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DIAGONAL SUMS OF DOUBLY SUBSTOCHASTIC MATRICES∗
LEI CAO†, ZHI CHEN‡, XUEFENG DUAN§, SELCUK KOYUNCU¶, AND HUILAN LI‖

Abstract. Let Ωn denote the convex polytope of all n × n doubly stochastic matrices, and ωn denote the convex polytope of all n × n doubly substochastic matrices. For a matrix A ∈ ωn, define the sub-defect of A to be the smallest integer k such that there exists an (n + k) × (n + k) doubly stochastic matrix containing A as a submatrix. Let ωn,k denote the subset of ωn which contains all doubly substochastic matrices with sub-defect k. For π a permutation of symmetric group of degree n, the sequence of elements a1π(1), a2π(2),..., anπ(n) is called the diagonal of A corresponding to π. Let h(A) and l(A) denote the maximum and minimum diagonal sums of A ∈ ωn,k, respectively. In this paper, existing results of h and l functions are extended from Ωn to ωn,k. In addition, an analogue of Sylvester’s law of the h function on ωn,k is proved.

Key words. Doubly substochastic matrices, Sub-defect, Maximum diagonal sum.

AMS subject classifications. 15A51, 15A83.

1. Introduction. An n by n real matrix A = [aij] is called a doubly stochastic matrix if

1. aij ≥ 0, and
2. ∑i aij = 1 and ∑j aij = 1 for all i and j.

One can define doubly substochastic matrices by replacing the equalities by inequalities ∑i aij ≤ 1 and ∑j aij ≤ 1 in (2). Doubly stochastic matrices and doubly substochastic matrices have been studied intensively by many mathematicians (see [3], [7], [9] and [11]). Denote Ωn and ωn the set of all n by n doubly stochastic matrices and the set of all n × n doubly substochastic matrices, respectively. It is clear that Ωn ⊆ ωn. For B ∈ ωn, denote the sum of all elements of B by σ(B), i.e

\[ \sigma(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}. \]  

Recently, Cao, Koyuncu and Parmer defined an interesting characteristic called sub-defect on the set ωn. For B ∈ ωn, the sub-defect of B is denoted by sd(B). It is the smallest integer k such that there exists an (n + k) × (n + k) doubly stochastic matrix containing B as a submatrix. It has been shown that the sub-defect can be calculated easily by taking the ceiling of the difference of the size of the matrix and the sum of all entries (see [4], [5] and [6]).

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**Theorem 1.1.** (Theorem 2.1 of [6]) Let $B = [b_{ij}]$ be an $n \times n$ doubly substochastic matrix. Then

$$sd(B) = [n - \sigma(B)],$$

where $[x]$ is the ceiling of $x$.

Let $\omega_{n,k}$ denote the set of matrices in $\omega_n$ with sub-defect equal to $k$. It is worth to point out that the sub-defect $k$ then provides a way to partition $\omega_n$ into $n+1$ convex subsets which are $\omega_{n,0} = \Omega_n, \omega_{n,1}, \ldots, \omega_{n,n}$. Namely,

(i) $\omega_{n,k}$ is convex for all $k$;
(ii) $\omega_{n,i} \cap \omega_{n,j} = \emptyset$ for $i \neq j$;
(iii) $\bigcup_{i=0}^{n} \omega_{n,i} = \omega_n$.

Let $A = [a_{ij}]$ be a real $n \times n$ matrix. Denote $S_n$ the symmetric group of degree $n$. For $\pi \in S_n$, the sequence of elements $a_{1\pi(1)}, a_{2\pi(2)}, \ldots, a_{n\pi(n)}$ is called the diagonal of $A$ corresponding to $\pi$ and will also be denoted by $\pi$. A diagonal $\pi$ of $A$ is a maximum (minimum) diagonal if $\sum_{i=1}^{n} a_{i\pi(i)}$ is a maximum (minimum) among all $n!$ diagonal sums. The value of the maximum and minimum diagonal sums of $A$ will be denoted by $h(A)$ and $l(A)$, respectively, and in case the matrix under consideration is fixed, simply by $h$ and $l$, respectively. For $X = [x_{ij}]$ an $n \times n$ real matrix, denote

$$\langle A, X \rangle = \sum_{i,j} a_{ij} x_{ij}.$$  

Note that $h(A)$ is also the support function of the assignment polytope $\Omega_n$, i.e.,

$$h(A) = \sup \{ \langle A, X \rangle : X \in \Omega_n \}.$$  

Similarly, $l(A)$ can be defined as

$$l(A) = \inf \{ \langle A, X \rangle : X \in \Omega_n \}.$$  

In [12], Wang investigated and conjectured some interesting properties when the domains of these two functions are restricted on $\Omega_n$. We extend the existing results of $h$ function and $l$ function on $\omega_n$.

