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Lei Cao
Georgian Court University, lcao@georgian.edu

Zhi Chen
Nanjing Agricultural University, chenzhi@njau.edu.cn

Xuefeng Duan
Guilin University of Electronic Technology, duanxuefeng@guet.edu.cn

Selcuk Koyuncu
University of North Georgia, Selcuk.Koyuncu@ung.edu

Huilan Li
Shandong Normal University, huilanli77@gmail.com

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DIAGONAL SUMS OF DOUBLY SUBSTOCHASTIC MATRICES*

LEI CAO†, ZHI CHEN‡, XUEFENG DUAN§, SELCUK KOYUNCU¶, AND HUILAN LI‖

Abstract. Let Ω_n denote the convex polytope of all $n \times n$ doubly stochastic matrices, and ω_n denote the convex polytope of all $n \times n$ doubly substochastic matrices. For a matrix $A \in \omega_n$, define the sub-defect of A to be the smallest integer k such that there exists an $(n+k) \times (n+k)$ doubly stochastic matrix containing A as a submatrix. Let $\omega_{n,k}$ denote the subset of ω_n which contains all doubly substochastic matrices with sub-defect k. For π a permutation of symmetric group of degree n, the sequence of elements $a_{1\pi(1)}, a_{2\pi(2)}, \ldots, a_{n\pi(n)}$ is called the diagonal of A corresponding to π. Let $h(A)$ and $l(A)$ denote the maximum and minimum diagonal sums of $A \in \omega_{n,k}$, respectively. In this paper, existing results of h and l functions are extended from Ω_n to $\omega_{n,k}$. In addition, an analogue of Sylvesters law of the h function on $\omega_{n,k}$ is proved.

Key words. Doubly substochastic matrices, Sub-defect, Maximum diagonal sum.

AMS subject classifications. 15A51, 15A83.

1. Introduction.

An n by n real matrix $A = [a_{ij}]$ is called a doubly stochastic matrix if

1. $a_{ij} \geq 0$, and
2. $\sum_i a_{ij} = 1$ and $\sum_j a_{ij} = 1$ for all i and j.

One can define doubly substochastic matrices by replacing the equalities by inequalities $\sum_i a_{ij} \leq 1$ and $\sum_j a_{ij} \leq 1$ in (2). Doubly stochastic matrices and doubly substochastic matrices have been studied intensively by many mathematicians (see [3], [7], [9] and [11]). Denote Ω_n and ω_n the set of all n by n doubly stochastic matrices and the set of all $n \times n$ doubly substochastic matrices, respectively. It is clear that $\Omega_n \subseteq \omega_n$. For $B \in \omega_n$, denote the sum of all elements of B by $\sigma(B)$, i.e

$$\sigma(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}. \tag{1.1}$$

Recently, Cao, Koyuncu and Parmer defined an interesting characteristic called sub-defect on the set ω_n. For $B \in \omega_n$, the sub-defect of B is denoted by $sd(B)$. It is the smallest integer k such that there exists an $(n+k) \times (n+k)$ doubly stochastic matrix containing B as a submatrix. It has been shown that the sub-defect can be calculated easily by taking the ceiling of the difference of the size of the matrix and the sum of all entries (see [4], [5] and [6]).

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†School of Mathematics and Statistics, Shandong Normal University, Shandong 250358, China, and Department of Mathematics, Georgian Court University, NJ 08701, USA (lcao@georgian.edu).

‡Department of Mathematics, Nanjing Agricultural University, Jiangsu 210095, China (chenzhi@njau.edu.cn). Supported by the National Natural Science Foundation of China (no. 11601233), the Fundamental Research Funds for the Central Universities (no. KJQN201718), and the Natural Science Foundation of Jiangsu Province (no. BK20160708).

§College of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, China (duanxuefeng@guet.edu.cn). Supported by the National Natural Science Foundation of China (no. 11561015) and the Natural Science Foundation of Guangxi Province (no. 2016GXNSFFA380009).

¶Department of Mathematics, University of North Georgia, GA 30566, USA (Selcuk.Koyuncu@ung.edu).

‖School of Mathematics and Statistics, Shandong Normal University, Shandong 250358, China (huilanli77@gmail.com).

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THEOREM 1.1. (Theorem 2.1 of [6]) *Let $B = [b_{ij}]$ be an $n \times n$ doubly substochastic matrix. Then*

$$sd(B) = \lceil n - \sigma(B) \rceil,$$

where $\lceil x \rceil$ is the ceiling of x.

