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THE LARGEST EIGENVALUE AND SOME HAMILTONIAN PROPERTIES OF GRAPHS*

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Abstract. In this note, sufficient conditions, based on the largest eigenvalue, are presented for some Hamiltonian properties of graphs.

Key words. The largest eigenvalue, Hamiltonian property.

AMS subject classifications. 05C50, 05C45.

1. Introduction. We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph G, we use n to denote its order $|V(G)|$. The complement a graph is denoted by G^c. A subset V_1 of the vertex set $V(G)$ is independent if no two vertices in V_1 are adjacent in G. The eigenvalues of a graph G, denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, are defined as the eigenvalues of its adjacency matrix $A(G)$. For a square matrix M, we use $det(M)$ to denote its determinant. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

In 2010, Fiedler and Nikiforov [3] obtained the following spectral conditions for the Hamiltonicity and traceability of graphs.

THEOREM 1.1. *Let G be a graph of order n.*

(1) *If $\lambda_1(G) \geq n-2$, then G contains a Hamiltonian path unless $G = K_{n-1} + v$; if strict inequality holds, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*
(2) *If $\lambda_1(G^c) \leq \sqrt{n-1}$, then G contains a Hamiltonian path unless unless $G = K_{n-1} + v$.*
(3) *If $\lambda_1(G^c) \leq \sqrt{n-2}$, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*

Motivated by the results of Fiedler and Nikiforov, a lot of authors obtained additional spectral conditions for the Hamiltonian properties of graphs. Some of them can be found in [11], [6], [8], [7], [10], and [1]. In this note, we present new spectral conditions based on the largest eigenvalue for the Hamiltonicity and traceability of graphs. The main results are as follows.

THEOREM 1.2. *Let G be a graph of order $n \geq 3$ with connectivity κ ($\kappa \geq 2$). If $\lambda_1 \leq \delta\sqrt{\frac{\kappa+1}{n-\kappa-1}}$, then G is Hamiltonian or G is $K_{\kappa,\,\kappa+1}$.*

THEOREM 1.3. *Let G be a graph of order $n \geq 12$ with connectivity κ ($\kappa \geq 1$). If $\lambda_1 \leq \delta\sqrt{\frac{\kappa+2}{n-\kappa-2}}$, then G is traceable or G is $K_{\kappa,\,\kappa+2}$.*

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2. Lemmas. We need the following results as our lemmas when we prove Theorems 1.2 and 1.3. Lemma 2.1 below is from [9].

Lemma 2.1. *Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If $d(x)+d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.*

Lemma 2.2 below is from [5].

Lemma 2.2. *Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex in A has degree at least k and each vertex in B has degree at least l, then G contains a cycle of length at least $2\min(|B|, k+l-1, 2k-2)$.*

3. Proof of Theorem 1.2. Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not Hamiltonian. Then $n \geq 2\kappa + 1$ (otherwise $\delta \geq \kappa \geq \frac{n}{2}$ and G is Hamiltonian). Since $\kappa \geq 2$, G has a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find s $(s \geq \kappa)$ pairwise disjoint (except for u_0) paths $P_1, P_2, \ldots, P_s$ between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of C. We use v_i^+ to denote the successor of v_i along the orientation of C, where $1 \leq i \leq s$. Since C is a longest cycle in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Set $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+\}$. Then S is independent. Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$. Set $T := V - S = \{ w_1, w_2, \ldots, w_{n-\kappa-1} \}$. We label the vertices of $u_0, u_1, \ldots, u_\kappa, w_1, w_2, \ldots, w_{n-\kappa-1}$ by $1, 2, \ldots, \kappa+1, \kappa+2, \ldots, n$, respectively. Let $d_1(w_i) = |N(w_i) \cap S|$ and $d_2(w_i) = |N(w_i) \cap T|$ for each i with $1 \leq i \leq n-\kappa-1$. Obviously, $\sum_{i=0}^{\kappa+1} d(u_i) = \sum_{i=1}^{n-\kappa-1} d_1(w_i)$.

Define a two by two matrix $B = (\, B_{i\,j} \,)_{2\times 2}$, where

$$B_{11} = 0, \;\; B_{12} = \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1}, \;\; B_{21} = \frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1}, \;\; B_{22} = \frac{\sum_{i=1}^{n-\kappa-1} d_2(w_i)}{n-\kappa-1}.$$

Then B is a quotient matrix of the adjacency matrix of G with partition S and T. Let $\mu_1 \geq \mu_2$ be the eigenvalues of B. Then, by Corollary 2.3 on Page 596 in [4], we have that $\lambda_1 \geq \mu_1$ and $\mu_2 \geq \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G.