The paper is organized as follows: In Section 2, we show some properties of $h$-function and $l$-function on $\omega_{n,k}$ with respect to the sub-defect $k$. In Section 3, we prove an analogue of the Sylvester's law of $h$ functions on $\omega_{n,k}$. In addition, we give an example to illustrate that the analogue of Frobenius inequalities of the rank function is not true on $\omega_n$. Throughout this paper, we denote by $J_n$ the $n \times n$ matrix whose all entries are 1.

2. The $h$-function and $l$-function on $\omega_{n,k}$. In this paper, we shall view $h$ and $l$ as two functions defined on $\omega_{n,k}$ in the natural way and study their properties. For $k = 0$, which is when restricted on $\Omega_n$, some interesting properties have been discussed and explored in [12]. For $k \geq 1$, one crucial difference between matrices in $\Omega_n$ and those in $\omega_{n,k}$ is the sum of all elements. That is actually how sub-defect is defined originally. If $A \in \omega_{n,k}$, then $\sigma(A)$ is inside the interval $[n-k, n-k+1]$. We explore and show properties of the $h$ and $l$ functions on $\omega_{n,k}$ with respect to the sub-defect $k$ or the sum of all elements of the matrices. We first notice that in $\omega_{n,k}$, the function $h$ is convex while the function $l$ is concave.

**Proposition 2.1.** (i) $h$ is a convex function;
(ii) $l$ is a concave function.
Proof. Let $A$ and $B$ be two nonnegative matrices and $\lambda \in [0, 1]$. It is clear that
\[
h(\lambda A + (1 - \lambda)B) \leq h(\lambda A) + h((1 - \lambda)B) = \lambda h(A) + (1 - \lambda)h(B)
\]
and
\[
l(\lambda A + (1 - \lambda)B) \geq l(\lambda A) + l((1 - \lambda)B) = \lambda l(A) + (1 - \lambda)l(B),
\]
and hence, the proposition holds.

Let $A \in \omega_n$. It is not hard to see the extreme values of $h(A)$ and $l(A)$ given by the following proposition.

**Proposition 2.2.** Let $A \in \omega_n$. Then
\[
0 \leq l(A) \leq \frac{\sigma(A)}{n} \leq h(A) \leq \sigma(A).
\]

**Proof.** It is clear that $l(A) \geq 0$ and $h(A) \leq \sigma(A)$. From the covering theorem (Theorem 2.1 in [12]), we can get $l(A) \leq \frac{\sigma(A)}{n} \leq h(A)$, which implies the proposition.

In (2.2), $l(A) = 0$ if and only if $A$ has a zero diagonal, such as partial permutation matrices. On the other hand, $h(A) = \sigma(A)$ if and only if $A$ has only one non-zero diagonal such that the sum of all entries of the diagonal is equal to $\sigma(A)$. For example, let $n - k \leq s < n - k + 1$ and $A$ an $n$ by $n$ matrix containing $\lceil s \rceil$ 1’s and an $s - \lceil s \rceil$ on the diagonal as follows.

\[
A = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}.
\]

It is easy to check that $h(A) = \sigma(A) = s$. For $A \in \omega_n$, denote $\sigma(A) = s$. Then $l(A) = \frac{s}{n} = h(A)$ if and only if $A$ is in the following form:

\[
A = \frac{s}{ln} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}.
\]

where $t \geq s$, a positive integer and the first $t$ rows of $A$ are filled up by $\frac{s}{tn}$.

**Corollary 2.3.** Let $B \in \omega_{n,k}$. Then
\[
\frac{n-k}{n} \leq h(B) < n-k+1.
\]

**Proof.** This is a direct consequence of Proposition 2.2 and Theorem 1.1, which implies that $n - k \leq \sigma(B) < n - k + 1$.\]
Remark 2.4. From [12], we know that for $A \in \Omega_n$, $h(A) \geq 1$ with equality if and only if $A = \frac{1}{n}J_n$. However, in $\omega_{n,k}$, such an $B$ satisfying $h(B) = \frac{n-k}{n}$ is not unique. For example, we can take $B_1 = \frac{n-k}{n^2}J_n$, and $B_2$ an $n$-square matrix with $n-k$ rows filled up by $\frac{1}{n}$'s, i.e.,

$$B_1 = \frac{n-k}{n^2} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By direct computation, we have $\sigma(B_1) = \sigma(B_2) = n-k$ and $h(B_1) = h(B_2) = \frac{n-k}{n}$.

