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Martin Z. Ljubenović
*Faculty of Mechanical Engineering, Department of Mathematics,University of Niš, Serbia*, martinljubenovic@gmail.com

Dragan S. Djordjevic
*Faculty of Sciences and Mathematics, Department of Mathematics, University of Niš, Serbia*, dragan@pmf.ni.ac.rs



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# BOUNDED LINEAR OPERATORS THAT PRESERVE THE WEAK SUPERMAJORIZATION ON $\ell^1(I)^+$, WHEN $I$ IS AN INFINITE SET*

MARTIN LJUBENOVIĆ† AND DRAGAN S. DJORDJEVIĆ‡

**Abstract.** Linear preservers of weak supermajorization which is defined on positive functions contained in the discrete Lebesgue space $\ell^1(I)$ are characterized. Two different classes of operators that preserve the weak supermajorization are formed. It is shown that every linear preserver may be decomposed as sum of two operators from the above classes, and conversely, the sum of two operators which satisfy an additional condition is a linear preserver. Necessary and sufficient conditions under which a bounded linear operator is a linear preserver of the weak supermajorization are given. It is concluded that positive linear preservers of the weak supermajorization coincide with preservers of weak majorization and standard majorization on $\ell^1(I)$.

**Key words.** Weak supermajorization, Linear preserver, Doubly superstochastic, Permutation.

**AMS subject classifications.** 47B60, 15A86, 15B51, 47B38.

**1. Introduction.** In recent years, the investigation in the field of the majorization theory is oriented toward the generalization of some well-known results in the matrix theory using the extension of the most useful majorization relations [33]. Applications of the infinite-dimensional majorization for generalization the Schur-Horn theorem are studied in [2, 5, 18, 28, 32]. We suggest the following papers [3, 4, 14, 19, 20, 31] for majorization theory in von Neumann and Jordan algebras. The best collection of existing results in the finite-dimensional majorization theory and its applications is the book [29], by Marshall, Olkin and Arnold.

In the finite-dimensional case, for two vectors $x, y \in \mathbb{R}^n$, vector $x$ is weakly supermajorized by $y$, if

$$\sum_{i=1}^{k} x_i^{\uparrow} \geq \sum_{i=1}^{k} y_i^{\uparrow} \quad (k = 1, 2, \ldots, n), \tag{1.1}$$

where $x_1^{\uparrow} \leq x_2^{\uparrow} \leq \cdots \leq x_n^{\uparrow}$ is the increasing rearrangement of components of a vector $x$. We denote it by $x \prec^{ws} y$. If additionally,

$$\sum_{i=1}^{n} x_i^{\uparrow} = \sum_{i=1}^{n} y_i^{\uparrow},$$

then $x$ is majorized by $y$, and denote it $x \prec y$.

We recall that a square matrix with non-negative real entries is called doubly stochastic, if each of its row sums and each of its column sums are equal 1. More general, $n \times n$ matrix $\widetilde{D} = (\widetilde{d}_{ij})$ with non-negative

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Corresponding Author: Martin Ljubenović.

†Faculty of Mechanical Engineering, Department of Mathematics, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia. (martinljubenovic@gmail.com). Supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 174007.

‡Faculty of Sciences and Mathematics, Department of Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia (dragan@pmf.ni.ac.rs). Supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 174007.

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real entries is called doubly superstochastic, if there is a doubly stochastic matrix $D = (d_{ij})$ such that

$$\widetilde{d}_{ij} \geq d_{ij} \tag{1.2}$$

for each $i$ and $j$. This implies that each of its row sums and each of its column sums are grater than or equal to 1. We note that the converse is not true, that is there are matrices which satisfy the last rows and columns conditions but there is no doubly stochastic matrix such that (1.2) holds.

There is several "alternative" definitions of majorization relations in finite dimensions (see [29, Theorems I.1.A.3–5]). We give the most operative equivalents for the standard majorization and weak supermajorization using doubly stochastic and superstochastic matrix, respectively. Hardy, Litlewood and Polya [16] provide that $x \prec y$ if and only if there is a doubly stochastic matrix $D$ such that

$$x = Dy. \tag{1.3}$$

For positive vectors $x, y \in (\mathbb{R}^n)^+$, relation $x \prec^{ws} y$ holds if and only if there is a doubly superstochastic matrix $\widetilde{D}$ such that

$$x = \widetilde{D}y. \tag{1.4}$$

The extension of the standard majorization relation by doubly stochastic operators on $\ell^p(I)$ is introduced in [6] based on definition (1.3), (see also [7, 10, 12, 21]). Precisely, a positive bounded linear operator $A : \ell^p(I) \to \ell^p(I)$ is called doubly stochastic if

$$(\forall j \in I) \ \sum_{i\in I} Ae_j(i) = 1 \quad \text{and} \quad (\forall i \in I) \ \sum_{j\in I} Ae_j(i) = 1. \tag{1.5}$$

The set of all doubly stochastic operators we will denote by $DS(\ell^p(I))$. The function $f \in \ell^p(I)$ is majorized by $g \in \ell^p(I)$ if there is $D \in DS(\ell^p(I))$ such that

$$g = Df.$$

The notion of the weak supermajorization on $\ell^p(I)^+$ which represents a generalization of the weak supermajorization on $\mathbb{R}^n$ using the alternative definition (1.4) for positive functions, is introduced in [25] using doubly superstochastic operators on $\ell^p(I)$. Let $\mathbb{A} = \{a_{ij} : i, j \in I\}$ be a family of real numbers, where $I$ is an arbitrary non-empty set. First of all, in the work [25] it has been shown that the conditions

$$M_2 := \sup_{j\in I} \sum_{i\in I} |a_{ij}| < \infty, \tag{1.6}$$

$$M_1 := \sup_{i\in I} \sum_{j\in I} |a_{ij}| < \infty \tag{1.7}$$

are sufficient for the family $\mathbb{A}$ to be considered as bounded linear operator $A$ on $\ell^p(I)$ for every $p \in [1, \infty]$. The operator $A$ is defined as standard matrix operator by

$$Af := \sum_{i\in I} \left( \sum_{j\in I} a_{ij} f(j) \right) e_i. \tag{1.8}$$

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Corollary 1.1. [25, Corollary 3.1] *Let $\mathbb{A} = \{a_{ij} : i, j \in I\}$ be a family of real numbers. If this family satisfies conditions* (1.6) *and* (1.7)*, then this family may be considered as a bounded linear operator $A$ on $\ell^p(I)$ defined by* (1.8)*, for every $p \in [1, \infty]$.*

Using the last corollary we may identify above families and appropriate operators defined by (1.8), so we use the same letter $A$ for both of them in order to simplify notation. Within this class of operators which satisfy conditions (1.6) and (1.7), in accordance with finite dimensional case, notions of row, column and doubly superstochastic are extended.

Definition 1.2. [25, Definition 3.1] Let $A = \{a_{ij} : i, j \in I\}$ be a family of positive real numbers, which satisfies (1.6) and (1.7), where $I$ is an arbitrary non-empty set. The family $A$ is called

- *row superstochastic*, if $\sum_{j\in I} a_{ij} \geq 1$, $\forall i \in I$;
- *column superstochastic*, if $\sum_{i\in I} a_{ij} \geq 1$, $\forall j \in I$;
- *doubly superstochastic*, if there is $\widetilde{A} \in DS(\ell^p(I))$ such that $a_{ij} \geq \langle \widetilde{A}e_j, e_i\rangle$, $\forall i, j \in I$.

In the sequel, in many situations stochastic families will be called stochastic operators, when we consider these families as matrix operators defined by (1.8). The set of all doubly superstochastic families (operators) on $\ell^p(I)$, $p \in [1, \infty)$ are denoted by $DSPS(\ell^p(I))$.

It is easy to see that every doubly superstochastic operator is both row and column superstochastic. However, the converse does not necessarily hold in general. In fact, the row and column superstochastic operator $A = \{a_{ij} : i, j \in \mathbb{N}\}$ defined by

$$A = \begin{bmatrix} 0 & \frac{3}{5} & \frac{3}{5} & 0 & 0 & \cdots \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 & \cdots \\ \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is not doubly superstochastic.

