Electronic Journal of Linear Algebra

Volume 33 *Volume 33: Special Issue for the International Conference on Matrix Analysis and its Applications, MAT TRIAD 2017* Article 13

2018

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Recommended Citation
Hladík, Milan. (2018), "AE Regularity of Interval Matrices", *Electronic Journal of Linear Algebra*, Volume 33, pp. 137-146.
DOI: https://doi.org/10.13001/1081-3810.3705

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 33, pp. 137-146, Janurary 2019.
http://repository.uwyo.edu/ela

AE REGULARITY OF INTERVAL MATRICES*

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Abstract. Consider a linear system of equations with interval coefficients, and each interval coefficient is associated with either a universal or an existential quantifier. The AE solution set and AE solvability of the system is defined by ∀∃-quantification. The paper deals with the problem of what properties must the coefficient matrix have in order that there is guaranteed an existence of an AE solution. Based on this motivation, a concept of AE regularity is introduced, which implies that the AE solution set is nonempty and the system is AE solvable for every right-hand side. A characterization of AE regularity is discussed, and also various classes of matrices that are implicitly AE regular are investigated. Some of these classes are polynomially decidable, and therefore give an efficient way for checking AE regularity. Eventually, there are also stated open problems related to computational complexity and characterization of AE regularity.

Key words. Interval computation, Quantified systems, Linear equations, Interval systems.

AMS subject classifications. 65G40, 15Bxx, 65F30

1. Introduction.

Solving systems of interval linear equations is a basic problem of interval computation [21, 22]. In the last decade, there was a particular interest in the so called AE solutions defined by ∀∃ quantification of interval parameters. AE solutions were studied not only for interval systems [4, 6, 17, 24, 25, 32], but also in the context of interval linear programming [7, 8, 16, 18, 19]. Both of these areas motivate us to introduce a novel concept of the so called AE regularity. In the context of interval linear systems, it is the property that guarantees AE solvability in a similar manner as a real nonsingular matrix guarantees solvability of linear equations. In the context of interval linear programming, it helps in identifying stable optimal basis and AE optimal solutions.

An interval matrix is defined as

$$\boldsymbol{A} := \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$ and $\overline{A}$, $\underline{A} \le \overline{A}$, are given matrices, and the inequality between matrices is understood entrywise. The corresponding midpoint and the radius matrices are defined respectively as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $m \times n$ interval matrices is denoted by $\mathbb{IR}^{m\times n}$, and intervals and interval vectors are considered as special cases of interval matrices.

Consider an interval system of linear equations $\boldsymbol{A}x = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Its solutions set is traditionally defined as the union of all solutions of realizations of interval coefficients, that is

$$\{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A},\ \exists b \in \boldsymbol{b} : Ax = b\}.$$

*Received by the editors on January 18, 2018. Accepted for publication on January 7, 2019. Handling Editor: Natalia Bebiano. Corresponding Author: Milan Hladík. This work was funded by the Czech Science Foundation Grant P403-18-04735S and also supported by Charles University project UNCE/SCI/004.

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We say that $\boldsymbol{A}$ is *regular* if every $A \in \boldsymbol{A}$ is nonsingular. Regularity of $\boldsymbol{A}$ implies that each system realization has a unique solution and the overall solution set is a bounded polyhedron. Checking whether an interval matrix is regular, however, is a co-NP-hard problem [2, 11, 23]. A survey of forty necessary and sufficient condition for regularity were summarized by Rohn [30].

Next, we say that some property $\mathcal{P}$ holds strongly (weakly) for an interval matrix $\boldsymbol{A}$ if it holds for every (some) matrix $A \in \boldsymbol{A}$.

Let us now consider a more general concept by using $\forall\exists$ quantification of interval parameters. Each interval entry of $\boldsymbol{A}$ and $\boldsymbol{b}$ is associated either with the universal, or with the existential quantifier. Thus, we can disjointly split the interval matrix as $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$, where $\boldsymbol{A}^{\forall}$ is the interval matrix comprising universally quantified coefficients, and $\boldsymbol{A}^{\exists}$ concerns existentially quantified coefficients. Similarly, we decompose the right-hand side vector $\boldsymbol{b} = \boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists}$. Now, $x \in \mathbb{R}^n$ is called an *AE solution* [32] if

$$\forall A^{\forall} \in \boldsymbol{A}^{\forall}, \forall b^{\forall} \in \boldsymbol{b}^{\forall}, \exists A^{\exists} \in \boldsymbol{A}^{\exists}, \exists b^{\exists} \in \boldsymbol{b}^{\exists} : (A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}.$$

The interval system $\boldsymbol{A}x = \boldsymbol{b}$ is called *AE solvable* [6] if for each realization of $\forall$-parameters there are realizations of $\exists$-parameters such that the resulting system has a solution. Formally, it is AE solvable if

$$\forall A^{\forall} \in \boldsymbol{A}^{\forall}, \forall b^{\forall} \in \boldsymbol{b}^{\forall}, \exists A^{\exists} \in \boldsymbol{A}^{\exists}, \exists b^{\exists} \in \boldsymbol{b}^{\exists} : (A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists} \text{ is solvable.}$$

Obviously, if the interval system has an AE solution, then it is AE solvable, but the converse implication does not hold in general [6].