Actually, if $A, B \in \omega_n$, then $AB \in \omega_n$ (Proposition 2.4 in [5]). We can evaluate the extreme values of $h(AB)$ and $l(AB)$.

**Theorem 2.5.** Let $A \in \omega_{n,k}$ and $B$ be an $n \times n$ real matrix with nonnegative entries. Then

(i) $h(AB) \leq h(B)$;
(ii) $l(B) \leq l(AB)$.

**Proof.** (i) For reader’s convenience, we first prove a special case when $k = 0$, which means $A \in \Omega_n$. The case that both $A$ and $B$ in $\Omega_n$ has been proved in [12].

Due to Birkhoff’s theorem (see [2], [3] and [10]), we can always write

$$A = \alpha_1 P_1 + \cdots + \alpha_m P_m,$$

where $P_1, \ldots, P_m$ are permutation matrices and $\alpha_1 + \cdots + \alpha_m = 1$. It is clear that $h(B) = h(PB)$ for an arbitrary permutation matrix $P$. Then we have

$$h(AB) = h(\alpha_1 P_1 B + \cdots + \alpha_m P_m B) \leq \alpha_1 h(P_1 B) + \cdots + \alpha_m h(P_m B) = \alpha_1 h(B) + \cdots + \alpha_m h(B) = (\alpha_1 + \cdots + \alpha_m) h(B) = h(B)$$

in which the inequality sign is due to the convexity of $h$.

Next we show the inequality holds for any integer $0 \leq k \leq n$ and all $A \in \omega_{n,k}$. Simply, let

$$\tilde{A} = \begin{bmatrix} A & X \\ Y & Z \end{bmatrix}$$

be a doubly stochastic matrix containing $A$ as a principal submatrix. (For instance, we can let $\tilde{A}$ be the minimal doubly stochastic completion obtained by the method described in the proof of Theorem 2.1 in [6].)

Write

$$\tilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$
with the same size as $\tilde{A}$. Since $\tilde{A}$ is a doubly stochastic matrix, we can apply (2.3) to $\tilde{A}$ and $\tilde{B}$ to get

$$h(AB) \leq h\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = h(\tilde{A}\tilde{B}) \leq h(\tilde{B}) = h(B).$$

(ii) For $A \in \Omega_n$, replacing $h$ function by $l$ function and using the concavity of $l$ in (2.3), we get

$$l(AB) \geq l\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = l(\tilde{A}\tilde{B}) \geq l(\tilde{B}) = l(B).$$

Then applying (2.6) to $\tilde{A}$ and $\tilde{B}$ defined in (2.4) and (2.5), respectively, we have

$$l(AB) \geq l\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = l(\tilde{A}\tilde{B}) \geq l(\tilde{B}) = l(B) \quad \Box$$

**Corollary 2.6.** Let $A, B \in \omega_{n,k}$. Then

(i) $h(AB) \leq \min\{h(A), h(B)\}$;
(ii) $l(AB) \geq \max\{l(A), l(B)\}$.

**Remark 2.7.** To determine whether the equality in (i) holds, simply let $A = B = \begin{bmatrix} I_n-k & 0 \\ 0 & 0 \end{bmatrix}$. Then we have $AB = A = B$, and therefore, $h(AB) = h(A) = h(B) = \min\{h(A), h(B)\}$.

**Remark 2.8.** In [12], Wang shows that for $A, B \in \Omega_n$, $h(AB) \leq h(A)h(B)$. However similar result does not hold for $A, B \in \omega_{n,k}$. To see this, simply choose

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \in \omega_{3,2}$$

and $B$ just the transpose of $A$, i.e., $B = A^t$. Since

$$AB = \begin{bmatrix} 3/16 & 3/16 & 3/16 \\ 3/16 & 3/16 & 3/16 \\ 0 & 0 & 0 \end{bmatrix},$$

we have $h(AB) = 3/8$. However $h(A) = h(B) = 1/2$, and hence,

$$3/8 = h(AB) > h(A)h(B) = 1/4.$$

**Corollary 2.9.** Let $A \in \omega_{n,k}$. Then

(i) $h(A^m) \leq h(A)$;
(ii) $l(A^m) \geq l(A)$.