Definition 1.3. [25, Definition 4.1] For two positive functions $f, g \in \ell^p(I)^+$, $f$ is *weakly supermajorized* by $g$, if there exists a doubly superstochastic operator $D \in DSPS(\ell^p(I))$, such that $f = Dg$, and denote it by $f \prec^{ws} g$.

Linear preservers, as operators which provide that for two elements in some relation their images are also in a relation, has been studied by many mathematicians. For a survey of linear preserver problems see [15, 26, 27]. Also, preservers represent very interesting topic in the majorization theory for the finite-dimensional case [1, 17, 30, 34] as well as various infinite-dimensional cases such as majorizations on discrete Lebesgue spaces $\ell^p(I)$, $\ell^\infty(I)$, $c_0$, $c$, etc. See for for details [6, 7, 8, 9, 11, 13, 23, 24].

The paper is organized as follows. Section 2 contains notations and some published results. At the beginning of Section 3, we provide by Theorem 3.4 that the bounded linear operator

$$T_1 = \sum_{k\in I_0} \lambda_k P_{\theta_k},$$

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which is defined by the one-to-one operator

$$P_{\theta_k}(f) := \sum_{i \in I} f(i) e_{\theta_k(i)},$$

represents a preserver of weak supermajorization, where the set $I_0$ is at most a countable subset of an infinite set $I$, sequence $(\lambda_k)_{i \in I_0}$ is in $\ell^1(I_0)^+$, and every $\theta_k$ belongs to a countable family $\Theta$ of one-to-one maps $\theta_k : I \to I$ with mutually disjoint ranges $\theta_k(I)$. The set of all such operators we will denote by $\mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$. On the other hand, Example 3.6 gives that there is another one class of operators which preserve weak supermajorization relation on $\ell^1(I)^+$, denoted by $\mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$. These operators are defined by

$$B_h(f) := h \sum_{i \in I} f(i), \quad \forall f \in \ell^1(I),$$

for some $h \in \ell^1(I)^+$. Example 3.7 shows that an arbitrary chosen sum of two operators from two above defined classes $\mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $\mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$ in not a linear preserver of the weak supermajorization in general. Theorem 3.8 gives sufficient conditions

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \ \forall i_2 \in I_2, \ \forall f \in \ell^1(I)^+, \tag{1.9}$$

that the sum of two operators $T = A + B$ represents a weak supermajorization preserver, when $I$ is an infinite set, where $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$. The aim of the rest of the paper is to prove the opposite, that is, every weak supermajorization preserver $T$ on $\ell^1(I)^+$ may be uniquely decomposed as sum of two operators $T = A + B$ which satisfies (1.9), where $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$. In order to provide the last result presented in Theorem 3.14, we characterize "rows" and "columns" of linear preservers. Namely, Lemma 3.10 shows that an arbitrary chosen "row" can not contain two distinct strictly positive elements, so we conclude that ether the "row" contains exactly one nonzero element or all elements in this "row" are mutually equal, by Lemma 3.11. Theorem 3.12 presents the necessary and sufficient conditions under which an operator belongs in first class of preservers $\mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$. As consequence of all provided results, Theorem 3.15 gives the necessary and sufficient conditions under which a bounded linear operator on $\ell^1(I)$ is a weak supermajorization preserver.

Corollary 3.16 shows that positive linear preservers of all three the most common majorization relations: standard majorization on $\ell^1(I)$, weak majorization and weak supermajorization on positive cone $\ell^1(I)^+$, are the same, when $I$ is an infinite set. On the other hand, this corollary is the infinite-dimensional version of [17, Theorem 2.1], provided by Hasani and Vali.

**2. Notations and preliminaries.** Let $I$ be an arbitrary non-empty set. The function $f : I \longrightarrow \mathbb{R}$ is summable if there exists a real number $\sigma$ with the following property: Given $\epsilon > 0$, we can find a finite set $J_0 \subseteq I$ such that

$$\left| \sigma - \sum_{j \in J} f(j) \right| \leq \epsilon$$

whenever $J$ is a finite set and $J_0 \subseteq J$. Then $\sigma$ is called the sum of $f$ and we denote it by $\sigma = \sum_{i \in I} f(i)$.

We denote by $\ell^1(I)$ the Banach space of all functions $f : I \longrightarrow \mathbb{R}$ such that $\|f\|_1 := \sum_{i \in I} |f(i)| < \infty$. Each function $f \in \ell^1(I)$ has a representation

$$f = \sum_{i \in I} f(i) e_i.$$

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Functions $e_i : I \longrightarrow \mathbb{R}$ are defined by Kronecker delta, i.e., $e_i(j) = \delta_{ij}$, $i, j \in I$.

For each $g \in \ell^\infty(I)$, where $\ell^\infty(I)$ is the Banach space with supremum norm, the rule $f \longrightarrow \langle f, g\rangle := \sum\limits_{i \in I} f(i)g(i)$ defines a bounded linear functional on $\ell^1(I)$. This map $\langle \cdot, \cdot \rangle : \ell^1(I) \times \ell^\infty(I) \longrightarrow \mathbb{R}$ is called the dual pairing.

Weak supermajorization relation is defined on the cone of positive functions which we denote by

$$\ell^1(I)^+ := \left\{ f \in \ell^1(I) : f(i) \geq 0, \forall i \in I \right\}.$$

Sometimes we will analyze two sets $I_f^0$ and $I_f^+$ as subsets of $I$ defined by

$$I_f^0 := \{i \in I : f(i) = 0\},$$

$$I_f^+ := \{i \in I : f(i) > 0\},$$

for any $f \in \ell^1(I)$.

Let $A : \ell^1(I) \longrightarrow \ell^1(I)$ be a bounded linear operator, where $I$ is a non-empty set. The operator $A$ is called:

- *positive*, if $Ag \in \ell^1(I)^+$, for each $g \in \ell^1(I)^+$;
- *a permutation*, if there exists a bijection $\theta : I \longrightarrow I$ for which $Ae_j = e_{\theta(j)}$, for each $j \in I$.

The sets of all permutations on $\ell^1(I)$ is denoted by $P(\ell^1(I))$, respectively.

Weak supermajorization relation $\prec^{ws}$ on $\ell^1(I)^+$ may be considered as partial order [25, Corollary 4.5], if we identify all function which are different up to the permutation. The last result is provided using the next very important theorem.

THEOREM 2.1. [25, Theorem 4.4] *For $f, g \in \ell^1(I)^+$, the following conditions are equivalent:*

*i) $f \prec^{ws} g$ and $g \prec^{ws} f$.*
*ii) There exists a permutation $P \in P(\ell^1(I))$ such that $f = Pg$.*

LEMMA 2.2. [24, Lemma 3.1] *Let $u = \{u_j\} \in \mathbb{R}^n$ and let $\{u_{ij} \,|\, i \in I_0,\, j = 1, \ldots, n\}$ be a family of real numbers, where $I_0$ is at most a countable set. If*

$$\sum_{j=1}^{n} \alpha_j u_j \in \left\{ \sum_{j=1}^{n} \alpha_j u_{ij} \,\middle|\, i \in I_0 \right\}, \tag{2.10}$$

*for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_j > 0$ for each $j = 1, \ldots, n$, then there exists $k \in I_0$ such that $u_j = u_{kj}$, for each $j = 1, \ldots, n$.*

**3. Linear preservers of weak supermajorization on $\ell^1(I)^+$.** Firstly, we give the definition of the linear preserver of the weak supermajorization on $\ell^1(I)^+$.

DEFINITION 3.1. A bounded linear operator $T : \ell^1(I) \to \ell^1(I)$ is called linear preserver of the weak supermajorization on $\ell^1(I)^+$, if $T$ preserves the weak supermajorization relation, that is, $Tf \prec^{ws} Tg$, whenever $f \prec^{ws} g$, where $f, g \in \ell^1(I)^+$. The set of all linear preservers of the weak supermajorization on $\ell^1(I)^+$ is denoted by $\mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$.

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The first result gives basic properties of linear preservers of weak supermajorization.

