**PROPOSITION 2.10.** *Let  $S = P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n)$  be defined as in Algorithm 2.9. Then the set  $S$  is non-empty if and only if  $a_1, \dots, a_m, b_1, \dots, b_n$  satisfy a finite set of linear inequalities.*

*Proof.* Because  $(a_1/r, \dots, a_m/r) \in P_r(a_1, \dots, a_m)$ , and  $P_r(a_1, \dots, a_m)$  is governed by a finite set of inequalities, it is a non-empty polyhedron. Thus, there are finitely many extreme points expressed as linear combinations of  $a_1, \dots, a_m$ . Now,  $Q_r(b_1, \dots, b_n)$  is determined a finite set of inequalities, say,  $v_j^t x \leq \beta_j$  for  $j = 1, \dots, N$ , where  $v_1, \dots, v_N \in \mathbb{R}^n$  with entries in  $\{b_1, \dots, b_n\}$  and  $\beta_1, \dots, \beta_N \in \mathbb{R}$ . If all the extreme points of  $P_r(a_1, \dots, a_m)$  lies in the complement of the half space defined by  $v_1^t x \leq \beta_1$ , then  $S$  is empty. Otherwise, the intersection of  $P_r(a_1, \dots, a_m)$  and the half space defined by  $v_1^t x \leq \beta_1$  is a non-empty polytope, and has a finite number of extreme points expressed as linear combinations of  $a_1, \dots, a_m, b_1, \dots, b_n$ . We can repeat the argument to this new polytope and the half space  $v_2^t x \leq \beta_2$ . We may conclude either the set and the half space has empty intersection or non-empty intersection. Repeating the process, we get a finite set of inequalities involving  $a_1, \dots, a_m, b_1, \dots, b_n$ , such that any one of them being violated will imply that  $S = \emptyset$ , and  $S \neq \emptyset$  if all the inequalities are satisfied.  $\square$

By the above proposition, one can determine whether

$$S = P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n) \neq \emptyset$$

by checking a finite set of inequalities in terms of  $a_1, \dots, a_m, b_1, \dots, b_n$ . Using this result, one may determine the set

$$\mathcal{S}_r(m : \sigma_2) = \{\sigma \in \mathcal{D}_m : \text{there is } \rho \in \mathcal{S}(\sigma, \sigma_2) \text{ with rank at most } r\}$$

for a given  $\sigma_2 \in \mathcal{D}_n$ ; and

$$\mathcal{S}_r(\sigma_1 : n) = \{\sigma \in \mathcal{D}_n : \text{there is } \rho \in \mathcal{S}(\sigma_1, \sigma) \text{ with rank at most } r\}$$

for a given  $\sigma_1 \in \mathcal{D}_m$ . We have the following.

**PROPOSITION 2.11.** *Suppose  $\sigma_2 \in \mathcal{D}_n$  has eigenvalues  $b_1 \geq \dots \geq b_n$ . Then  $\sigma \in \mathcal{S}_r(m : \sigma_2)$  if and only if  $\sigma$  has eigenvalues  $a_1 \geq \dots \geq a_m$  such that  $P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n) \neq \emptyset$ .*

*Suppose  $\sigma_1 \in \mathcal{D}_m$  has eigenvalues  $a_1 \geq \dots \geq a_m$ . Then  $\sigma \in \mathcal{S}_r(\sigma_1 : n)$  if and only if  $\sigma$  has eigenvalues  $b_1 \geq \dots \geq b_m$  such that  $P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n) \neq \emptyset$ .*

Although one can determine whether the set  $S = P_r(a_1, \dots, a_m) \cap Q_r(b_1, \dots, b_n)$  is non-empty by checking a finite set of inequalities, the number of inequalities involved may be very large. For low dimension

case, the inequalities may be reduced to a smaller set after the redundant inequalities are removed. We illustrate this in the following proposition with a direct proof. It would be nice if one can give a description of non-redundant inequalities governing the eigenvalues of the reduced states of a bipartite state with prescribed rank.

Given  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ , we say that  $\mathbf{a}$  is majorized by  $\mathbf{b}$ , denoted by  $\mathbf{a} \prec \mathbf{b}$ , if for  $1 \leq k \leq n - 1$  the sum of the  $k$  largest components of  $\mathbf{a}$  is less than or equal to that of  $\mathbf{b}$ . By Horn's result [3],  $\mathbf{a}$  is the diagonal of some  $B \in H_n$  with eigenvalues  $b_1, \dots, b_n$  if and only if  $\mathbf{a} \prec \mathbf{b}$ .