**Lemma 2.10.** Let $A, B \in \omega_n$. Then

$$0 \leq l(AB) \leq \frac{\sigma(A)\sigma(B)}{n^2} \leq h(AB).$$

**Proof.** The leftmost inequality is trivial.
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To show
\[ \frac{\sigma(A)\sigma(B)}{n^2} \leq h(AB), \]
let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be two matrices in \( \omega_n \). First note that \( AB = [\sum_{k=1}^n a_{ik}b_{kj}] \). Without loss of generality, assume
\[ h(AB) = \sum_{i,j=1}^n a_{ij}b_{ji}. \]

We need to find the minimum value of \( h(AB) \) subject to the conditions
\[ \sum_{i,j=1}^n a_{ij} = \sigma(A) \]
and
\[ \sum_{i,j=1}^n b_{ij} = \sigma(B). \]

We introduce two Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \), and then construct the Lagrange function \( H \) as follows.
\[ H = h(AB) - \lambda_1 \left( \sum_{i,j} a_{ij} - \sigma(A) \right) - \lambda_2 \left( \sum_{i,j} b_{ij} - \sigma(B) \right). \]

Using Lagrange multiplier method, we have
\[ \frac{\partial H}{\partial a_{ij}} = b_{ji} - \lambda_1 = 0, \]
\[ \frac{\partial H}{\partial b_{ij}} = a_{ji} - \lambda_2 = 0, \]
\[ \frac{\partial H}{\partial \lambda_1} = \sum_{i,j} a_{ij} - \sigma(A) = 0, \]
\[ \frac{\partial H}{\partial \lambda_2} = \sum_{i,j} b_{ij} - \sigma(B) = 0. \]

Solving the system of equations above, we get
\[ \sum_{i,j} a_{ij} = n^2 \lambda_2 = \sigma(A), \quad \sum_{i,j} b_{ij} = n^2 \lambda_1 = \sigma(B), \]
\[ a_{ij} = \lambda_2 = \frac{\sigma(A)}{n^2}, \quad b_{ij} = \lambda_1 = \frac{\sigma(B)}{n^2}. \]

Due to the convexity of the function \( h \), we know that
\[ h_{\text{min}}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}. \]

Similarly, by the method of Lagrange multipliers and the concavity of \( l \) function, we can prove that
\[ l_{\text{max}}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}. \]

Since both \( l \) function and \( h \) function are well defined on the set of all \( n \times n \) real matrices, although \( A + B \) is not necessarily in \( \omega_n \) for \( A, B \in \omega_{n,k} \), both \( l(A + B) \) and \( h(A + B) \) are well defined and we have the following result.
PROPOSITION 2.11. Let \(A, B \in \omega_{n,k}\). Then

(i) \(0 \leq h(A) + h(B) - h(A + B) \leq \min\{h(A), h(B)\} < n - k + 1\);

(ii) \(l(A + B) - l(A) - l(B) < \frac{2(n - k + 1)}{n}\).

Proof. (i) Since \(h(A + B) \leq h(A) + h(B)\), it is clear that

\[0 \leq h(A) + h(B) - h(A + B) \leq \min\{h(A), h(B)\} < n - k + 1,\]

where the equality implies \(h(A + B) = \max\{h(A), h(B)\}\). To see the upper bound is sharp, one can choose such \(A\) and \(B\) that both contain \(n - k\) 1’s and an \(\epsilon\) as follows:

\[
A = \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \vdots \\ & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \vdots & \vdots \\ & \vdots & 0 & \epsilon \\ & & & 0 \end{bmatrix},
\]

where \(0 \leq \epsilon < 1\). Then \(h(A) = h(B) = h(A + B) = n - k + \epsilon\), letting \(\epsilon \to 1\) and we get \(\sup_{A,B \in \omega_{n,k}} \{h(A) + h(B) - h(A + B)\} = n - k + 1\).