LEMMA 3.2. *Let* $\lambda \in \mathbb{R}, \lambda \geq 0$. *Then*

i) $f \prec g$ *implies* $f \prec^{ws} g$, *for every* $f, g \in \ell^1(I)^+$;
ii) $\lambda K \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$, *for each* $K \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$;
iii) $K_1K_2 \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$, *for each* $K_1, K_2 \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$;
iv) *if* $K \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$, *then* $Ke_j(i) \geq 0$, $\forall i, j \in I$.

*Proof.* Statement $i)$ is straightforward.

Let $K \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$ and let $\lambda \in \mathbb{R}$, $\lambda \geq 0$. Then $f \prec^{ws} g$ implies $Kf \prec^{ws} Kg$, i.e., there is $D \in DSPS(\ell^1(I))$ such that $Kf = DKg$. Since $\lambda Kf = \lambda DKg = D(\lambda Kg)$, we get

$$(\lambda K)f \prec^{ws} (\lambda K)g,$$

so $\lambda K \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$. Further, relation $f \prec^{ws} g$ implies $K_2 f \prec^{ws} K_2 g$, which implies $K_1K_2 f \prec^{ws} K_1K_2 g$, thus $K_1K_2 \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$.

To prove $iv)$, we suppose contrary that there is at least one pair $i_0, j_0 \in I$ such that for the preserver $K$ we have $\langle Ke_{j_0}, e_{i_0}\rangle < 0$. Since $e_{j_0} \prec^{ws} e_k$ does not imply $Ke_{j_0} \prec^{ws} Ke_k$ because $Ke_{j_0} \notin \ell^1(I)^+$, hence we get that $K$ is not a linear preserver which is impossible. Thus, $iv)$ holds. ∎

In the sequel, will consider the bounded linear operator $P_\theta : \ell^1(I) \to \ell^1(I)$ defined by

$$P_\theta(f) := \sum_{j \in I} f(j) e_{\theta(j)}, \tag{3.11}$$

for every $f = \sum_{j \in I} f(j) e_j \in \ell^1(I)$, where $\theta : I \to I$ is a one-to-one function. If $\theta$ is surjection, it is easy to see that $P$ is a permutation. Clearly, $\|P\| = 1$.

Our first result shows that for a doubly superstochastic operator $Q$ on $\ell^1(I)$ and for a family of operators $P_\theta$ defined by (3.11), which are determined by one-to-one functions $\theta$ with mutually disjoint images, there is at least one doubly superstochastic operator $D$ such that $DP_\theta = P_\theta Q$, for every $\theta$. Using this result, we will find the sufficient conditions that an arbitrary bounded linear operator on $\ell^1(I)$ is a preserver of weak supermajorization, when $I$ is an infinite set. In the second part of this paper, we provide that they are actually necessary and sufficient conditions.

THEOREM 3.3. *Let* $Q \in DSPS(\ell^1(I))$. *Suppose that*

$$\Theta := \{\theta_j : I \xrightarrow{1-1} I \mid j \in I_0, \ \ \theta_i(I) \cap \theta_j(I) = \emptyset, \ \ i \neq j\} \tag{3.12}$$

*is a family of one-to-one maps on* $I$ *with mutually disjoint images, where* $I_0$ *is at most a countable set. Then, there is at least one* $D \in DSPS(\ell^1(I))$ *such that* $P_\theta Q = DP_\theta$, $\forall \theta \in \Theta$.

*Proof.* Let $D = \{d_{ij} \mid i, j \in I\}$ be a family defined by

$$d_{ij} := \begin{cases} \langle Qe_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I), \text{ for some } \ \theta \in \Theta, \\ b, & i, j \notin \bigcup_{\theta \in \Theta} \theta(I) \text{ and } j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{3.13}$$

where $b \geq 1$ is arbitrary chosen.

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The family $D$ satisfies conditions (1.6) and (1.7). More precisely, if $j \in \theta(I)$ is arbitrary chosen, for some $\theta \in \Theta$, then using definition (3.13), we get

$$\begin{aligned}\sum_{i\in I}|d_{ij}| &= \sum_{i\in\theta(I)}|d_{ij}| + \sum_{i\in I\setminus\theta(I)}|d_{ij}|\\ &= \sum_{i\in\theta(I)}\langle Qe_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle \le \sup_{j\in I}\sum_{i\in I}|q_{ij}| < \infty,\end{aligned}$$

because operator $Q$ satisfies conditions (1.6) and (1.7). If $j \in I \setminus \bigcup_{\theta\in\Theta}\theta(I)$ then $\sum_{i\in I}|d_{ij}| = b \ge 1$, thus it follows that

$$\sup_{j\in I}\sum_{i\in I}|d_{ij}| < \infty.$$

Similarly, we may conclude that

$$\sup_{i\in I}\sum_{j\in I}|d_{ij}| < \infty.$$

Thus, the family $D$ may be considered as bounded linear operator on $\ell^p(I)$ for each $p \in [1,\infty]$, defined by (1.8), by Corollary 1.1. Using (3.13) we obtain that

$$\sum_{j\in I} d_{ij} \ge 1, \quad \forall i \in I$$

and

$$\sum_{i\in I} d_{ij} \ge 1, \quad \forall j \in I.$$

Thus, $D$ is row and column superstochastic, by Definition 1.2.

We claim that $D$ is a doubly superstochastic. We have to show that there is a doubly stochastic operator $\widetilde{D} \in DS(\ell^1(I))$ such that $d_{ij} \ge \langle \widetilde{D}e_j, e_i\rangle$, $\forall i,j \in I$. Since $Q \in DSPS(\ell^1(I))$, hence there is an operator $\widetilde{Q} \in DS(\ell^1(I))$ with $\langle Qe_j, e_i\rangle \ge \langle \widetilde{Q}e_j, e_i\rangle$, $\forall i,j \in I$.

Similarly as above, we define a family $\widetilde{D} = \{\widetilde{d}_{ij} \mid i,j \in I\}$ to be

$$\widetilde{d}_{ij} := \begin{cases} \langle \widetilde{Q}e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i,j \in \theta(I), \text{ for some } \theta \in \Theta,\\ 1, & i,j \notin \bigcup_{\theta\in\Theta}\theta(I) \text{ and } j = i,\\ 0, & \text{otherwise,}\end{cases} \tag{3.14}$$

Obviously, using Corollary 1.1, we get that the family $\widetilde{D}$ defines a doubly stochastic operator on $\ell^1(I)$ defined by (1.8), because it is easy to see that

$$(\forall i \in I)\ \sum_{j\in I}\widetilde{d}_{ij} = 1 \quad \text{and} \quad (\forall j \in I)\ \sum_{i\in I}\widetilde{d}_{ij} = 1.$$

Next, if $i,j \in \theta(I)$ for some $\theta \in \Theta$, then by above definitions (3.13) and (3.14) we have

$$d_{ij} = \langle Qe_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle \ge \langle \widetilde{Q}e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle = \widetilde{d}_{ij},$$

and

$$d_{ii} = b \ge 1 = \widetilde{d}_{ii},$$

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when $i \notin \bigcup_{\theta\in\Theta} \theta(I)$. Thus, $D \in DSPS(\ell^1(I))$.