PROPOSITION 2.12. Suppose  $\sigma_1 \in \mathcal{D}_3$  has eigenvalues  $a_1 \geq a_2 \geq a_3$  and  $\sigma_2 \in \mathcal{D}_6$  has eigenvalues  $b_1 \geq \dots \geq b_6$ . Then  $\sigma_1 \in \mathcal{S}_2(3 : \sigma_2)$  if and only if  $a_1, a_2, a_3$  satisfying  $\sum_{i=1}^3 a_i = \sum_{j=1}^6 b_j$  and the following inequalities:

$$(2.6) \quad \begin{aligned} & b_3 + b_6, b_4 + b_5 \leq a_1 \leq b_1 + b_2, \\ & \frac{b_3 + b_4 + b_5 + b_6}{2} \leq a_2 \leq \frac{b_1 + b_2 + b_3 + b_4}{2}, \\ & b_5 + b_6 \leq a_3 \leq b_1 + b_4, b_2 + b_3. \end{aligned}$$

Proof. Suppose  $\sigma_2 \in \mathcal{D}_6$  has eigenvalues  $b_1 \geq \dots \geq b_6$  and  $\sigma_1 \in \mathcal{S}_2(3 : \sigma_2)$  has eigenvalues  $a_1 \geq a_2 \geq a_3$ . Then by Corollary 2.8, there exists a unitary matrix  $U \in M_6$  such that  $U^* \sigma_2 U = (C_{ij})_{1 \leq i, j \leq 2}$  with

$$\lambda(C_{11}) = \lambda(C_{22}) = (c_1, c_2, c_3) \quad \text{and} \quad \lambda(C_{11} + C_{22}) = (a_1, a_2, a_3).$$

Then there exist  $c_1 \geq c_2 \geq c_3$  and  $3 \times 3$  unitary matrices  $V_1$  and  $V_2$  such that  $\text{diag}(V_i^* C_{ii} V_i) = (c_1, c_2, c_3)$  for  $i = 1, 2$ . Hence,  $(c_1, c_1, c_2, c_2, c_3, c_3) \prec (b_1, \dots, b_6)$  [3], and we have:

1.  $a_1 \leq 2c_1 \leq b_1 + b_2$ .
2.  $a_2 \leq c_1 + c_2 \leq \frac{b_1 + b_2 + b_3 + b_4}{2}$ .
3.  $a_3 \leq 2c_2 = (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - 2(c_1 + c_3) \leq (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - (b_2 + b_3 + b_5 + b_6) = b_1 + b_4$ ,  
and  $(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - (b_1 + b_4 + b_5 + b_6) = b_2 + b_3$ .

Here,  $2(c_1 + c_3) \geq b_2 + b_3 + b_5 + b_6$ ,  $b_1 + b_4 + b_5 + b_6$  follows from the fact that [2]

$$(\{2, 3, 6, 7\}, \{1, 3, 4, 5\}, \{1, 3, 4, 5\}), (\{1, 4, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 4, 5\}) \in LR(6, 2).$$

The other inequalities can be deduced by looking at  $2\mu I_3 - \text{diag}(a_1, a_2, a_3)$  and  $\mu I_6 - \text{diag}(b_1, \dots, b_6)$ , where  $\mu = (b_1 + \dots + b_6)/6$ .

Given  $b_1 \geq \dots \geq b_6$ , let  $S$  be the set of  $(a_1, a_2, a_3)$  satisfying  $a_1 \geq a_2 \geq a_3$ ,  $\sum_{i=1}^3 a_i = \sum_{i=1}^6 b_i = 6\mu$  and (2.6). Then  $S$  is a convex polyhedron in  $\mathbb{R}^3$ . If  $S$  is non-empty, we can choose an extreme point  $(a_1, a_2, a_3)$  of  $S$ . Therefore, among the inequalities  $a_1 \geq a_2 \geq a_3$  and (2.6), at least two equalities hold. If  $a_1 = a_2 = a_3 = 2\mu$ , then from (2.6) we have

$$\begin{aligned} & b_3 + b_6, b_4 + b_5 \leq 2\mu \leq b_1 + b_4, b_2 + b_3, \\ & b_5 \leq 2\mu - b_4 \leq b_1 \quad \text{and} \quad b_6 \leq 2\mu - b_3 \leq b_2. \end{aligned}$$