Related to AE solvability, we introduce the following natural concept of regularity.

DEFINITION 1.1. *An interval matrix $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$ is called* AE regular *if $\forall A^{\forall} \in \boldsymbol{A}^{\forall}, \exists A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $A = A^{\forall} + A^{\exists}$ is nonsingular.*

AE regularity generalizes regularity of $\boldsymbol{A}$ (which is the case when there are no $\exists$- parameters), so it is also co-NP-hard to check this property. It is also a question how to characterize AE regularity. In the following section, we will approach to this problem and show several classes of matrices that are implicitly AE regular.

REMARK 1.1. *Another kind of motivation comes from the real eigenvalue interval problem [9, 10, 13, 27]. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and define the real eigenvalue set as*

$$\Lambda := \{\lambda \in \mathbb{R};\ \exists A \in \boldsymbol{A} : \det(A - \lambda I_n) = 0\}.$$

Let $\boldsymbol{\lambda} \in \mathbb{IR}$. Then $\boldsymbol{\lambda}$ includes at least one eigenvalue from each $A \in \boldsymbol{A}$ if and only if

$$\forall A \in \boldsymbol{A}\ \exists \lambda \in \boldsymbol{\lambda} : A - \lambda I_n \text{ is singular.}$$

Similarly, the whole interval $\boldsymbol{\lambda}$ is included in the eigenvalue set, that is, $\boldsymbol{\lambda} \subseteq \Lambda$, if and only if

$$\forall \lambda \in \boldsymbol{\lambda}\ \exists A \in \boldsymbol{A} : A - \lambda I_n \text{ is singular.}$$

We see that eigenvalue inner and outer estimates relate to $\forall\exists$ quantification and regularity or singularity of interval matrices. Notice that the problem is more complicated by the fact that the same parameter λ concerns all the diagonal entries. This is called a linear dependency and makes the problem yet more difficult.

REMARK 1.2. *The problem of AE regularity becomes even more complicated if we allow both quantifiers in some matrix entries, that is, the split* $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$ *is not disjoint.*

In this case, we can reduce both matrices $\boldsymbol{A}^{\forall}$ *and* $\boldsymbol{A}^{\exists}$ *to matrices* $\tilde{\boldsymbol{A}}^{\forall}$ *and* $\tilde{\boldsymbol{A}}^{\exists}$ *such that* $\boldsymbol{A} = \tilde{\boldsymbol{A}}^{\forall} + \tilde{\boldsymbol{A}}^{\exists}$ *is a disjoint splitting. Let us show it on an example of one interval* $\boldsymbol{a} = \boldsymbol{a}^{\forall} + \boldsymbol{a}^{\exists}$*. Suppose that* $a^{\forall}_{\Delta} \le a^{\exists}_{\Delta}$*. Then define* $\tilde{\boldsymbol{a}}^{\forall} := 0$ *and* $\tilde{\boldsymbol{a}}^{\exists} := [\overline{a}^{\forall} + \underline{a}^{\exists}, \underline{a}^{\forall} + \overline{a}^{\exists}]$*, which is a proper interval. Now, if there exists a required* $\tilde{a}^{\exists} \in \tilde{\boldsymbol{a}}^{\exists}$*, then for every* $a^{\forall} \in \boldsymbol{a}^{\forall}$ *we can take* $a^{\exists} := \tilde{a}^{\exists} - a^{\forall} \in \boldsymbol{a}^{\exists}$ *and we have* $a^{\forall} + a^{\exists} = \tilde{a}^{\exists}$*. Analogously for the case* $a^{\forall}_{\Delta} > a^{\exists}_{\Delta}$*. Thus AE regularity of* $\tilde{\boldsymbol{A}}^{\forall} + \tilde{\boldsymbol{A}}^{\exists}$ *implies AE regularity of* $\boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$*.*

Notice that this reduction is not equivalent. Consider for example $\boldsymbol{A}^{\forall} = [-1, 1]$ *and* $\boldsymbol{A}^{\exists} = [-1, 1]$*. Then* $\tilde{\boldsymbol{A}}^{\forall} + \tilde{\boldsymbol{A}}^{\exists} = 0$ *is not AE regular, but the original matrices obviously are.*

2. Strong singularity. An $n \times n$ interval matrix in the form $\boldsymbol{A} = 0 + \boldsymbol{A}^{\exists}$, that is, with no $\forall$-quantified interval parameters, is AE regular if and only if there is at least one nonsingular matrix in $\boldsymbol{A}$. The negation of this property is an interval matrix containing only singular matrices. Even though this property is not the typical case, it may happen. If we want to characterize AE regularity for the general case, we have to inspect also this particular situation.