We will show that $P_\theta Q = DP_\theta$, $\forall\theta \in \Theta$. Choose an arbitrary function $\theta \in \Theta$. We get

$$DP_\theta(e_j) = D(e_{\theta(j)}) = \sum_{l\in\theta(I)} d_{l,\theta(j)} e_l + \sum_{l\notin\theta(I)} d_{l,\theta(j)} e_l.$$

Using the definition of the operator $D$, we have that $d_{l,\theta(j)} = 0$, when $l \notin \theta(I)$, so we conclude that

$$DP_\theta(e_j) = \sum_{l\in\theta(I)} d_{l,\theta(j)} e_l = \sum_{l\in\theta(I)} \langle Qe_j, e_{\theta^{-1}(l)}\rangle e_l = \sum_{i\in I} Qe_j(i) e_{\theta(i)}. \tag{3.15}$$

Further, using $Q(e_j) = \sum\limits_{i\in I} Qe_j(i)e_i$, we obtain

$$P_\theta Q(e_j) = \sum_{i\in I} Qe_j(i) P_\theta(e_i) = \sum_{i\in I} Qe_j(i) e_{\theta(i)}. \tag{3.16}$$

Combining (3.15) and (3.16), we get $DP_\theta(e_j) = P_\theta Q(e_j)$, $\forall j \in I$. Then

$$\begin{aligned} DP_\theta(f) &= DP_\theta\left(\sum_{j\in I} f(j)e_j\right) = \left(\sum_{j\in I} f(j) DP_\theta(e_j)\right) \\ &= \left(\sum_{j\in I} f(j) P_\theta Q(e_j)\right) = P_\theta Q(f), \end{aligned}$$

for arbitrary choosen function $f = \sum\limits_{j\in I} f(j)e_j \in \ell^1(I)$. ◻

If $f \prec^{ws} g$, that is $f = Qg$, for some $Q \in DSPS(\ell^1(I))$, then we get $P_\theta f = P_\theta Q g = DP_\theta g$, for some $D \in DSPS(\ell^1(I))$, by Theorem 3.3. It follows that $P_\theta f \prec^{ws} P_\theta g$. Thus, $P_\theta \in \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$. In particular, it follows that $P(\ell^1(I)) \subset \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$.

Theorem 3.4. *Let $I_0$ be at most countable subset of an infinite set $I$. Suppose that $\Theta$ is a family of one-to-one maps on $I$ with disjoint images, defined by (3.12). If $\lambda \in \ell^1(I_0)^+$, then*

$$T := \sum_{j\in I_0} \lambda_j P_{\theta_j} \in \mathcal{M}^{ws}_{pr}(\ell^1(I)^+). \tag{3.17}$$

*Proof.* Let $T = \sum\limits_{j\in I_0} \lambda_j P_{\theta_j}$ be a linear operator on $\ell^1(I)$. Firstly we show that $T$ is bounded. Since the family $\Theta$ is contains only functions with disjoint images, we get

$$\|Tf\| = \sum_{j\in I_0}\sum_{i\in I} (\lambda_j f(i)) = \|f\| \sum_{j\in I_0} \lambda_j = \|\lambda\|\|f\|.$$

We conclude that $T$ is a bounded linear operator on $\ell^1(I)$ with norm $\|T\| = \|\lambda\|$. Suppose that $f \prec^{ws} g$. It follows that $f = Qg$ for some $Q \in DSPS(\ell^1(I))$. There is an operator $D \in DSPS(\ell^1(I))$ such that

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$P_\theta Q = DP_\theta$, for each $\theta \in \Theta$, by Theorem 3.3. Now, using linearity and continuity of operator $D$, we obtain

$$\begin{aligned} Tf &= \sum_{j\in I_0} \lambda_j P_{\theta_j}(f) = \sum_{j\in I_0} \lambda_j P_{\theta_j}(Qg) = \sum_{j\in I_0} \lambda_j DP_{\theta_j}(g) \\ &= D\left(\sum_{j\in I_0} \lambda_j P_{\theta_j}(g)\right) = D(Tg). \end{aligned} \tag{3.18}$$

which implies that $Tf \prec^{ws} Tg$. $\square$

Example 3.5. Let $1 < m \in \mathbb{N}$ and let $\Theta := \{\theta_1, \theta_2, \ldots, \theta_m\}$ be a family of one-to-one maps $\theta_n : \mathbb{N} \to \mathbb{N}$, defined by

$$\theta_n(k) = m^k + n, \quad \forall k \in \mathbb{N}, \ \ n = 1, 2, \ldots, m.$$

We define the operator $T := \sum\limits_{n=1}^{m} \frac{1}{n^2} P_{\theta_n}$. Using the definition of the family $\Theta$, we can represent $T$ in the following way:

$$Tf = \left[\underbrace{0,\ldots,0}_{m\text{-times}}, \underbrace{f(1), \frac{f(1)}{2^2}, \ldots, \frac{f(1)}{m^2}}_{\text{m-times}}, \underbrace{0,\ldots,0}_{\substack{m^2-2m\\ \text{times}}}, \underbrace{f(2), \frac{f(2)}{2^2}, \ldots, \frac{f(2)}{m^2}}_{\text{m-times}}, \underbrace{0,\ldots,0}_{\substack{m^3-m^2-m\\ \text{times}}}, \underbrace{f(3), \frac{f(3)}{2^2}, \ldots, \frac{f(3)}{m^2}}_{\text{m-times}}, \ldots\right]^T$$

We get that $T \in \mathcal{M}^{ws}_{pr}(\ell^1(\mathbb{N})^+)$, by Theorem 3.4. Moreover, we conclude that $T$ is bounded

$$\|Tf\| = \sum_{n=1}^{m}\sum_{k=1}^{\infty} |\frac{1}{n^2} f(k)| \le \sum_{n=1}^{m} \frac{1}{n^2}\|f\| \le \|f\| \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\|f\|$$

for every $f \in \ell^1(I)$.

Previously, we showed that operators defined by (3.17) are linear preservers of weak supermajorization on $\ell^1(I)^+$, by Theorem 3.4. The set of all such operators we will denote by $\mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$. However, we can find linear preservers of weak supermajorization relation which do not have form (3.17), that is, which are not contained in $\mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$.

Example 3.6. We define an operator $B_h$ to be

$$B_h(f) := h \sum_{i\in I} f(i), \quad \forall f \in \ell^1(I). \tag{3.19}$$

where $h \in \ell^1(I)^+$ is arbitrary fixed. Obviously, $B_h$ is a bounded linear operator with norm $\|B_h\| = \|h\|$.

Suppose that $f \prec^{ws} g$, where $f, g \in \ell^1(I)^+$, that is, $f = Dg$, for some $D \in DSPS(\ell^1(I))$. Using the definition of operator $D$, we have that there is an operator $\widetilde{D} \in DS(\ell^1(I))$ such that $De_j(i) \ge \widetilde{D}e_j(i)$, so changing the order of summation, we get

$$\begin{aligned} \|f\| &= \sum_{i\in I} f(i) = \sum_{i\in I}\sum_{j\in I} g(j) De_j(i) \\ &\ge \sum_{i\in I}\sum_{j\in I} g(j)\widetilde{D}e_j(i) = \sum_{j\in I} g(j) \sum_{i\in I} \widetilde{D}e_j(i) = \|g\|. \end{aligned} \tag{3.20}$$

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Thus, $B_h(f) = h\|f\| \geq h\|g\| = B_h(g)$. Let $\alpha = \frac{\|f\|}{\|g\|} \geq 1$. Let $Q := \alpha\mathcal{I} \in DSPS(\ell^1(I))$, where $\mathcal{I}$ stands for the identity operator. Since

$$QB_h(g) = \|g\|Qh = \alpha\|g\|h = \|f\|h = B_h(f),$$

hence $B_h(f) \prec^{ws} B_h(g)$, that is, $B_h \in \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$.

The set of all linear preservers of weak supermajorization on $\ell^1(I)^+$ introduced in (3.19) we will denote by $\mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$. Obviously, $\mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ is a convex cone and $\mathcal{A}^{ws}_{pr}(\ell^1(I)^+) \bigcap \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ contains only null operator. Also, the sum of two operators from two different classes $\mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $\mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ is not a linear preserver in general, which is presented in the next example.

Example 3.7. Fix $j, k \in I$ such that $j \neq k$. Suppose that operator $B_{e_k}$ is defined by (3.19). Thus, "columns" $B_{e_k}e_r$, $r \in I$ of the operator $B_{e_k}$ are mutually equal and they are actually equal with function $e_k$, so we have $B_{e_k}(e_j) = e_k$. Let $P \in P(\ell^1(I)) \subset \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ be a permutation which satisfies

$$P(e_j) = e_k.$$

We will show that $P + B_{e_k} \notin \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$. Clearly, $e_j \prec^{ws} e_k$. On the other hand,

$$(P + B_{e_k})(e_k) = e_i + e_k,$$

where have to be $i \neq k$, and

$$(P + B_{e_k})(e_j) = 2e_k.$$

We claim that

$$(P + B_{e_k})(e_j) = 2e_k \not\prec^{ws} e_i + e_k = (P + B_{e_k})(e_k).$$

Suppose contrary that there is an operator $D \in DSPS(\ell^1(I))$ such that $D(e_i + e_k) = 2e_k$. It follows that $De_i(k) + De_k(k) = 2$. Since $e_k(t) = 0$, for each $t \neq k$, hence we get

$$De_i(t) = De_k(t) = 0, \quad \text{for every } t \in I \setminus \{k\}. \tag{3.21}$$

Since the operator $D$ is column superstochastic, it has to be

$$De_i(k) = De_k(k) = 1. \tag{3.22}$$

However, using (3.21) and (3.22) we conclude that there is no $\widetilde{D} \in DS(\ell^1(I))$ such that $De_j(i) \geq \widetilde{D}e_j(i)$, $\forall i, j \in I$, so we have a contradiction, thus $D \notin DSPS(\ell^1(I))$, so $P + B_k \notin \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$.