(ii) Since \(\frac{1}{2}(A + B) \in \omega_{n,k}\), we have \(l(A + B) < \frac{n - k + 1}{n}\) or \(l(A + B) < \frac{2(n - k + 1)}{n}\). With \(l(A), l(B) \geq 0\), we get \(l(A + B) - l(A) - l(B) < \frac{2(n - k + 1)}{n}\).

\[\square\]

3. The analogue of the Sylvester’s law of the maximum diagonals of matrices in \(\omega_{n,k}\). The Sylvester’s law of the rank function (2.17.8 in [8]) says that if \(A\) is an \(m \times t\) real matrix and \(B\) an \(t \times n\) real matrix, then

\[
\max\{\text{rank}(A), \text{rank}(B)\} \leq \text{rank}(A) + \text{rank}(B) - \text{rank}(AB) \leq n.
\]

In [12], Wang conjectured the analogue of Sylvester’s law of \(h\) function on \(\Omega_n\), and later on it was proved by Balasubramanian for a more general case using the statement \(\text{tr}(A) + \text{tr}(B) - \text{tr}(AB) \leq n\). For further use, we state the result as follows.

THEOREM 3.1. (Main Theorem of [1]) If \(A, B\) are \(n \times n\) real matrices with all elements in the closed interval \([0, 1]\), then

\[
(3.7) \quad h(A) + h(B) - h(AB) \leq n.
\]

Also, Balasubramanian gave the conditions for which the equality holds. Based on this theorem, we give two analogues of (3.7) as follows.

LEMMA 3.2. Let \(A \in \Omega_n\) and \(B \in \omega_n\). Then

\[1 \leq h(A) + h(B) - h(AB) \leq n,
\]

where both the equalities can be tight.
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Proof. The inequality involving the upper bound is due to Theorem 3.1. To get the equality of the upper bound, simply take $A$ to be any permutation matrix and any $B \in \omega_n$. In this case, $h(AB) = h(B)$, and therefore, $h(A) + h(B) - h(AB) = h(A) = n$.

For the lower bound, it is due to the combination of $h(A) \geq 1$ and Theorem 2.5 (i). Thus, we have

$$h(A) + h(B) - h(AB) \geq h(A) \geq 1.$$ 

The equality for the lower bound holds when $A = \frac{1}{n} J_n \in \Omega_n$ and $B = \frac{s}{n^2} J_n \in \omega_n$, where $0 < s < n$. In this case, $AB = B$ and then $h(B) = h(AB) = \frac{s}{n}$, which implies that

$$h(A) + h(B) - h(AB) = h(A) = 1.$$ 

Let $A, B \in \omega_{n,k}$. Then, due to Theorem 3.1 and Corollary 2.6 (i), we have

$$\max\{h(A), h(B)\} \leq h(A) + h(B) - h(AB) \leq n.$$ 

When $k = 0$, i.e., $A, B \in \Omega_n$, both upper bound and lower bound are tight. However, when $k$ is close to $n$, the upper bound is not tight anymore. In addition, it seems that the lower bound can be more precise with respect to the sub-defect $k$. So, we explore the role of $k$ and obtain the following theorem for the doubly substochastic matrix case, which is stronger than (3.8).

**Theorem 3.3.** Let $A, B \in \omega_{n,k}$. Then

$$\frac{n - k}{n} \leq h(A) + h(B) - h(AB) \leq \min\{n, 2(n - k + 1)\}.$$ 

In particular when $k \geq \frac{n}{2} + 1$,

$$\sup_{A, B \in \omega_{n,k}} \{h(A) + h(B) - h(AB)\} = 2(n - k + 1).$$

In order to prove Theorem 3.3, we need the following lemma.

**Lemma 3.4.** Let $A \in \omega_{n,k}$. Then we have $h(A) < n - k + 1$ and

$$\sup_{A \in \omega_{n,k}} \{h(A)\} = n - k + 1.$$ 

Proof. Since $A \in \omega_{n,k}$, $\sigma(A) < n - k + 1$. It is clear that $h(A) \leq \sigma(A) < n - k + 1$. So, $n - k + 1$ is an upper bound. To show $n - k + 1$ is the least upper bound, one can construct the following diagonal matrix:

$$A_\epsilon = \begin{bmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & \epsilon \\
& & & 0 \\
& & & & \ddots \\
\end{bmatrix}$$

which contains $n - k$ 1’s and an $\epsilon$ on the diagonal. For $0 \leq \epsilon < 1$, $A_\epsilon \in \omega_{n,k}$. Note that

$$\lim_{\epsilon \to 1^-} h(A_\epsilon) = n - k + 1,$$
which means that
\[ \sup_{A \in \omega_{n,k}} \{ h(A) \} = n - k + 1. \]

**Corollary 3.5.** Let \( A \in \omega_{n,k} \). Then there exists an \( 0 \leq \epsilon < 1 \), such that
\[ h(A) \leq h(A_\epsilon). \]

**Proof.** It is clear that
\[ h(A_\epsilon) = \max \{ h(A) : \sigma(A) = \sigma(A_\epsilon), A \in \omega_{n,k} \}. \]
Therefore, the corollary holds.