Thus,  $\text{diag}(b_1, b_5)$  is unitarily similar to  $B_1$  with diagonal entries  $2\mu - b_4, b$  and  $\text{diag}(b_2, b_6)$  is unitarily similar to  $B_2$  with diagonal entries  $2\mu - b_3, c$ ; see [3]. Thus,  $B = \text{diag}(b_1, \dots, b_6)$  is unitarily similar

to  $\text{diag}(b_3, b_4) \oplus B_1 \oplus B_2$ . There exists a permutation matrix  $P$  such that  $P^*BP = (D_{ij})$  with  $D_{11} = \text{diag}(b_3, 2\mu - b_4, b)$  and  $D_{22} = \text{diag}(2\mu - b_3, b_4, c)$ . By the trace condition, we see that  $b + c = 2\mu$ . The result follows from Corollary 2.8.

Suppose either  $a_1 > a_2$  or  $a_2 > a_3$ . Then at least one of the equalities in (2.6) holds. Consider the following cases:

1.  $a_1 = b_1 + b_2$ . Then we have  $\frac{b_3 + b_4 + b_5 + b_6}{2} \leq a_2 = (b_3 + b_4 + b_5 + b_6) - a_3 \leq b_3 + b_4$ . Therefore,  $(a_2, a_3) \prec (b_3 + b_4, b_5 + b_6)$ . So there exists a  $2 \times 2$  unitary matrix  $U_1$  such that  $U_1^* \text{diag}(b_3 + b_4, b_5 + b_6) U_1 = (a_2, a_3)$ . Let  $U_2 = U_1 \text{diag}(1, -1)$ . Let

$$B_1 = \frac{1}{2} \begin{bmatrix} b_3 + b_4 & b_3 - b_4 \\ b_3 - b_4 & b_3 + b_4 \end{bmatrix} \quad \text{and} \quad B_2 = \frac{1}{2} \begin{bmatrix} b_5 + b_6 & b_5 - b_6 \\ b_5 - b_6 & b_5 + b_6 \end{bmatrix}.$$

Then  $B = \text{diag}(b_1, b_2) \oplus B_1 \oplus B_2$  has eigenvalues  $b_1, \dots, b_6$ . There exists a permutation matrix  $P$  such that  $P^*BP = (D_{ij})$  with

$$D_{11} = \frac{1}{2} \text{diag}(2b_1, b_3 + b_4, b_5 + b_6) \quad \text{and} \quad D_{22} = \frac{1}{2} \text{diag}(2b_2, b_3 + b_4, b_5 + b_6).$$

Let  $U = P([1] \oplus U_1 \oplus [1] \oplus U_2)$ . Then  $U^*BU$  will satisfy condition (2) in Corollary 2.8.

2.  $a_2 = \frac{b_1 + b_2 + b_3 + b_4}{2}$ . Then  $a_1 + a_2 \geq 2a_2 = b_1 + b_2 + b_3 + b_4 \Rightarrow a_3 \leq b_5 + b_6$ . Therefore,  $a_3 = b_5 + b_6$  and  $(a_1, a_2) \prec (b_1 + b_2, b_3 + b_4)$ . Thus, the result follows as in the previous case.
3.  $a_3 = b_1 + b_4$ . Then

$$\begin{aligned} (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) &= a_1 + a_2 + a_3 \geq 3a_3 \\ &= 3(b_1 + b_4) \geq (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \\ \Rightarrow a_1 = a_2 = a_3 &= b_1 + b_4 = b_2 + b_5 = b_3 + b_6 \\ \Rightarrow C = \text{diag}(b_1, \dots, b_6) &\text{ will satisfy (2) in Corollary 2.8} \end{aligned}$$

4.  $a_3 = b_2 + b_3$ . Then  $a_1 + a_2 = b_1 + b_4 + b_5 + b_6$  and  $a_2 \geq a_3 \geq b_5 + b_6$ . Therefore,  $(a_1, a_2) \prec (b_1 + b_4, b_5 + b_6)$ . Thus, the result follows as in the Case 2.

The proof for the other equalities are similar. □

Note that the same set of inequalities (2.6) will determine whether  $\sigma \in \mathcal{D}_6$  with eigenvalues  $b_1 \geq \dots \geq b_6$  lying in  $\mathcal{S}_2(\sigma_1 : 6)$  for a given  $\sigma_1 \in \mathcal{D}_3$  with eigenvalues  $a_1 \geq a_2 \geq a_3$ .