In the proofs below, we use some ideas in the proof of Theorem 3.3 in [4]. We, from Perron-Frobenius theorem, have that $|\lambda_n| \leq \lambda_1$. Thus,

$$\begin{aligned}
\lambda_1^2 &\geq -\lambda_1\lambda_n \geq -\mu_1\mu_2 = -det(B) = B_{12}\, B_{21} \\
&= \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1}\;\frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1} = \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1}\;\frac{\sum_{i=0}^{k} d(u_i)}{\kappa+1}\;\frac{\kappa+1}{n-\kappa-1} \\
&\geq \frac{\delta^2(\kappa+1)}{n-\kappa-1} \geq \lambda_1^2.
\end{aligned}$$

Therefore, $\lambda_1 = -\lambda_n$, $\lambda_1 = \mu_1$, $\lambda_n = \mu_2$, and $d(u_i) = \delta$ for each i with $0 \leq i \leq \kappa$. Since $0 = \lambda_1 + \lambda_n = \mu_1 + \mu_2 = B_{22}$, $d_2(w_i) = |N(w_i) \cap T| = 0$ for each i with $1 \leq i \leq n-\kappa-1$. Thus, G is a bipartite graph with partition sets S and T.

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Notice that

$$\delta = \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1} = \frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1}\ \frac{n-\kappa-1}{\kappa+1}$$
$$= \frac{\sum_{i=1}^{n-\kappa-1} d(w_i)}{n-\kappa-1}\ \frac{n-\kappa-1}{\kappa+1} \geq \delta\ \frac{n-\kappa-1}{\kappa+1}.$$

Therefore, $n \leq 2\kappa+2$. Since $n \geq 2\kappa+1$. We have $n = 2\kappa+1$ or $n = 2\kappa+2$.

When $n = 2\kappa+1$, then $n-\kappa-1 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for i with $0 \leq i \leq \kappa$, $u_i w_j \in E$ for each i with $0 \leq i \leq \kappa$ and for each j with $1 \leq j \leq n-\kappa-1$. Hence, G is $K_{\kappa,\,\kappa+1}$.

When $n = 2\kappa+2$, then $n-\kappa-1 = \kappa+1$ and G is a balanced bipartite graph. From Lemma 2.1, we have G is Hamiltonian, a contradiction.

This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3. Let G be a graph satisfying the conditions in Theorem 1.3. Suppose, to the contrary, that G is not traceable. Then $n \geq 2\kappa+2$ (otherwise $\delta \geq \kappa \geq \frac{n-1}{2}$ and G is traceable). Choose a longest path P in G and give an orientation on P. Let x and y be the two end vertices of P. Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths $P_1, P_2, \ldots, P_s$ between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of P. Since P is a longest path in G, $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use v_i^+ to denote the successor of v_i along the orientation of P, where $1 \leq i \leq s$. Since P is a longest path in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s-1$. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_s^+, x\}$ is independent (otherwise G would have paths which are longer than P). Set $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+, x\}$. Then S is independent. Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$ and $u_{\kappa+1} = x$. Set $T := V - S = \{\, w_1, w_2, \ldots, w_{n-\kappa-2} \,\}$. We label the vertices of $u_0, u_1, \ldots, u_\kappa, u_{\kappa+1}, w_1, w_2, \ldots, w_{n-\kappa-2}$ by $1, 2, \ldots, \kappa+1, \kappa+2, \ldots, n$, respectively. Let $d_1(w_i) = |N(w_i) \cap S|$ and $d_2(w_i) = |N(w_i) \cap T|$ for each i with $1 \leq i \leq n-\kappa-2$. Obviously, $\sum_{i=0}^{\kappa+1} d(u_i) = \sum_{i=1}^{n-\kappa-2} d_1(w_i)$.

Define a two by two matrix $B = (\, B_{i\,j}\,)_{2\times 2}$, where

$$B_{11} = 0,\ \ B_{12} = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2},\ \ B_{21} = \frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2},\ \ B_{22} = \frac{\sum_{i=1}^{n-\kappa-2} d_2(w_i)}{n-\kappa-2}.$$

Then B is a quotient matrix of the adjacency matrix of G with partition S and T. Let $\mu_1 \geq \mu_2$ be the eigenvalues of B. Then, by Corollary 2.3 on Page 596 in [4], we have that $\lambda_1 \geq \mu_1$ and $\mu_2 \geq \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G.