Let $\omega_{n,k}$ denote the set of matrices in ω_n with sub-defect equal to k. It is worth to point out that the sub-defect k then provides a way to partition ω_n into $n+1$ convex subsets which are $\omega_{n,0} = \Omega_n, \omega_{n,1}, \ldots, \omega_{n,n}$. Namely,

(i) $\omega_{n,k}$ is convex for all k;
(ii) $\omega_{n,i} \cap \omega_{n,j} = \emptyset$ for $i \neq j$;
(iii) $\bigcup_{i=0}^{n} \omega_{n,i} = \omega_n$.

Let $A = [a_{ij}]$ be a real $n \times n$ matrix. Denote S_n the symmetric group of degree n. For $\pi \in S_n$, the sequence of elements $a_{1\pi(1)}, a_{2\pi(2)}, \ldots, a_{n\pi(n)}$ is called the diagonal of A corresponding to π and will also be denoted by π. A diagonal π of A is a maximum (minimum) diagonal if $\sum_{i=1}^{n} a_{i\pi(i)}$ is a maximum (minimum) among all $n!$ diagonal sums. The value of the maximum and minimum diagonal sums of A will be denoted by $h(A)$ and $l(A)$, respectively, and in case the matrix under consideration is fixed, simply by h and l, respectively. For $X = [x_{ij}]$ an $n \times n$ real matrix, denote

$$\langle A, X \rangle = \sum_{i,j} a_{ij} x_{ij}.$$

Note that $h(A)$ is also the support function of the assignment polytope Ω_n, i.e.,

$$h(A) = \sup\{\langle A, X \rangle : X \in \Omega_n\}.$$

Similarly, $l(A)$ can be defined as

$$l(A) = \inf\{\langle A, X \rangle : X \in \Omega_n\}.$$

In [12], Wang investigated and conjectured some interesting properties when the domains of these two functions are restricted on Ω_n. We extend the existing results of h function and l function on ω_n.

The paper is organized as follows: In Section 2, we show some properties of h-function and l-function on $\omega_{n,k}$ with respect to the sub-defect k. In Section 3, we prove an analogue of the Sylvesters law of h functions on $\omega_{n,k}$. In addition, we give an example to illustrate that the analogue of Frobenius inequalities of the rank function is not true on ω_n. Throughout this paper, we denote by J_n the $n \times n$ matrix whose all entries are 1.

2. The h-function and l-function on $\omega_{n,k}$. In this paper, we shall view h and l as two functions defined on $\omega_{n,k}$ in the natural way and study their properties. For $k = 0$, which is when restricted on Ω_n, some interesting properties have been discussed and explored in [12]. For $k \geq 1$, one crucial difference between matrices in Ω_n and those in $\omega_{n,k}$ is the sum of all elements. That is actually how sub-defect is defined originally. If $A \in \omega_{n,k}$, then $\sigma(A)$ is inside the interval $[n-k, n-k+1)$. We explore and show properties of the h and l functions on $\omega_{n,k}$ with respect to the sub-defect k or the sum of all elements of the matrices. We first notice that in $\omega_{n,k}$, the function h is convex while the function l is concave.

PROPOSITION 2.1. *(i) h is a convex function;*
(ii) l is a concave function.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Proof. Let A and B be two nonnegative matrices and $\lambda \in [0,1]$. It is clear that

$$h(\lambda A + (1-\lambda)B) \leq h(\lambda A) + h((1-\lambda)B) = \lambda h(A) + (1-\lambda)h(B)$$

and

$$l(\lambda A + (1-\lambda)B) \geq l(\lambda A) + l((1-\lambda)B) = \lambda l(A) + (1-\lambda)l(B),$$

and hence, the proposition holds. ∎

Let $A \in \omega_n$. It is not hard to see the extreme values of $h(A)$ and $l(A)$ given by the following proposition.