The next result gives sufficient conditions that the sum of two arbitrary chosen bounded linear operators form two disjoint classes $\mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $\mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ is a linear preserver of weak supermajorization on $\ell^1(I)^+$.

Theorem 3.8. *Let $I$ be an infinite set. If $A \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ are chosen to be*

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \ \forall i_2 \in I_2, \ \forall f \in \ell^1(I)^+, \tag{3.23}$$

*where $I_1, I_2 \subset I$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$, then $A + B \in \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$.*

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*Proof.* Choose an arbitrary operators $A \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$, such that (3.23) holds. It follows that the operator $A$ has the following form:

$$A = \sum_{j \in I_0} \lambda_j P_{\theta_j},$$

where family $\Theta$ is presented in (3.12). Without lose of generality, we suppose that $\lambda_j > 0$, $\forall j \in I_0$. Now,

$$\begin{aligned} Af(i_2) &= \sum_{i \in I} f(i) A e_i(i_2) = \sum_{i \in I} f(i) \sum_{j \in I_0} \lambda_j P_{\theta_j} e_i(i_2) \\ &= \sum_{i \in I} f(i) \sum_{j \in I_0} \lambda_j e_{\theta_j(i)}(i_2), \quad \forall f \in \ell^1(I). \end{aligned}$$

Using theorem assumption $Af(i_2) = 0$, we get that

$$e_{\theta_j(i)}(i_2) = \langle e_{\theta_j(i)}, e_{i_2} \rangle = 0, \quad \forall i \in I, \ \forall j \in I_0, \ \forall i_2 \in I_2,$$

so

$$\left( \bigcup_{j \in I_0} \theta_j(I) \right) \cap I_2 = \emptyset.$$

Suppose that $f \prec^{ws} g$, for some $f, g \in \ell^1(I)^+$. It follows that $f = Qg$, where $Q \in DSPS(\ell^1(I))$. There is at least one operator $D \in DSPS(\ell^1(I))$ such that $P_{\theta_j} Q = D P_{\theta_j}$, $\forall j \in I_0$ by Theorem 3.3, and $AQ = DA$, by (3.18) . Actually, the operator $D$ is not unique, which is obvious by its definition:

$$d_{ij} := \begin{cases} \langle Q e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)} \rangle, & i, j \in \theta(I), \text{ for some } \ \theta \in \Theta, \\ b, & i, j \notin \bigcup_{\theta \in \Theta} \theta(I) \text{ and } j = i, \\ 0, & \text{otherwise}, \end{cases}$$

(see proof of Theorem 3.3). Using (3.20) and choosing $b := \frac{\|f\|}{\|g\|} \geq 1$, we get

$$BQg = Bf = h \sum_{j \in I} f(j) = bh \sum_{j \in I} g(j) = bBg,$$

for some $h \in \ell^1(I)^+$. Also $De_j = be_j$, $\forall j \in I_2$, by definition of $D$, so using (3.23) we obtain

$$DB\tilde{f} = \sum_{j \in I_2} B\tilde{f}(j) De_j = bB\tilde{f}$$

for every $\tilde{f} \in \ell^1(I)^+$. Now, we conclude

$$\begin{aligned} D(A+B)g &= DAg + DBg = DAg + bBg = AQg + BQg \\ &= (A+B)Qg = (A+B)f. \end{aligned}$$

It follows that $(A+B)f \prec^{ws} (A+B)g$, thus $A + B \in \mathcal{M}^{ws}_{pr}(\ell^1(I)^+)$. ∎

In the rest of the paper, our aim is to prove the opposite direction of the last theorem, that is, to prove that every linear preserver of weak supermajorization on $\ell^1(I)^+$ may be represent by the sum of two operators from classes $A \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$, which satisfies conditions (3.23). For this purpose, we need the following results.

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LEMMA 3.9. *Let $T:\ell^1(I)\to\ell^1(I)$ be a linear preserver of weak supermajorization on $\ell^1(I)^+$. Suppose that $J$ is a finite subset of $I$ and let $\delta: J\to J$ be a bijection. For every $u\in I$ there exists $v\in I$ such that*

$$Te_m(u)=Te_{\delta(m)}(v),\quad \forall m\in J. \tag{3.24}$$

*Proof.* Let $card(J)=n\in\mathbb{N}$. Since $T\in\mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$, hence

$$\sum_{m\in J}a_me_m\prec^{ws}\sum_{m\in J}a_me_{\delta(m)}\quad\text{and}\quad\sum_{m\in J}a_me_{\delta(m)}\prec^{ws}\sum_{m\in J}a_me_m$$

implies

$$\sum_{m\in J}a_mTe_m\prec^{ws}\sum_{m\in J}a_mTe_{\delta(m)}\quad\text{and}\quad\sum_{m\in J}a_mTe_{\delta(m)}\prec^{ws}\sum_{m\in J}a_mTe_m$$

for every $a=(a_{m_1},a_{m_2},\dots,a_{m_n})$, where $a_{m_j}>0$ for each $m_j$. Now, using Theorem 2.1, we obtain that functions $h:=\sum\limits_{m\in J}a_mTe_m$ and $h_\delta:=\sum\limits_{m\in J}a_mTe_{\delta(m)}$ are different up to the permutation, that is,

$$\sum_{m\in J}a_mTe_m(u)\in\left\{\sum_{m\in J}a_mTe_{\delta(m)}(k)\mid k\in I_0\right\},$$

where $I_0:=I_{h_\delta}^+\cup\{r\}$ and $r\in I_{h_\delta}^0$. Since $I_{h_\delta}^+$ is a countable set, hence using Lemma 2.2, we get that there is a $v\in I$ such that (3.24) holds. ∎

LEMMA 3.10. *Let $T:\ell^1(I)\to\ell^1(I)$ be a linear preserver of weak supermajorization on $\ell^1(I)^+$, where $I$ is an infinite set. If there are two distinct $k,l\in I$ such that $Te_k(i)>0$ and $Te_l(i)>0$, for some $i\in I$, then $Te_k(i)=Te_l(i)$.*

*Proof.* Let $\tilde{k}:=Te_k(i)>0$ and $\tilde{l}:=Te_l(i)>0$ and suppose contrary, $\tilde{k}\neq\tilde{l}$. Let

$$\mathcal{K}:=\{i\in I:Te_k(i)=\tilde{k}\}, \tag{3.25}$$

$$\mathcal{L}:=\{i\in I:Te_k(i)=\tilde{l}\}.$$

Clearly, $\mathcal{K}\subset I$ is a non-empty set. Since $Te_k\in\ell^1(I)^+$, hence

$$card(\mathcal{K})<\aleph_0\quad\text{and}\quad card(\mathcal{L})<\aleph_0. \tag{3.26}$$

Let $a_1,a_2>0$ and let $m\in I\setminus\{k,l\}$ be arbitrary chosen. Clearly,

$$a_1e_k+a_2e_l\prec^{ws}a_1e_k+a_2e_m\quad\text{and}\quad a_1e_k+a_2e_m\prec^{ws}a_1e_k+a_2e_l.$$

Since $T$ is a linear preserver of weak supermajorization, we get

$$a_1Te_k+a_2Te_l\prec^{ws}a_1Te_k+a_2Te_m\quad\text{and}\quad a_1Te_k+a_2Te_m\prec^{ws}a_1Te_k+a_2Te_l.$$

Using Theorem 2.1, we obtain

$$a_1Te_k(i)+a_2Te_l(i)\in\{a_1Te_k(j)+a_2Te_m(j)\mid j\in I\}.$$

Since the above set is at most countable because $Te_k,Te_m\in\ell^1(I)^+$, hence using Lemma 2.2, we get that there is $j\in I$ such that $Te_k(i)=Te_k(j)=\tilde{k}$ and $Te_l(i)=Te_m(j)=\tilde{l}$. It follows that $j\in\mathcal{K}$. Since

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$m \in I$ is arbitrary chosen and $card(I) > card(\mathcal{K})$, hence there is $s \in \mathcal{K}$ and for this $s$ there is a sequence $(m_j)_{j\in\mathbb{N}}$ of distinct elements $m_j \in I$ such that $Te_{m_j}(s) = \tilde{l} > 0$, $\forall j \in \mathbb{N}$.