Recall that an interval matrix $\boldsymbol{A}$ is *strongly singular* if every $A \in \boldsymbol{A}$ is singular. In the following, we use the term *a vertex* matrix of $\boldsymbol{A}$, which is any matrix $A \in \boldsymbol{A}$ such that $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$ for all i, j.

THEOREM 2.1. $\boldsymbol{A}$ *is strongly singular if and only if each vertex matrix is singular.*

Proof. $\boldsymbol{A}$ is strongly singular if and only if and only if $\det(A) = 0$ for every $A \in \boldsymbol{A}$. Due to linearity of the determinant with respect to the i, jth entry, the largest and the smallest values of the determinants are attained for vertex matrices. ∎

Next, we show a couple of properties of strongly singular interval matrices.

PROPOSITION 2.2. *If* $\boldsymbol{A}$ *is strongly singular, then* $[-A_{\Delta}, A_{\Delta}]$ *is strongly singular and in particular* A_{Δ} *is singular.*

Proof. Let $A \in \boldsymbol{A}$ be arbitrarily chosen, let $A^1 \in \boldsymbol{A}$ be a matrix that results from A by replacing the first row by $\overline{A}_{1*}$, and let $\tilde{A}^1$ be a matrix that results from A by replacing the first row by $(A_{\Delta})_{1*}$. Since $\det(A^1) = \det(A) = 0$ and by row linearity of determinants, the matrix $\tilde{A}^1$ is singular. Therefore the interval matrix $\tilde{\boldsymbol{A}}$ that results from $\boldsymbol{A}$ by replacing the first row by $(A_{\Delta})_{1*}$ is strongly singular. Proceeding similarly for the next rows of $\tilde{\boldsymbol{A}}$, we arrive at singularity of A_{Δ}.

Replacing $\overline{A}$ by any other matrix $A \in \boldsymbol{A}$ in the above considerations, we obtain singularity of any matrix in $[-A_{\Delta}, A_{\Delta}]$. ∎

PROPOSITION 2.3. *If* $\boldsymbol{A}$ *is strongly singular, then it remains strongly singular even if we replace any of its intervals with positive width by any other interval or a real number.*

Proof. Let i, j be arbitrary such that $(A_{\Delta})_{ij} > 0$. Denote by $\boldsymbol{A}^{ij}$ the matrix $\boldsymbol{A}$ after removing the ith row and the jth column. By the Laplace expansion of the determinant of $A \in \boldsymbol{A}$ by ith row we see that $\det(A^{ij}) = 0$ for any $A^{ij} \in \boldsymbol{A}^{ij}$, whence $\boldsymbol{A}^{ij}$ must be strongly singular. Therefore, the determinant of A remains zero even if we replace $\boldsymbol{A}_{ij}$ by any other interval or a real number. ∎

PROPOSITION 2.4. *If* $\boldsymbol{A}$ *is strongly singular, then the rows and columns of* A_{Δ} *can be permuted in such a way that we get a block upper triangular matrix of type* $\left(\begin{smallmatrix} B & C \\ 0_{k,\ell} & D \end{smallmatrix}\right)$*, where the sum of the number of rows and columns of the zero submatrix is in satisfies* $k + \ell > n$*.*

In the following, we will use $A_{[i:j]}$ to denote the principal submatrix of A indexed by $i, i+1, \dots, j$.

Proof. Since $\boldsymbol{A}$ is strongly singular, for each permutation p there is $i \in \{1, \dots, n\}$ such that $(A_\Delta)_{i,p(i)} = 0$. This is easy to see since otherwise we can choose a nonsingular $A \in [-A_\Delta, A_\Delta]$ by putting $A_{i,p(i)} := (A_\Delta)_{i,p(i)}$ and zero otherwise.