Proposition 2.2. *Let $A \in \omega_n$. Then*

$$0 \leq l(A) \leq \frac{\sigma(A)}{n} \leq h(A) \leq \sigma(A). \tag{2.2}$$

Proof. It is clear that $l(A) \geq 0$ and $h(A) \leq \sigma(A)$. From the covering theorem (Theorem 2.1 in [12]), we can get $l(A) \leq \frac{\sigma(A)}{n} \leq h(A)$, which implies the proposition. ∎

In (2.2), $l(A) = 0$ if and only if A has a zero diagonal, such as partial permutation matrices. On the other hand, $h(A) = \sigma(A)$ if and only if A has only one non-zero diagonal such that the sum of all entries of the diagonal is equal to $\sigma(A)$. For example, let $n-k \leq s < n-k+1$ and A an n by n matrix containing $\lfloor s \rfloor$ 1's and an $s - \lfloor s \rfloor$ on the diagonal as follows.

$$A = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & s-\lfloor s \rfloor & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}.$$

It is easy to check that $h(A) = \sigma(A) = s$. For $A \in \omega_n$, denote $\sigma(A) = s$. Then $l(A) = \frac{s}{n} = h(A)$ if and only if A is in the following form:

$$A = \frac{s}{tn}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $t \geq s$, a positive integer and the first t rows of A are filled up by $\frac{s}{tn}$.

Corollary 2.3. *Let $B \in \omega_{n,k}$. Then*

$$\frac{n-k}{n} \leq h(B) < n-k+1.$$

Proof. This is a direct consequence of Proposition 2.2 and Theorem 1.1, which implies that $n-k \leq \sigma(B) < n-k+1$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
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http://repository.uwyo.edu/ela

REMARK 2.4. From [12], we know that for $A \in \Omega_n$, $h(A) \geq 1$ with equality if and only if $A = \frac{1}{n}J_n$. However, in $\omega_{n,k}$, such an B satisfying $h(B) = \frac{n-k}{n}$ is not unique. For example, we can take $B_1 = \frac{n-k}{n^2}J_n$, and B_2 an n-square matrix with $n-k$ rows filled up by $\frac{1}{n}$'s, i.e.,

$$B_1 = \frac{n-k}{n^2}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By direct computation, we have $\sigma(B_1) = \sigma(B_2) = n-k$ and $h(B_1) = h(B_2) = \frac{n-k}{n}$.

Actually, if $A, B \in \omega_n$, then $AB \in \omega_n$ (Proposition 2.4 in [5]). We can evaluate the extreme values of $h(AB)$ and $l(AB)$.

THEOREM 2.5. *Let $A \in \omega_{n,k}$ and B be an $n \times n$ real matrix with nonnegative entries. Then*

(i) $h(AB) \leq h(B)$;
(ii) $l(B) \leq l(AB)$.

Proof. (i) For reader's convenience, we first prove a special case when $k = 0$, which means $A \in \Omega_n$. The case that both A and B in Ω_n has been proved in [12].

Due to Birkhoff's theorem (see [2], [3] and [10]), we can always write

$$A = \alpha_1 P_1 + \cdots + \alpha_m P_m,$$

where $P_1, \dots, P_m$ are permutation matrices and $\alpha_1 + \cdots + \alpha_m = 1$. It is clear that $h(B) = h(PB)$ for an arbitrary permutation matrix P. Then we have

$$\begin{aligned} h(AB) &= h(\alpha_1 P_1 B + \cdots + \alpha_m P_m B) \\ &\leq \alpha_1 h(P_1 B) + \cdots + \alpha_m h(P_m B) \\ &= \alpha_1 h(B) + \cdots + \alpha_m h(B) \\ &= (\alpha_1 + \cdots + \alpha_m) h(B) \\ &= h(B) \end{aligned} \tag{2.3}$$

in which the inequality sign is due to the convexity of h.

Next we show the inequality holds for any integer $0 \leq k \leq n$ and all $A \in \omega_{n,k}$. Simply, let

$$\tilde{A} = \begin{bmatrix} A & X \\ Y & Z \end{bmatrix} \tag{2.4}$$

be a doubly stochastic matrix containing A as a principal submatrix. (For instance, we can let $\tilde{A}$ be the minimal doubly stochastic completion obtained by the method described in the proof of Theorem 2.1 in [6].) Write

$$\tilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \tag{2.5}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 35, pp. 42-52, February 2019.
http://repository.uwyo.edu/ela

with the same size as $\tilde{A}$. Since $\tilde{A}$ is a doubly stochastic matrix, we can apply (2.3) to $\tilde{A}$ and $\tilde{B}$ to get

$$h(AB) \leq h\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = h(\tilde{A}\tilde{B}) \leq h(\tilde{B}) = h(B).$$

(ii) For $A \in \Omega_n$, replacing h function by l function and using the concavity of l in (2.3), we get

$$l(AB) \geq l(B). \tag{2.6}$$

Then applying (2.6) to $\tilde{A}$ and $\tilde{B}$ defined in (2.4) and (2.5), respectively, we have

$$l(AB) \geq l\left(\begin{bmatrix} AB & 0 \\ YB & 0 \end{bmatrix}\right) = l(\tilde{A}\tilde{B}) \geq l(\tilde{B}) = l(B). \qquad \square$$

Corollary 2.6. *Let $A, B \in \omega_{n,k}$. Then*

(i) $h(AB) \leq \min\{h(A), h(B)\}$;
(ii) $l(AB) \geq \max\{l(A), l(B)\}$.