We define the a family $\{\Phi_j\}_{j\in\mathbb{N}}$ where $\Phi_j := \{m_1, m_2, \dots m_j\}$ for every $j \in \mathbb{N}$. Also, we form bijections $\phi_j : \Phi_j \cup \{k\} \to \Phi_j \cup \{k\}$ in the following way:

$$\phi_j(x) := \begin{cases} m_j, & x = k, \\ k, & x = m_j \\ x, & x \in \Phi_j \setminus \{m_j\}. \end{cases}$$

Now, using Lemma 3.9 we can find $s_j \in I$ such that

$$Te_{m_j}(s_j) = Te_{\phi_j(k)}(s_j) = Te_k(s) = \tilde{k}, \tag{3.27}$$

$$Te_k(s_j) = Te_{\phi_j(m_j)}(s_j) = Te_{m_j}(s) = \tilde{l} \tag{3.28}$$

and

$$Te_x(s_j) = Te_{\phi_j(x)}(s_j) = Te_x(s) = \tilde{l}, \quad \forall x \in \Phi_j \setminus \{m_j\}. \tag{3.29}$$

Suppose that there are distinct $a, b \in \mathbb{N}$ such that $s_a = s_b$. Without lose of generality, suppose that $a < b$. Now, using above expression (3.27), since $\phi_a(k) = m_a$ we get that

$$Te_{m_a}(s_a) = \tilde{k}.$$

However, since $b > a$, hence $m_a \in \Phi_b \setminus \{m_b\}$ so using $\phi_b(m_a) = m_a$ and (3.29), we get

$$Te_{m_a}(s_b) = \tilde{l}.$$

Using above facts and the assumption $s_a = s_b$, we conclude

$$\tilde{l} = Te_{m_a}(s_b) = Te_{m_a}(s_a) = \tilde{k},$$

which is contradiction with $\tilde{k} \neq \tilde{l}$, so it has to be $s_a \neq s_b$, whenever $a \neq b$. Since $s_j \in \mathcal{L}$, $\forall j \in \mathbb{N}$ by (3.28), hence $\mathcal{L}$ is an infinite set, which is a contradiction with (3.26). ∎

Further, we prove that if there are two strictly positive elements in one "row" of linear preserver of weak supermajorization on $\ell^1(I)^+$, when $I$ is an infinite set, then all elements in this "row" are the same.

Lemma 3.11. *Let $T : \ell^1(I) \to \ell^1(I)$ be a linear preserver of weak supermajorization on $\ell^1(I)^+$, where $I$ is an infinite set. If there are two distinct $k, l \in I$ such that $Te_k(i) > 0$ and $Te_l(i) > 0$, for some $i \in I$, then the set $\{Te_j(i) \mid j \in I\}$ is a singleton.*

*Proof.* Firstly, it easy to see that if $Te_j(i) > 0$, for some $j \in I$, then $Te_j(i) = Te_k(i) = Te_l(i)$, by Lemma 3.10.

Suppose contrary that there is a $m \in I$ such that $Te_m(i) = 0$. Let

$$M := \{j \in I \,|\, Te_k(j) = Te_l(j) = \tilde{k}\},$$

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where $\tilde{k} := Te_k(i) = Te_l(i)$. Obviously, the set $M$ is a finite nonempty set. We chose an arbitrary $n \in I \setminus \{k, l, m\}$, and define a bijection

$$\delta_n(x) := \begin{cases} k, & x = k, \\ l, & x = l, \\ n, & x = m, \\ m, & x = n. \end{cases}$$

Now, by Lemma 3.9, there is $i_n \in I$ such that

$$Te_n(i_n) = Te_{\delta_n(m)}(i_n) = Te_m(i) = 0,$$

$$Te_k(i_n) = Te_{\delta_n(k)}(i_n) = Te_k(i) = \tilde{k}$$

and

$$Te_l(i_n) = Te_{\delta_n(l)}(i_n) = Te_l(i) = \tilde{k}.$$

Now, it is easy to conclude that $i_n \in M \subset I$. Since $card(M) < \aleph_0$, we get that there exists $s \in M$ and for this $s$ there is a sequence of distinct elements $(m_j)_{j\in\mathbb{N}}$ such that $Te_{m_j}(s) = 0$, $\forall j \in \mathbb{N}$. Similarly as in the proof of the last lemma, we define bijections

$$\phi_j : \Phi_j \cup \{k, l\} \to \Phi_j \cup \{k, l\},$$

correspond to sets $\Phi_j := \{m_1, m_2, \dots m_j\}$, $j \in \mathbb{N}$, defined by

$$\phi_j(x) := \begin{cases} k, & x = k, \\ l, & x = m_j, \\ m_j & x = l, \\ x, & x \in \Phi_j \setminus \{m_j\}. \end{cases}$$

Again using Lemma 3.9 for each $j \in I$, we can find $s_j \in I$ such that

$$Te_k(s_j) = Te_{\phi_j(k)}(s_j) = Te_k(s) = \tilde{k},$$

$$Te_{m_j}(s_j) = Te_{\phi_j(l)}(s_j) = Te_l(s) = \tilde{k}, \tag{3.30}$$

$$Te_l(s_j) = Te_{\phi_j(m_j)}(s_j) = Te_{m_j}(s) = 0$$

and

$$Te_x(s_j) = Te_{\phi_j(x)}(s_j) = Te_x(s) = 0, \quad \forall x \in \Phi_j \setminus \{m_j\}. \tag{3.31}$$

If we assume that there exist integers $a < b$ such that $s_a = s_b$, then using bijection $\phi_a$, we get

$$Te_{m_a}(s_a) = \tilde{k}, \quad \text{by (3.30)},$$

and using bijection $\phi_b$ we obtain

$$Te_{m_a}(s_a) = Te_{m_a}(s_b) = 0, \quad \text{by (3.31)},$$

which is contradiction with $\tilde{k} > 0$. Thus, $s_a \neq s_b$, whenever $a \neq b$. If we define set $\mathcal{K}$ as in (3.25), we get that $s_j \in \mathcal{K}$, $\forall j \in I$, which implies that $\mathcal{K}$ is an infinite set, which is impossible by (3.26). ∎

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In the next theorem, we present necessary and sufficient conditions under which $T \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$.

THEOREM 3.12. *Let $A : \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where $I$ is an infinite set. Then, $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ if and only if $Ae_j \prec^{ws} Ae_k$ and $Ae_k \prec^{ws} Ae_j$, $\forall k, j \in I$, and for each $i \in I$ there is at most one $j \in I$ such that $Ae_j(i) > 0$.*

*Proof.* Firstly, suppose that operator $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ is defined by (3.17). Since $A$ is the supermajorization preserver by Theorem 3.4, we get that $e_j \prec^{ws} e_k$ and $e_k \prec^{ws} e_j$ implies $Ae_j \prec^{ws} Ae_k$ and $Ae_k \prec^{ws} Ae_j$, $\forall k, j \in I$. Since the family $\Theta$ defined by (3.12) contains maps with disjoint images, hence if $s \notin \bigcup_{j\in I_0}(\theta_j(I))$, then we get $P_{\theta_j} e_l(s) = 0$, $\forall j \in I_0$, thus $Ae_l(s) = 0$, for every $l \in I$. If $s \in \bigcup_{j\in I_0}(\theta_j(I))$, then there is exactly one ordered pair $(j_s, r_s)$, where $j_s \in I_0$ and $r_s \in I$, such that $\theta_{j_s}(r_s) = s$, and $\theta_{j_0}(r_0) \neq s$ for each pair $(j_0, r_0)$ with $(j_0, r_0) \neq (j_s, r_s)$. Hence,

$$\begin{aligned} Ae_r(s) &= \sum_{j\in I_0} \lambda_j P_{\theta_j} e_r(s) \\ &= \sum_{j\in I_0} \lambda_j e_{\theta_j(r)}(s) = \lambda_{j_s} e_{\theta_{j_s}(r)}(s) = 0, \end{aligned}$$

when $r \neq r_s$. Thus, each "row" contains at most one non-zero element, so the second part is valid.