We will proceed by induction for those matrices satisfying the above mentioned property. If the last row of A_Δ is zero, we are done. So suppose without loss of generality that $(A_\Delta)_{nn} > 0$. By induction applied on $(A_\Delta)_{[1:(n-1)]}$, there is a permutation of rows and columns that brings $(A_\Delta)_{[1:(n-1)]}$ into the demanded form. By moving the last row into the first row, the resulting matrix A'_Δ now is block upper triangular. Suppose that left top block has size k. If there is a permutation p on $\{1, \dots, k\}$ such that $(A'_\Delta)_{i,p(i)} > 0$ for every $i = 1, \dots, k$, then apply the induction on $(A'_\Delta)_{[(k+1):n]}$; otherwise apply it on $(A'_\Delta)_{[1:k]}$. □

Matrices of type $A_{yz} = A_c - \operatorname{diag}(y) A_\Delta \operatorname{diag}(z)$, where $y, z \in \{\pm 1\}^n$ and $\operatorname{diag}(y)$ denotes the diagonal matrix with y on the diagonal, are often used in verifying various properties of interval matrices such as positive definiteness, and in some sense also regularity [2]. Notice however that strong singularity cannot be checked by inspecting only these matrices as the following counterexample shows. The matrix

$$\boldsymbol{A} = \begin{pmatrix} [-1,1] & [-1,1] \\ [-1,1] & [-1,1] \end{pmatrix}$$

is not strongly singular, but all matrices of type A_{yz} are singular.

We leave two important open questions here:

- Is there a simpler (computationally cheaper) characterization of strong singularity?
- What is the computational complexity of checking strong singularity. Is it a polynomial or an NP-hard problem?

As an open problem we also state the following conjecture. The "if" part is obvious, but the converse is the open and hard one.

CONJECTURE 1. *$\boldsymbol{A}$ is strongly singular if and only if it has a submatrix of size $k \times \ell$ that is real and has the rank of $k + \ell - n - 1$.*

3. AE regularity.

Now, we consider AE regularity in the general form. We first show that this property is useful for AE solvability of interval systems.

THEOREM 3.1. *If $\boldsymbol{A}$ is AE regular, then $\boldsymbol{A}x = \boldsymbol{b}$ is AE solvable for each $\boldsymbol{b}$. The converse is not true in general.*

Proof. For each ∀-realization, we find ∃-realization such that A is nonsingular, whence solvability of $Ax = b$ follows.

The counter-example for the converse direction is

$$\boldsymbol{A}^\forall = 0, \quad \boldsymbol{A}^\exists = \begin{pmatrix} 0 & [-1,1] \\ 0 & [-1,1] \end{pmatrix}.$$

Then $\boldsymbol{A}$ is not AE regular, but $\boldsymbol{A}x = \boldsymbol{b}$ is AE solvable for each $\boldsymbol{b}$. □

Obviously, AE regularity implies nonemptiness of the AE solution set. On the other hand, AE regularity does not imply boundedness of the AE solution set; counter-example: $\boldsymbol{A}^\forall = 0$, $\boldsymbol{A}^\exists = ([-1,1])$, $\boldsymbol{b} = 0$.

In order to characterize AE regularity, we utilize the following master interval linear system with linear dependencies given by multiple appearance of $A^{\forall}$

$$(A^{\forall} + A_v^{\exists})x_v = 0, \quad [-e, e]^T x_v = 1, \quad v \in V, \tag{3.1}$$

where $A^{\forall} \in \boldsymbol{A}^{\forall}$ and $A_v^{\exists}$, $v \in V$, are all vertex matrices of $\boldsymbol{A}^{\exists}$.

PROPOSITION 3.2. *$\boldsymbol{A}$ is not AE regular if and only if (3.1) is solvable for some $A^{\forall} \in \boldsymbol{A}^{\forall}$.*

Proof. $\boldsymbol{A}$ is not AE regular if and only if there is $A^{\forall} \in \boldsymbol{A}^{\forall}$ such that $A^{\forall} + \boldsymbol{A}^{\exists}$ strongly singular. By Theorem 2.1, this is equivalent to the condition that all matrices $A^{\forall} + A_v^{\exists}$, $v \in V$, are singular. The system (3.1) then formulates singularity of these matrices. ∎

By the above theorem, we reduced AE regularity to solvability of a parametric system. Parametric systems are hard to solve and characterize, even for particular case; see [5, 20, 24]. This suggests that also AE solvability is a very hard problem in general.

4. Special classes. This section presents several classes of interval matrices that are inherently AE regular.

4.1. M-matrix. A real matrix $A \in \mathbb{R}^{n \times n}$ is an M-matrix if the off-diagonal entries are non-positive and there is $x > 0$ such that $Ax > 0$. Interval M-matrices in the strong sense (i.e., with $\forall$-quantification) were investigated, e.g., in [1, 22]. We first discuss $\exists$-quantified version, and then extend it to the general $\forall\exists$ case.

We say that $\boldsymbol{A}$ is weakly an M-matrix if there is $A \in \boldsymbol{A}$ being an M-matrix. Weak M-matrices are characterized as follows.