Remark 2.7. To determine whether the equality in (i) holds, simply let $A = B = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}$. Then we have $AB = A = B$, and therefore, $h(AB) = h(A) = h(B) = \min\{h(A), h(B)\}$.

Remark 2.8. In [12], Wang shows that for $A, B \in \Omega_n$, $h(AB) \leq h(A)h(B)$. However similar result does not hold for $A, B \in \omega_{n,k}$. To see this, simply choose

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \in \omega_{3,2}$$

and B just the transpose of A, i.e., $B = A^t$. Since

$$AB = \begin{bmatrix} \frac{3}{16} & \frac{3}{16} & 0 \\ \frac{3}{16} & \frac{3}{16} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have $h(AB) = \frac{3}{8}$. However $h(A) = h(B) = \frac{1}{2}$, and hence,

$$\frac{3}{8} = h(AB) > h(A)h(B) = \frac{1}{4}.$$

Corollary 2.9. *Let $A \in \omega_{n,k}$. Then*

(i) $h(A^m) \leq h(A)$;
(ii) $l(A^m) \geq l(A)$.

Lemma 2.10. *Let $A, B \in \omega_n$. Then*

$$0 \leq l(AB) \leq \frac{\sigma(A)\sigma(B)}{n^2} \leq h(AB).$$

Proof. The leftmost inequality is trivial.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 35, pp. 42-52, February 2019.
http://repository.uwyo.edu/ela

To show

$$\frac{\sigma(A)\sigma(B)}{n^2} \leq h(AB),$$

let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices in ω_n. First note that $AB = [\sum_{k=1}^n a_{ik}b_{kj}]$. Without loss of generality, assume

$$h(AB) = \sum_{i,j=1}^n a_{ij}b_{ji}.$$

We need to find the minimum value of $h(AB)$ subject to the conditions

$$\sum_{i,j=1}^n a_{ij} = \sigma(A)$$

and

$$\sum_{i,j=1}^n b_{ij} = \sigma(B).$$

We introduce two Lagrange multipliers λ_1 and λ_2, and then construct the Lagrange function H as follows.

$$H = h(AB) - \lambda_1 \left(\sum_{i,j} a_{ij} - \sigma(A) \right) - \lambda_2 \left(\sum_{i,j} b_{ij} - \sigma(B) \right).$$

Using Lagrange multiplier method, we have

$$\frac{\partial H}{\partial a_{ij}} = b_{ji} - \lambda_1 = 0,$$
$$\frac{\partial H}{\partial b_{ij}} = a_{ji} - \lambda_2 = 0,$$
$$\frac{\partial H}{\partial \lambda_1} = \sum_{i,j} a_{ij} - \sigma(A) = 0,$$
$$\frac{\partial H}{\partial \lambda_2} = \sum_{i,j} b_{ij} - \sigma(B) = 0.$$

Solving the system of equations above, we get

$$\sum_{i,j} a_{ij} = n^2\lambda_2 = \sigma(A), \quad \sum_{i,j} b_{ij} = n^2\lambda_1 = \sigma(B),$$
$$a_{ij} = \lambda_2 = \frac{\sigma(A)}{n^2}, \quad b_{ij} = \lambda_1 = \frac{\sigma(B)}{n^2}.$$

Due to the convexity of the function h, we know that

$$h_{min}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}.$$

Similarly, by the method of Lagrange multipliers and the concavity of l function, we can prove that

$$l_{max}(AB) = \frac{\sigma(A)\sigma(B)}{n^2}. \qquad \square$$

Since both l function and h function are well defined on the set of all $n \times n$ real matrices, although $A + B$ is not necessarily in ω_n for $A, B \in \omega_{n,k}$, both $l(A + B)$ and $h(A + B)$ are well defined and we have the following result.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 35, pp. 42-52, February 2019.
http://repository.uwyo.edu/ela

Proposition 2.11. *Let $A, B \in \omega_{n,k}$. Then*

(i) $0 \leq h(A) + h(B) - h(A+B) \leq \min\{h(A), h(B)\} < n - k + 1$*;*

(ii) $l(A+B) - l(A) - l(B) < \dfrac{2(n-k+1)}{n}$.