Let $A : \ell^1(I) \to \ell^1(I)$ be a bounded linear operator. If $A := 0$ then $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$, obviously. Let $A \neq 0$. It follows that there is $k, l \in I$ such that $Ae_k(l) > 0$ which implies that $Ae_j \neq 0$ for every $j$, using the theorem assumptions that "columns" of the operator $A$ is mutually weakly supermajorized so they are different up to the permutation by Theorem 2.1. More precisely, there exist permutations $P_j \in P(\ell^1(I))$ corresponding to bijections $\omega_j : I \longrightarrow I$ such that

$$P_j A e_k = A e_j,$$

for every $j \in I$.

We define a family $\Theta$ of maps $\theta_j$, $j \in I_0$ defined by $\theta_j(i) = \omega_i(j)$, $\forall i \in I$, that is,

$$\Theta := \{\theta_j : I \to I \mid j \in I_0\},$$

where $I_0 := I^+_{Ae_k}$, is at most a countable set. Clearly, $\theta_i$ are one-to-one maps. To show that maps $\theta_i$ have mutually disjoint images $\theta_i(I)$, assume that for some $a, b \in I_0$, $a \neq b$ there exist $j_a, j_b \in I$ such that $i_0 := \theta_a(j_a) = \theta_b(j_b)$, so $\omega_{j_a}(a) = \omega_{j_b}(b)$. Since $a, b \in I^+_{Ae_k}$, it is $Ae_k(a) > 0$ and $Ae_k(b) > 0$. Also,

$$Ae_{j_a}(i_0) = \langle Ae_{j_a}, e_{\omega_{j_a}(a)}\rangle = \langle Ae_{j_a}, P_{j_a} e_a\rangle = \langle P_{j_a}^{-1} Ae_{j_a}, e_a\rangle = Ae_k(a) > 0.$$

Similarly,

$$Ae_{j_b}(i_0) = Ae_k(b) > 0,$$

which is a contradiction with theorem's assumptions.

We claim that operator $A$ has the form (3.17). If we set $\lambda_i := Ae_k(i)$, $\forall i \in I_0$, and fixing $g = \sum\limits_{j\in I} g(j)e_j \in$

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$\ell^1(I)^+$, then we obtain

$$
\begin{aligned}
Ag &= \sum_{j\in I} g(j)Ae_j = \sum_{i\in I} g(j)P_jAe_k \\
&= \sum_{j\in I} g(j)\left(\sum_{i\in I_0} Ae_k(i)P_je_i\right) \\
&= \sum_{j\in I} g(j)\sum_{i\in I_0}\lambda_i e_{\omega_j(i)} = \sum_{j\in I} g(j)\sum_{i\in I_0}\lambda_i e_{\theta_i(j)}. \qquad (3.32)
\end{aligned}
$$

Further,

$$
P_{\theta_i}(g) = \sum_{j\in I} g(j)P_{\theta_i}e_j = \sum_{j\in I} g(j)e_{\theta_i(j)}, \qquad (3.33)
$$

by (3.11).

There exists a finite set $J_0 \subset I_0$ such that for each finite set $\widetilde{I_0} \supset J_0$, we have that

$$
\sum_{j\in I_0\setminus\widetilde{I_0}} \lambda_j \le \epsilon,
$$

where $\epsilon > 0$ is arbitrarily chosen, so combining (3.32) and (3.33), we conclude

$$
\begin{aligned}
\left\|Ag - \sum_{i\in\widetilde{I_0}}\lambda_iP_{\theta_i}(g)\right\| &= \left\|\sum_{j\in I} g(j)\sum_{i\in I_0\setminus\widetilde{I_0}}\lambda_i e_{\theta_i(j)}\right\| \\
&= \sum_{j\in I}\sum_{i\in I_0\setminus\widetilde{I_0}} g(j)\lambda_i \\
&= \|g\|\sum_{i\in I_0\setminus\widetilde{I_0}}\lambda_i \le \epsilon\|g\|,
\end{aligned}
$$

so we get $A = \sum_{i\in I_0}\lambda_iP_{\theta_i}$. ∎

Example 3.13. Let $k \in \mathbb{N}\setminus\{1\}$ and let $\theta_1, \theta_k : \mathbb{N}\to\mathbb{N}$ be one-to-one functions defined by

$$
\theta_1(j) = kj \ \text{ and } \ \theta_k(j) = k^j, \quad \forall j\in\mathbb{N},
$$

for some $k \in \mathbb{N}\setminus\{1\}$, and suppose that $F(f) := P_{\theta_1}(f) + P_{\theta_k}(f)$.

Now, $P_{\theta_1}(e_k) = e_{\theta_1(k)} = e_{k^2}$ and $P_{\theta_k}(e_2) = e_{\theta_k(2)} = e_{k^2}$, so

$$
F(e_2)(k^2) = \langle F(e_2), e_{k^2}\rangle = \langle e_{2k} + e_{k^2}, e_{k^2}\rangle = 1,
$$

and

$$
F(e_k)(k^2) = \langle F(e_k), e_{k^2}\rangle = \langle e_{k^2} + e_{k^k}, e_{k^2}\rangle = 1.
$$

Suppose that $i \in \mathbb{N}\setminus\{2,k\}$. Now, $\theta_1(e_i) \ne k^2$ and $\theta_k(e_i) \ne k^2$, so $\langle F(e_i), e_{k^2}\rangle = 0$, which implies that $F \notin \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$, by Theorem 3.12.

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The above example shows that $\mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ is neither a vector space nor a convex cone, so the same holds for $\mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$.

Now, we may prove the most important result in this paper that arbitrary chosen linear preserver of weak supermajorization may be decomposed as sum of two unique operators defined by (3.17) and (3.19).

THEOREM 3.14. *Let $I$ be an infinite set. If $T \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$ then there are unique operators $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$ such that $T = A + B$. Moreover, these operators $A$ and $B$ satisfy condition* (3.23).

*Proof.* Let $T \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$. We define two sets $I_1, I_2 \subset I$ such that $I_1$ contains each $i \in I$ such that $\langle Te_j, e_i\rangle > 0$ for at most one $j \in I$ and $I_2 := I \setminus I_1$. In the other words, using Lemma 3.11, we get that the set $I_2$ contains all $k \in I$ for which $Te_j(k) = c > 0$, $\forall j \in I$.

Now, we define operators $A, B : \ell^1(I) \to \ell^1(I)$ by

$$Af(i) := \begin{cases} Tf(i) & i \in I_1, \\ 0, & i \in I_2 \end{cases}$$

and

$$Bf(i) := \begin{cases} Tf(i) & i \in I_2, \\ 0, & i \in I_1. \end{cases}$$

The operators $A$ and $B$ are bounded linear operators and $A + B = T$, obviously.

Next, we will show that $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$.

Suppose that $I_2 = \emptyset$. It follows that there is no $i \in I$ such that

$$Te_{j_1}(i) > 0 \quad\text{and}\quad Te_{j_2}(i) > 0,$$

and because for a preserver always holds

$$Ae_j \prec^{ws} Ae_k \;\text{ and }\; Ae_k \prec^{ws} Ae_j, \quad \forall j, k \in I,$$

we obtain using Theorem 3.12 that $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$. Obviously, $B = 0 \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$.