Now, we are ready to prove Theorem 3.3.

**Proof of Theorem 3.3. Upper bound.** On the one hand, due to Theorem 3.1, \( A \) and \( B \) satisfy
\[ h(A) + h(B) - h(AB) \leq n. \]

Since when \( 0 \leq k < \frac{n}{2} + 1 \) we have \( 2(n - k + 1) > n \), and therefore, the right hand side inequality in Theorem 3.3 holds. For \( k \geq \frac{n}{2} + 1 \), we have
\[ 2(n - k + 1) = 2 \left( n - \left( \frac{n}{2} + 1 \right) + 1 \right) \leq n. \]

Thus, we need to show that when \( k \geq \frac{n}{2} + 1 \), \( h(A) + h(B) - h(AB) \leq 2(n - k + 1) \). To see this, let \( A_\epsilon \) be as in (3.9) and \( B_\eta \) be the matrix as follows.

\[ B_\eta = \begin{bmatrix}
0 & & \\
& \ddots & \vdots \\
& 0 & \eta \\
& & 1 \\
& & & \ddots \\
& & & & 1
\end{bmatrix} \]

which contains \( n - k \) 1’s and a nonnegative real number \( 0 \leq \eta < 1 \). Since \( k \geq \frac{n}{2} + 1 \), \( A_\epsilon B_\eta = 0 \), and hence, \( h(A_\epsilon B_\eta) = 0 \). In addition, due to Corollary 3.5, we have both
\[ \max_{A \in \omega_{n,k}} h(A) \leq \lim_{\epsilon \to 1^-} h(A_\epsilon) = n - k + 1, \]
and
\[ \max_{B \in \omega_{n,k}} h(B) \leq \lim_{\eta \to 1^-} h(B_\eta) = n - k + 1. \]

Therefore, we claim that
\[ h(A) + h(B) - h(AB) \leq \lim_{\epsilon \to 1^-} h(A_\epsilon) + \lim_{\eta \to 1^-} h(B_\eta) = 2(n - k + 1). \]

**Lower bound.** Due to Corollary 2.3 and Corollary 2.6, we have
\[ h(A) + h(B) - h(AB) \geq \max \{ h(A), h(B) \} \geq \frac{n - k}{n}, \]
which means that $\frac{n-k}{n}$ is a lower bound. It is tight because one can always let

$$
A_0 = \frac{1}{n} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \in \omega_{n,k}
$$

such that all elements in the first $n-k$ rows are $\frac{1}{n}$ and 0 otherwise. Let $B_0 = A_0^t$. Then

$$
A_0 B_0 = \frac{1}{n} \begin{bmatrix}
J_{n-k} & 0 \\
0 & 0
\end{bmatrix}.
$$

So, we have

$$
h(A_0) = h(B_0) = h(A_0 B_0) = \frac{n-k}{n},
$$

and hence,

$$
h(A_0) + h(B_0) - h(A_0 B_0) = \frac{n-k}{n}. \quad \Box
$$

In [12], the authors also conjectured the analogue of Frobenius inequalities of the rank function (see page 27 in [8]).

**Conjecture 3.6.** (Conjecture 5.2 of [12]) Let $A, B, C \in \Omega_n$. Then

$$
h(AB) + h(BC) - h(ABC) \leq h(B).
$$

Note that (3.7) is a special case of Conjecture 3.6 by letting $B$ be the identity matrix. Although the Sylvester’s law of $h$ function is true and Conjecture 3.6 still remains mysterious to us, it is not true if we replace $\Omega_n$ by $\omega_n$ in Conjecture 3.6. Here is an example.

**Example 3.7.** Let

$$
A = \frac{1}{5} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$B = A^t$ and $C = A$. Then

$$
AB = \frac{1}{5} \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad BC = \frac{3}{25} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
$$
and
\[
ABC = \frac{3}{25}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

So, \( h(A) = h(B) = h(AB) = h(BC) = \frac{3}{5} \) and \( h(ABC) = \frac{9}{25} \), and hence,
\[
\frac{21}{25} = h(AB) + h(BC) - h(ABC) > h(B) = \frac{3}{5}.
\]

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REFERENCES