Proof. (i) Since $h(A+B) \leq h(A) + h(B)$, it is clear that

$$0 \leq h(A) + h(B) - h(A+B) \leq \min\{h(A), h(B)\} < n - k + 1,$$

where the equality implies $h(A+B) = \max\{h(A), h(B)\}$. To see the upper bound is sharp, one can choose such A and B that both contain $n-k$ 1's and an ϵ as follows:

$$A = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \epsilon & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 0 & & \\ & & & & & \ddots & 0 \\ \epsilon & & & & & & 0 \end{bmatrix},$$

where $0 \leq \epsilon < 1$. Then $h(A) = h(B) = h(A+B) = n - k + \epsilon$, letting $\epsilon \to 1$ and we get $\sup_{A,B \in \omega_{n,k}}\{h(A) + h(B) - h(A+B)\} = n - k + 1$.

(ii) Since $\frac{1}{2}(A+B) \in \omega_{n,k}$, we have $l(\frac{A+B}{2}) < \frac{n-k+1}{n}$ or $l(A+B) < \frac{2(n-k+1)}{n}$. With $l(A), l(B) \geq 0$, we get $l(A+B) - l(A) - l(B) < \frac{2(n-k+1)}{n}$. ∎

3. The analogue of the Sylvesters law of the maximum diagonals of matrices in $\omega_{n,k}$. The Sylverster's law of the rank function (2.17.8 in [8]) says that if A is an $m \times t$ real matrix and B an $t \times n$ real matrix, then

$$\max\{\mathrm{rank}(A), \mathrm{rank}(B)\} \leq \mathrm{rank}(A) + \mathrm{rank}(B) - \mathrm{rank}(AB) \leq n.$$

In [12], Wang conjectured the analogue of Sylvester's law of h function on Ω_n, and later on it was proved by Balasubramanian for a more general case using the statement $\mathrm{tr}(A) + \mathrm{tr}(B) - \mathrm{tr}(AB) \leq n$. For further use, we state the result as follows.

Theorem 3.1. (Main Theorem of [1]) *If A, B are $n \times n$ real matrices with all elements in the closed interval $[0,1]$, then*

$$h(A) + h(B) - h(AB) \leq n. \tag{3.7}$$

Also, Balasubramanian gave the conditions for which the equality holds. Based on this theorem, we give two analogues of (3.7) as follows.

Lemma 3.2. *Let $A \in \Omega_n$ and $B \in \omega_n$. Then*

$$1 \leq h(A) + h(B) - h(AB) \leq n,$$

where both the equalities can be tight.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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http://repository.uwyo.edu/ela

Proof. The inequality involving the upper bound is due to Theorem 3.1. To get the equality of the upper bound, simply take A to be any permutation matrix and any $B \in \omega_n$. In this case, $h(AB) = h(B)$, and therefore, $h(A) + h(B) - h(AB) = h(A) = n$.

For the lower bound, it is due to the combination of $h(A) \geq 1$ and Theorem 2.5 (i). Thus, we have

$$h(A) + h(B) - h(AB) \geq h(A) \geq 1.$$

The equality for the lower bound holds when $A = \frac{1}{n} J_n \in \Omega_n$ and $B = \frac{s}{n^2} J_n \in \omega_n$, where $0 < s < n$. In this case, $AB = B$ and then $h(B) = h(AB) = \frac{s}{n}$, which implies that

$$h(A) + h(B) - h(AB) = h(A) = 1. \qquad \square$$

Let $A, B \in \omega_{n,k}$. Then, due to Theorem 3.1 and Corollary 2.6 (i), we have

$$\max\{h(A), h(B)\} \leq h(A) + h(B) - h(AB) \leq n. \tag{3.8}$$

When $k = 0$, i.e., $A, B \in \Omega_n$, both upper bound and lower bound are tight. However, when k is close to n, the upper bound is not tight anymore. In addition, it seems that the lower bound can be more precise with respect to the sub-defect k. So, we explore the role of k and obtain the following theorem for the doubly substochastic matrix case, which is stronger than (3.8).