PROPOSITION 4.1. *Define $\tilde{A} \in \boldsymbol{A}$ as follows*

$$\tilde{a}_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } i = j, \\ \arg\min\{|a_{ij}|;\ a_{ij} \in \boldsymbol{a}_{ij}\} & \text{if } i \neq j. \end{cases} \tag{4.2}$$

Then $\boldsymbol{A}$ is weakly an M-matrix if and only if $\tilde{A}$ is an M-matrix.

Proof. "If." This is obvious as $\tilde{A} \in \boldsymbol{A}$.

"Only if." Let $A \in \boldsymbol{A}$ be an M-matrix. Then there is a vector $v > 0$ such that $Av > 0$. Further, from $a_{ij} \leq 0$ for $i \neq j$ we have that $A \leq \tilde{A}$ and $\tilde{a}_{ij} \leq 0$ for $i \neq j$. Eventually, from $\tilde{A}v \geq Av > 0$ it follows that $\tilde{A}$ is an M-matrix, too. ∎

We say that $\boldsymbol{A}^{\forall\exists}$ is an AE M-matrix if $\forall A^{\forall} \in \boldsymbol{A}^{\forall}\, \exists A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $A^{\forall} + A^{\exists}$ is an M-matrix. Obviously, an AE M-matrix is AE regular.

THEOREM 4.2. *Denote by $\tilde{A} = \underline{A}^{\forall} + \tilde{A}^{\exists}$ the matrix from (4.2) corresponding to the interval matrix $\underline{A}^{\forall} + \boldsymbol{A}^{\exists}$. Then $\boldsymbol{A}^{\forall\exists}$ is an AE M-matrix if and only if $\tilde{A}$ is an M-matrix and $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} \leq 0$ for all $i \neq j$.*

Proof. "If." If $\tilde{A}$ is an M-matrix, then there is a vector $v > 0$ such that $\tilde{A}v > 0$. Hence for every $A^{\forall} \in \boldsymbol{A}^{\forall}$ we can take $\tilde{A}^{\exists}$, and for the matrix $A = A^{\forall} + \tilde{A}^{\exists}$ we have $Av \geq \tilde{A}v > 0$. Since $A_{ij} \leq (\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} \leq 0$, this part is proved.

"Only if." By the assumption, $\underline{A}^{\forall} + \boldsymbol{A}^{\exists}$ is weakly an M-matrix, and hence by Proposition 4.1, $\tilde{A}$ is an M-matrix. The condition $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} \leq 0$ for all $i \neq j$ holds also from the assumption since from the disjunction of interval parameters we have either $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} = \overline{A}^{\forall}_{ij}$ or $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} = \tilde{A}^{\exists}_{ij}$ and for both cases it is true. ∎

4.2. H-matrix. A matrix $A \in \mathbb{R}^{n\times n}$ is called an H-matrix, if the so called comparison matrix $\langle A\rangle$ is an M-matrix, where $\langle A\rangle_{ii} = |a_{ii}|$ and $\langle A\rangle_{ij} = -|a_{ij}|$ for $i \neq j$. Interval H-matrices (corresponding to $\forall$-quantification) were investigated, e.g., in [1, 22]. We again first discuss $\exists$-quantified version, which will be then extended to the general $\forall\exists$ case.

We say that $\boldsymbol{A}$ is weakly an H-matrix if there is $A \in \boldsymbol{A}$ being an H-matrix. This class of matrices is characterized in the following theorem. Recall that for an interval $\boldsymbol{a} \in \mathbb{IR}$ its magnitude and mignitude are respectively defined as

$$\operatorname{mag}(\boldsymbol{a}) = \max\{|a|;\, a \in \boldsymbol{a}\} = |a_c| + a_\Delta,$$

$$\operatorname{mig}(\boldsymbol{a}) = \min\{|a|;\, a \in \boldsymbol{a}\} = \begin{cases} 0 & \text{if } 0 \in \boldsymbol{a}, \\ \min(|\underline{a}|, |\overline{a}|) & \text{otherwise.} \end{cases}$$

Proposition 4.3. *Define $\tilde{A} \in \boldsymbol{A}$ as follows*

$$\tilde{a}_{ij} = \begin{cases} \operatorname{mag}(\boldsymbol{a}_{ij}) & \text{if } i = j, \\ -\operatorname{mig}(\boldsymbol{a}_{ij}) & \text{if } i \neq j. \end{cases} \tag{4.3}$$

Then $\boldsymbol{A}$ is weakly an H-matrix if and only if $\tilde{A}$ is an M-matrix.

Proof. "If." This is obvious as the matrix $A \in \boldsymbol{A}$ corresponding to $\tilde{A}$ (i.e., the entries of A are attained as magnitudes and mignitudes in (4.3)) is an H-matrix.