Let $I_1, I_2 \neq \emptyset$. Next, we define

$$I_{Te_m}^1 := \left\{j \in I_{Te_m}^+ : Te_m(j) = max\{Te_m(r) : r \in I_{Te_m}^+\}\right\}$$

and

$$I_{Te_m}^k := \left\{ j \in I_{Te_m}^+ : Te_m(j) = max\left\{ Te_m(r) : r \in I_{Te_m}^+ \setminus \bigcup_{i=1}^{k-1} I_{Te_m}^i \right\}\right\}$$

when $k \geq 2$. Fix $m, n \in I$. Since $T \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$, hence $Te_m \prec^{ws} Te_n$ and $Te_n \prec^{ws} Te_m$, so functions $Te_n$ and $Te_m$ are different up to the permutation that is, there is a permutation $P \in P(\ell^1(I))$ corresponding to a bijection $\omega : I \to I$ with

$$\langle Te_m, e_i\rangle = \langle Te_n, e_{\omega(i)}\rangle, \quad \forall i \in I,$$

by Theorem 2.1. Since

$$card(I_{Te_m}^k) = card(I_{Te_n}^k) \;\text{ and }\; card(I_{Te_m}^0) = card(I_{Te_n}^0), \quad \forall k \in \mathbb{N},$$

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hence, it is easy to conclude that bijection $\omega$ is determined by bijections

$$\begin{aligned}\omega_0 : I^0_{Te_m} &\longrightarrow I^0_{Te_n}, \text{ if } I^0_{Te_m} \neq \emptyset,\\ \omega_k : I^k_{Te_m} &\longrightarrow I^k_{Te_n}, \text{ if } I^k_{Te_m} \neq \emptyset, \ \ k \in \mathbb{N},\end{aligned}$$

in the following way:

$$\omega(i) := \begin{cases} \omega_k(i), & i \in I^k_{Te_m},\\ \omega_0(i), & i \in I^0_{Te_m}.\end{cases}$$

Because for each $i \in I_2$, $Te_m(i) = Te_n(i)$, hence

$$card(I^k_{Te_m} \setminus I_2) = card(I^k_{Te_n} \setminus I_2), \ \ \forall k \in \mathbb{N},$$

so we may define bijections

$$\widetilde{\omega}_k : I^k_{Te_m} \setminus I_2 \longrightarrow I^k_{Te_n} \setminus I_2, \ \text{ if } \ I^k_{Te_m} \setminus I_2 \neq \emptyset, \ \ k \in \mathbb{N}$$

by

$$\widetilde{\omega}_k(i) = \omega_k(i), \quad \forall i \in I^k_{Te_m} \setminus I_2.$$

We form a bijection

$$\widetilde{\omega} : I \to I$$

defined by

$$\widetilde{\omega}(i) := \begin{cases} \widetilde{\omega}_k(i), & i \in I^k_{Te_m} \setminus I_2,\\ \omega_0(i), & i \in I^0_{Te_m},\\ i, & i \in I_2.\end{cases}$$

It follows that the permutation $\widetilde{P} \in P(\ell^1(I))$, which correspond to the bijection $\widetilde{\omega}$ defined by $Pe_i = e_{\widetilde{\omega}(i)}$, $\forall i \in I$ satisfies $PAe_m = Ae_n$. It follows that $Ae_m \prec^{ws} Ae_n$ and $Ae_n \prec^{ws} Ae_m$, $\forall m, n \in I$, so $A \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ by Theorem 3.12.

To show that $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$, firstly we get

$$\begin{aligned}Be_m &= \sum_{i\in I} Be_m(i)e_i = \sum_{i\in I_1} Be_m(i)e_i + \sum_{i\in I_2} Be_m(i)e_i\\ &= \sum_{i\in I_2} Be_n(i)e_i = \sum_{i\in I} Be_n(i)e_i = Be_n,\end{aligned} \tag{3.34}$$

for fixed $m, n \in I$. Using (3.34) and defining $h := Be_r$ for some $r \in I$, we obtain

$$Bf = B\left(\sum_{j\in I} f(j)e_j\right) = h\left(\sum_{j\in I} f(j)\right), \tag{3.35}$$

thus $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$.

If $I_1 = \emptyset$, then using statements (3.34) and (3.35), when $I_2 = I$, we obtain that $B = T \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ and $A = 0 \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$.

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Assume that there is another one pair $A_1, B_1$ such that $T = A_1 + B_1$, where $A_1 \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B_1 \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$. We get $A - A_1 = B_1 - B$. For operators $B$ and $B_1$ we know that $Be_m = Be_n$ and $B_1e_m = B_1e_n$, $\forall m, n \in I$. On the other hand, using Theorem 3.12, since for each $i \in I$, there is at most one $s \in I$ such that $Ae_s(i) > 0$, hence there is at least one $j_s \in I$ such that $Ae_{j_s}(i) = A_1e_{j_s}(i) = 0$. Using above arguments, we get

$$0 = (A - A_1)e_{j_s}(i) = (B_1 - B)e_{j_s}(i) = (B_1 - B)e_j(i), \quad \forall j \in I,$$

and thus, $B = B_1$ and $A = A_1$. $\square$

Now, it is clear why the sum of two operators $B_{e_k}$ and $P$, which satisfies $P(e_j) = e_k$, from Example 3.7, is not a linear preserver of weak supermajorization. Precisely, for $k \in I$, we have $P(e_j)(k) = 1 = B_{e_k}e_j(k)$, which is not possible for preservers of weak supermajorization on $\ell^1(I)^+$ by above theorem.

All results provided above are collected below.

THEOREM 3.15. *Let $T : \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where $I$ is an infinite set. The following statements are equivalent:*

*i)* $T \in \mathcal{M}_{pr}^{ws}(\ell^1(I)^+)$.
*ii) There are operators $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$ and disjoint sets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$ such that $T = A + B$ where $A, B$ are chosen to be*

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \ \ \forall i_2 \in I_2, \ \ \forall f \in \ell^1(I)^+.$$

*iii) There is an at most a countable set $I_0 \subset I$ and there is a family*

$$\Theta := \{\theta_j : I \xrightarrow{1-1} I \mid j \in I_0, \ \ \theta_i(I) \cap \theta_j(I) = \emptyset, \ \ i \neq j\}$$

*of one-to-one maps, $\theta_j \in \Theta$, $\forall j \in I_0$, and $(\lambda_j)_{j\in I_0} \in \ell^1(I_0)^+$ such that*

$$T = \sum_{j\in I_0} \lambda_j P_{\theta_j} + B_h,$$

*where $B_h(f) := h\sum_{i\in I_0} f(i)$, for $h \in \ell^1(I)^+$ with $h(i) = 0$, $\forall i \in \theta_j(I)$, for each $j \in I_0$.*
*iv) $Te_j \prec^{ws} Te_k$ and $Te_k \prec^{ws} Te_j$, $\forall k, j \in I$, and for each $i \in I$, either there exists exactly one $j \in I$ with $Te_j(i) > 0$ or the set $\{Te_j(i) \mid j \in I\}$ is a singleton.*

*Proof.* We have $i) \to iv) \to ii)$, by Lemma 3.11 and Theorem 3.14. Also, Statement $ii)$ implies $i)$ by Theorem 3.8. Statements $iii)$ and $iv)$ are equivalent by Theorem 3.12. $\square$

Weak and standard majorization relations and their linear preservers on $\ell^1(I)$ are studied in [6, 22, 23, 24]. Linear preservers of majorization, weak majorization and weak supermajorization on $\ell^1(I)^+$ are the same, if we consider only positive operators.

COROLLARY 3.16. *Let $I$ be an infinite set. Suppose that $T : \ell^1(I) \to \ell^1(I)$ is a positive bounded linear operator. The following statements are equivalent:*

*i) $T$ is a linear preserver of majorization relation ($\prec$);*
*ii) $T$ is a linear preserver of weak majorization relation ($\prec_w$);*
*iii) $T$ is a linear preserver of weak supermajorization relation ($\prec^{ws}$).*

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*Proof.* Statements i) and ii) are equivalent by [24, Corollary 3.1]. Statements i) is equivalent with iii) by [6, Proposition 5.9] and by Theorem 3.15. ◻

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