In case  $a_3 = 0$ , then  $b_5 = b_6 = 0$ , and the set of inequalities reduce to:

$$(b_3 + b_4)/2 \leq a_2 \quad \text{and} \quad a_1 \leq b_1 + b_2.$$

These inequalities will determine  $\sigma \in \mathcal{S}_2(\sigma_1 : 4)$  with eigenvalues  $b_1 \geq \dots \geq b_4$  for a given  $\sigma_1 \in \mathcal{D}_2$  with eigenvalues  $a_1 \geq a_2$ . The same set of inequalities will also determine  $\sigma \in \mathcal{S}_2(2 : \sigma_2)$  with eigenvalues  $a_1 \geq a_2$  for a given  $\sigma_2 \in \mathcal{D}_4$  with eigenvalues  $b_1 \geq \dots \geq b_4$ .

Note that  $\mathbf{a}$  satisfies (2.6) if and only if  $(c_1, c_2, c_3) \prec \mathbf{a} \prec (b_1 + b_2, b_3 + b_4, b_5 + b_6)$ , where

$$\mathbf{c} = \begin{cases} \left( b_4 + b_5, \frac{b_1 + b_2 + b_3 + b_6}{2}, \frac{b_1 + b_2 + b_3 + b_6}{2} \right) & \text{if } \frac{1}{3} \leq b_4 + b_5, \\ \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) & \text{if } b_4 + b_5 \leq \frac{1}{3} \leq b_2 + b_3, \\ \left( \frac{b_1 + b_4 + b_5 + b_6}{2}, \frac{b_1 + b_4 + b_5 + b_6}{2}, b_2 + b_3 \right) & \text{if } b_2 + b_3 \leq \frac{1}{3}. \end{cases}$$

**3. Quantum channels.** Recall that quantum channels are completely positive linear maps  $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$  that admit the operators sum representation

$$(3.7) \quad \phi(A) = \sum_{j=1}^r F_j A F_j^*,$$

for some  $n \times m$  matrices  $F_1, \dots, F_j$  such that  $\sum_{j=1}^r F_j^* F_j = I_m$ ; see [1, 6]. By the result in [1],  $\Phi$  is a quantum channel if and only if the Choi matrix  $C(\Phi) = (\Phi(E_{ij})) \in \mathcal{M}_m(\mathcal{M}_n)$  is positive semidefinite and  $\text{tr} \Phi(E_{ij}) = \delta_{ij}$ . Thus, the set of quantum channels can be identified with the set

$$\begin{aligned} QC(m, n) &= \{P = (P_{ij}) \in \mathcal{M}_m(\mathcal{M}_n) : P \text{ is positive semidefinite, } (\text{tr}(P_{ij})) = I_m\} \\ &= \{m\rho \in \mathcal{D}_{mn} : \text{tr}_2(\rho) = I_m/m\}. \end{aligned}$$

Consequently, the set of quantum channels  $\Phi : M_m \rightarrow M_n$  satisfying  $\Phi(I_m/m) = \rho_2 \in \mathcal{D}_n$  can be identified with  $\mathcal{S}(I_m/m, \rho_2)$ . In particular,  $\mathcal{S}(I_n/n, I_n/n)$  can be identified with the set of unital quantum channels from  $M_n$  to  $M_n$ .

For a quantum channel  $\Phi$ , its Choi rank is defined as the rank of its Choi matrix  $C(\Phi)$ . Moreover, it is known that  $\Phi$  has Choi rank  $r$  if and only if  $r$  is the minimum number of matrices  $F_1, \dots, F_r$  required in the operator sum representation of  $\Phi$ . By Theorem 2.3, we have the following.

**PROPOSITION 3.1.** *There is  $\rho \in \mathcal{S}(I_n/n, I_n/n)$  of rank  $k$  if and only if  $1 \leq k \leq n^2$ . Equivalently, there is a unital quantum channel with Choi rank  $k$  if and only if  $1 \leq k \leq n^2$ .*

By the result in the previous section, we have the following.

**PROPOSITION 3.2.** *Let  $\rho_2 \in \mathcal{D}_n$ . There is a quantum channel  $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$  with Choi rank  $k$  and  $\Phi(I_m/m) = \rho_2$  if and only if there is a rank  $k$  element in  $\mathcal{S}(I_m/m, \rho_2)$ . As a result, the value  $k$  can be any value between the minimum value  $r$  determined by Algorithm 2.9 and the maximum value  $\text{rank}(\rho) \cdot m$ .*

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