"Only if." Let $A \in \boldsymbol{A}$ be an H-matrix. Then $\langle A\rangle$ is an M-matrix, and there is a vector $v > 0$ such that $\langle A\rangle v > 0$. Therefore $\tilde{A}v \geq \langle A\rangle v > 0$. ∎

We say that $\boldsymbol{A}^{\forall\exists}$ is an AE H-matrix if $\forall A^{\forall} \in \boldsymbol{A}^{\forall}\ \exists A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $A^{\forall} + A^{\exists}$ is an H-matrix. Obviously, an AE H-matrix is AE regular.

Theorem 4.4. *Define the matrix $\tilde{A}$ as follows*

$$\tilde{a}_{ij} = \begin{cases} \operatorname{mig}(\boldsymbol{a}^{\forall}_{ij}) + \operatorname{mag}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i = j, \\ -\operatorname{mag}(\boldsymbol{a}^{\forall}_{ij}) - \operatorname{mig}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}^{\forall\exists}$ is an AE H-matrix if and only if $\tilde{A}$ is an M-matrix.

Proof. "If." Let $\tilde{A}^{\forall} \in \boldsymbol{A}^{\forall}$ and $\tilde{A}^{\exists} \in \boldsymbol{A}^{\exists}$ be the corresponding matrices, for which $\tilde{A}$ is attained. Then $\tilde{A} = \tilde{A}^{\forall} + \tilde{A}^{\exists}$ since each entry of $\boldsymbol{A}^{\forall\exists}$ is associated with at most one quantifier. Let $A^{\forall} \in \boldsymbol{A}^{\forall}$ be arbitrary and define $A := A^{\forall} + \tilde{A}^{\exists}$. Since $\tilde{A}$ is an M-matrix, there is a vector $v > 0$ such that $\tilde{A}v > 0$. From $\langle A\rangle v \geq \tilde{A}v > 0$ we conclude that A is an H-matrix.

"Only if." By the assumption, there exists $A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $\tilde{A}^{\forall} + A^{\exists}$ is an H-matrix. That is, there is a vector $v > 0$ such that $\langle \tilde{A}^{\forall} + A^{\exists}\rangle v > 0$. From $\tilde{A}v \geq \langle \tilde{A}^{\forall} + A^{\exists}\rangle v > 0$ we have that $\tilde{A}$ is an M-matrix. ∎

It is known that if A_c is an M-matrix, then $\boldsymbol{A}$ is regular if and only if $\boldsymbol{A}$ is strongly H-matrix [22, Prop. 4.1.7]. For generalized quantification, this statement is no longer valid. Consider, for example, the interval matrix

$$\boldsymbol{A}^{\forall\exists} = \begin{pmatrix} [0.8, 1]^{\exists} & -[0, 1]^{\forall} \\ -1 & 1 \end{pmatrix}.$$

Then A_c is an M-matrix and $\boldsymbol{A}^{\forall\exists}$ is AE regular, but it is not an AE H-matrix since

$$\begin{pmatrix} [0.8, 1]^{\exists} & -1 \\ -1 & 1 \end{pmatrix}.$$

is not weakly an H-matrix.

4.3. Inverse nonnegative matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is inverse nonnegative if $A^{-1} \geq 0$. Strong inverse nonnegativity of an interval matrix $\boldsymbol{A}$ was studied in [15, 26, 31], among others. For this class a simple characterization exists since $\boldsymbol{A}$ is inverse nonnegative if and only if $\underline{A}$ and $\overline{A}$ are inverse nonnegative.

In our $\forall\exists$ quantification, we say that $\boldsymbol{A}^{\forall\exists}$ is AE inverse nonnegative if $\forall A^{\forall} \in \boldsymbol{A}^{\forall}$, $\exists A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $A := A^{\forall} + A^{\exists}$ is inverse nonnegative. Obviously, an AE inverse nonnegative matrix is AE regular.

In contrast to $\forall$-quantified case, for the general AE inverse nonnegativity it seems there is no simple characterization. As a sufficient condition obtained by reversing the order of quantifiers, we obtain that $\boldsymbol{A}^{\forall\exists}$ is AE inverse nonnegative if $\exists A^{\exists} \in \boldsymbol{A}^{\exists}$, $\forall A^{\forall} \in \boldsymbol{A}^{\forall}$ the matrix $A^{\forall} + A^{\exists}$ is inverse nonnegative. This can be characterized in the following manner: $\exists A^{\exists} \in \boldsymbol{A}^{\exists}$ such that both matrices $\underline{A}^{\forall} + A^{\exists}$ and $\overline{A}^{\forall} + A^{\exists}$ are inverse nonnegative. How to choose a suitable $A^{\exists} \in \boldsymbol{A}^{\exists}$ is, however, an open question.