We, from Perron-Frobenius theorem, have that $|\lambda_n| \leq \lambda_1$. Thus,

$$\lambda_1^2 \geq -\lambda_1\lambda_n \geq -\mu_1\mu_2 = -det(B) = B_{12}\, B_{21}$$
$$= \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2}\ \frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2} = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2}\ \frac{\sum_{i=0}^{k+1} d(u_i)}{\kappa+2}\ \frac{\kappa+2}{n-\kappa-2}$$
$$\geq \frac{\delta^2(\kappa+2)}{n-\kappa-2} \geq \lambda_1^2.$$

Therefore, $\lambda_1 = -\lambda_n$, $\lambda_1 = \mu_1$, $\lambda_n = \mu_2$, and $d(u_i) = \delta$ for each i with $0 \leq i \leq \kappa+1$. Since $0 = \lambda_1 + \lambda_n = \mu_1 + \mu_2 = B_{22}$, $d_2(w_i) = |N(w_i) \cap T| = 0$ for each i with $1 \leq i \leq n-\kappa-2$. Thus, G is a bipartite graph with partition sets S and T.

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Notice that

$$\delta = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2} = \frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2} \; \frac{n-\kappa-2}{\kappa+2}$$
$$= \frac{\sum_{i=1}^{n-\kappa-2} d(w_i)}{n-\kappa-2} \; \frac{n-\kappa-2}{\kappa+2} \geq \delta \; \frac{n-\kappa-2}{\kappa+2}.$$

Therefore, $n \leq 2\kappa+4$. Since $n \geq 2\kappa+2$. We have $n = 2\kappa+2$, $n = 2\kappa+3$, or $n = 2\kappa+4$.

When $n = 2\kappa+2$, then $n-\kappa-2 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for i with $0 \leq i \leq \kappa+1$, $u_i w_j \in E$ for each i with $0 \leq i \leq \kappa+1$ and for each j with $1 \leq j \leq n-\kappa-2$. Hence, G is $K_{\kappa,\,\kappa+2}$.

When $n = 2\kappa+3$, then $n-\kappa-2 = \kappa+1$. Notice that $\kappa \geq 5$ since $n = 2\kappa+3 \geq 12$. Notice further that each vertex in S or T has degree at least $\delta \geq \kappa$. From Lemma 2.2, we have G has a cycle of length $2\kappa+2$. Since $n = 2\kappa+3$ and $\kappa \geq 5$, G has a path containing all the vertices of G. Namely, G is traceable, a contradiction.

When $n = 2\kappa+4$, then $n-\kappa-2 = \kappa+2$. Notice that $\kappa \geq 4$ since $n = 2\kappa+4 \geq 12$. Notice further that each vertex in S or T has degree at least $\delta \geq \kappa$. From Lemma 2.2, we have G has a cycle of length $2\kappa+4$, which implies that G is traceable, a contradiction.

This completes the proof of Theorem 1.3. □

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REFERENCES

[1] V. Benediktovich. Spectral condition for Hamiltonicity of a graph. *Linear Algebra and its Applications*, 494:70–79, 2016.
[2] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications.* Macmillan, London and Elsevier, New York, 1976.
[3] M. Fiedler and V. Nikiforov. Spectral radius and Hamiltonicity of graphs. *Linear Algebra and its Applications*, 432:2170–2173, 2010.
[4] W. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra and its Applications*, 227/228: 593–616, 1995.
[5] B. Jackson. Long cycles in bipartite graphs. *Journal of Combinatorial Theory, Series B*, 38:118–131, 1985.
[6] R. Li. Eigenvalues, Laplacian eigenvalues and some Hamiltonian properties of graphs. *Utilitas Mathematica*, 88:247–257, 2012.
[7] R. Liu, W.C. Shiu, and J. Xue. Sufficient spectral conditions on Hamiltonian and traceable graphs. *Linear Algebra and its Applications*, 467:254–266, 2015.
[8] M. Lu, H. Liu, and F. Tian. , Spectral radius and Hamiltonian graphs. *Linear Algebra and its Applications*, 437:1670–1674, 2012.
[9] J. Moon and L. Moser. On Hamiltonian bipartite graphs. *Israel Journal of Mathematics*, 1:163–165, 1963.
[10] B. Ning and J. Ge. Spectral radius and Hamiltonian properties of graphs. *Linear and Multilinear Algebra*, 63:1520–1530, 2015.
[11] B. Zhou. Signless Laplacian spectral radius and Hamiltonicity. *Linear Algebra and its Applications*, 432:566–570, 2010.