4.4. Positive definite matrices. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval matrix such that both $\underline{A}$ and $\overline{A}$ are symmetric. Then the symmetric counterpart to $\boldsymbol{A}$ is defined as the set of all symmetric matrices in $\boldsymbol{A}$:

$$\boldsymbol{A}_S := \{A \in \boldsymbol{A};\ A = A^T\}.$$

Then $\boldsymbol{A}_S$ is called strongly positive definite if every $A \in \boldsymbol{A}_S$ is positive definite (this concept corresponds to $\forall$-quantification). It is known [11, 14, 28] that checking strong positive definiteness of $\boldsymbol{A}_S$ is NP-hard. The finite reduction by Rohn [29] states that $\boldsymbol{A}_S$ is strongly positive definite if and only if the matrix $A_{yy} := A_c - \operatorname{diag}(y) A_{\Delta} \operatorname{diag}(y)$ is positive definite for each $y \in \{\pm 1\}^n$. On the other hand, $\exists$-quantified case is polynomial: Checking, whether $\boldsymbol{A}_S$ contains at least one positive definite matrix can be solved by means of semidefinite programming [12].

We say that $\boldsymbol{A}_S^{\forall\exists}$ is AE positive definite if $\forall A^{\forall} \in \boldsymbol{A}_S^{\forall}$, $\exists A^{\exists} \in \boldsymbol{A}_S^{\exists}$ such that $A := A^{\forall} + A^{\exists}$ is positive definite. AE positive definite matrices thus form another class of AE regular matrices.

Obviously, interchanging the order of quantifiers leads to a stronger condition. Utilizing the above Rohn's characterization of positive definiteness of interval matrices, we obtain the following.

PROPOSITION 4.5. *$\boldsymbol{A}^{\forall\exists}$ is AE positive definite provided there is $A^{\exists} \in \boldsymbol{A}_S^{\exists}$ such that $A^{\exists} + A_{yy}^{\forall}$ is positive definite for all $y \in \{\pm 1\}^n$.*

This condition can be checked by semidefinite programming [3]. In this model, $A^{\exists} \in \boldsymbol{A}_S^{\exists}$ is a linear constraint in variables $A^{\exists}$. Positive definiteness of $A^{\exists} + A_{yy}^{\forall}$ can be checked by maximizing α such that

$A^{\exists} + A^{\forall}_{yy} - \alpha I_n$ is positive semidefinite for each $y \in \{\pm 1\}^n$, where I_n is the identity matrix. The resulting semidefinite program, however, has an exponential size. This motivates us to propose a sufficient condition of polynomial time complexity. By Rohn [29], $\boldsymbol{A}_S$ is strongly positive definite provided $\lambda_{\min}(A_c) > \rho(A_\Delta)$, that is, the minimum eigenvalue of A_c is greater than the spectral radius of A_Δ. Utilizing this sufficient condition, we arrive at the following sufficient condition of AE positive definiteness.

PROPOSITION 4.6. *$\boldsymbol{A}^{\forall\exists}$ is AE positive definite provided there is $A^{\exists} \in \boldsymbol{A}^{\exists}_S$ such that $\lambda_{\min}(A^{\forall}_c + A^{\exists}) > \rho(A^{\forall}_\Delta)$.*

Notice that this condition can be checked efficiently by semidefinite programming since we are looking for $A^{\exists} \in \boldsymbol{A}^{\exists}_S$ such that $A^{\forall}_c + A^{\exists} - \rho(A^{\forall}_\Delta) I_n$ is positive definite. Thus the resulting semidefinite program draws

$$\max\ \alpha \ \text{ subject to } \ A^{\forall}_c + A^{\exists} - (\rho(A^{\forall}_\Delta) + \alpha) I_n \succeq 0,\ A^{\exists} \in \boldsymbol{A}^{\exists}_S.$$

The condition stated in Proposition 4.6 is then satisfied if and only if the optimal value of the semidefinite program is positive.

4.5. Structured quantifiers position. Real matrices are often somehow structured. Interval matrices can have a specific structure of interval parameters in addition. In this section we focus on interval matrices with particular structures.

The following theorem characterizes strong singularity for the case when intervals are situated in one row or one column only. By a suitable permutation of rows and columns, the form of (4.4) can easily be achieved, where $\boldsymbol{b} \in \mathbb{IR}^k$ or c can possibly be empty.

THEOREM 4.7. *The square interval matrix*

$$\boldsymbol{A} = \begin{pmatrix} B & \boldsymbol{b} \\ C & c \end{pmatrix} \tag{4.4}$$

with $b_\Delta > 0$ is strongly singular if and only if $(B^T\ C^T)$ or $(C\ c)$ has not full row rank.

Proof. “If”. Obvious.

“Only if”. Suppose that $(B^T\ C^T)$ has full row rank. Then c has length at least 1 since otherwise we could choose $b \in \boldsymbol{b}$ such that A would be nonsingular. Since $\boldsymbol{A}$ is strongly singular and $(B^T\ C^T)$ has full row rank, the vector (b^T, c^T) is linearly dependent on the rows of $(B^T\ C^T)$ for each $b \in \boldsymbol{b}$. In particular, c^T is linearly dependent on the rows of C^T.

Now, suppose to the contrary that $(C\ c)$ has full row rank. Then also C has full row rank since otherwise c^T wouldn't be linearly dependent on the rows of C^T. Thus we can extend C^T to a nonsingular square submatrix of $(B^T\ C^T)$. Without loss of generality assume that it is the right part of $(B^T\ C^T)$. Consider the Laplace expansion of A of the last column. Then the coefficient by a_{1n} (i.e., by b_1) is nonzero, and therefore by varying $b_1 \in \boldsymbol{b}_1$, the determinant of A cannot be constantly zero. A contradiction with strong singularity of $\boldsymbol{A}$. ∎

The following theorem characterizes AE regularity for the case when $\exists$-intervals are situated in one row or one column only. By a suitable permutation of rows and columns, we can always achieve the form

$$\boldsymbol{A}^{\forall\exists} = \begin{pmatrix} \boldsymbol{B}^{\forall} & \boldsymbol{b}^{\exists} \\ \boldsymbol{C}^{\forall} & \boldsymbol{c}^{\forall} \end{pmatrix} \tag{4.5}$$

with $b^{\exists}_{\Delta} > 0$.

THEOREM 4.8. *The square interval matrix* (4.5) *is AE regular if and only if the matrices* $(\boldsymbol{B}^T\ \boldsymbol{C}^T)$ *and* $(\boldsymbol{C}\ \boldsymbol{c})$ *have strongly full row rank.*

Proof. By negation, $\boldsymbol{A}^{\forall\exists}$ is not AE regular if and only if there are $B^{\forall} \in \boldsymbol{B}^{\forall}$, $C^{\forall} \in \boldsymbol{C}^{\forall}$ and $c^{\forall} \in \boldsymbol{c}^{\forall}$ such that

$$\begin{pmatrix} B^{\forall} & \boldsymbol{b}^{\exists} \\ C^{\forall} & c^{\forall} \end{pmatrix}$$

is strongly singular. By Theorem 4.7, $(B^T\ C^T)$ or $(B\ c)$ has not full row rank. Therefore, $\boldsymbol{A}^{\forall\exists}$ is not AE regular if and only if $(\boldsymbol{B}^T\ \boldsymbol{C}^T)$ or $(\boldsymbol{B}\ \boldsymbol{c})$ has not full row rank. ∎

As a related structured matrix, we have the following:

THEOREM 4.9. *Let* $\boldsymbol{B} \in \mathbb{IR}^{n\times k}$ *and* $\boldsymbol{C} \in \mathbb{IR}^{n\times(n-k)}$ *with* $C_{\Delta} > 0$. *Then* $(\boldsymbol{B}^{\forall}\, \boldsymbol{C}^{\exists})$ *is AE regular if and only if* $\boldsymbol{B}^{\forall}$ *has strongly full column rank.*

Proof. "If". By negation, suppose there is $B^{\forall} \in \boldsymbol{B}^{\forall}$ such that $(B^{\forall}\, \boldsymbol{C}^{\exists})$ is strongly singular. Since $C_{\Delta} > 0$, the matrix $B^{\forall}$ must have linearly dependent columns (otherwise there is $C^{\forall} \in \boldsymbol{C}^{\forall}$ such that $(B^{\forall}\, C^{\exists})$ is nonsingular).

"Only if". Obvious. ∎

As an open problem, we leave a generalization of the above two results.

CONJECTURE 2. *The square interval matrix*

$$\begin{pmatrix} \boldsymbol{B}^{\forall} & \boldsymbol{D}^{\exists} \\ \boldsymbol{C}^{\forall} & \boldsymbol{E}^{\forall} \end{pmatrix},$$

where $D^{\exists}_{\Delta} > 0$, *is AE regular if and only if* $(\boldsymbol{B}^T\ \boldsymbol{C}^T)$ *and* $(\boldsymbol{C}\ \boldsymbol{E})$ *have strongly full row rank.*

5. Conclusion. We introduced a generalized concept of regularity of interval matrices based on ∀∃ quantification. Characterization of the general case turned out to be a very difficult problem, and we stated several open question. On the other hand, we identified a couple of polynomially recognizable sub-classes such as M-matrices, H-matrices or matrices with a structured quantifier position.

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