THEOREM 3.3. *Let $A, B \in \omega_{n,k}$. Then*

$$\frac{n-k}{n} \leq h(A) + h(B) - h(AB) \leq \min\{n, 2(n-k+1)\}.$$

In particular when $k \geq \frac{n}{2} + 1$,

$$\sup_{A,B \in \omega_{n,k}} \{h(A) + h(B) - h(AB)\} = 2(n-k+1).$$

In order to prove Theorem 3.3, we need the following lemma.

LEMMA 3.4. *Let $A \in \omega_{n,k}$. Then we have $h(A) < n-k+1$ and*

$$\sup_{A \in \omega_{n,k}} \{h(A)\} = n-k+1.$$

Proof. Since $A \in \omega_{n,k}$, $\sigma(A) < n-k+1$. It is clear that $h(A) \leq \sigma(A) < n-k+1$. So, $n-k+1$ is an upper bound. To show $n-k+1$ is the least upper bound, one can construct the following diagonal matrix:

$$A_\epsilon = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \epsilon & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \tag{3.9}$$

which contains $n-k$ 1's and an ϵ on the diagonal. For $0 \leq \epsilon < 1$, $A_\epsilon \in \omega_{n,k}$. Note that

$$\lim_{\epsilon \to 1^-} h(A_\epsilon) = n-k+1,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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which means that

$$\sup_{A\in\omega_{n,k}}\{h(A)\} = n-k+1. \qquad \square$$

COROLLARY 3.5. *Let $A \in \omega_{n,k}$. Then there exists an $0 \le \epsilon < 1$, such that*

$$h(A) \le h(A_\epsilon).$$

Proof. It is clear that

$$h(A_\epsilon) = \max\{h(A) : \sigma(A) = \sigma(A_\epsilon), A \in \omega_{n,k}\}.$$

Therefore, the corollary holds. $\square$

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Upper bound. On the one hand, due to Theorem 3.1, A and B satisfy

$$h(A) + h(B) - h(AB) \le n.$$

Since when $0 \le k < \frac{n}{2} + 1$ we have $2(n-k+1) > n$, and therefore, the right hand side inequality in Theorem 3.3 holds. For $k \ge \frac{n}{2}+1$, we have

$$2(n-k+1) \le 2\left(n - (\frac{n}{2}+1) + 1\right) = n.$$

Thus, we need to show that when $k \ge \frac{n}{2}+1$, $h(A) + h(B) - h(AB) \le 2(n-k+1)$. To see this, let A_ϵ be as in (3.9) and B_η be the matrix as follows.

$$B_\eta = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \eta & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

which contains $n-k$ 1's and a nonnegative real number $0 \le \eta < 1$. Since $k \ge \frac{n}{2}+1$, $A_\epsilon B_\eta = \mathbf{0}$, and hence, $h(A_\epsilon B_\eta) = 0$. In addition, due to Corollary 3.5, we have both

$$\max_{A\in\omega_{n,k}} h(A) \le \lim_{\epsilon\to 1^-} h(A_\epsilon) = n-k+1,$$

and

$$\max_{B\in\omega_{n,k}} h(B) \le \lim_{\eta\to 1^-} h(B_\eta) = n-k+1.$$

Therefore, we claim that

$$h(A) + h(B) - h(AB) \le \lim_{\epsilon\to 1^-} h(A_\epsilon) + \lim_{\eta\to 1^-} h(B_\eta) = 2(n-k+1).$$

Lower bound. Due to Corollary 2.3 and Corollary 2.6, we have

$$h(A) + h(B) - h(AB) \ge \max\{h(A), h(B)\} \ge \frac{n-k}{n},$$

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which means that $\frac{n-k}{n}$ is a lower bound. It is tight because one can always let

$$A_0 = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \omega_{n,k}$$

such that all elements in the the first $n-k$ rows are $\frac{1}{n}$ and 0 otherwise. Let $B_0 = A_0^t$. Then

$$A_0B_0 = \frac{1}{n}\begin{bmatrix} J_{n-k} & 0 \\ 0 & 0 \end{bmatrix}.$$

So, we have

$$h(A_0) = h(B_0) = h(A_0B_0) = \frac{n-k}{n},$$

and hence,

$$h(A_0) + h(B_0) - h(A_0B_0) = \frac{n-k}{n}. \qquad \square$$

In [12], the authors also conjectured the analogue of Frobenius inequalities of the rank function (see page 27 in [8]).

Conjecture 3.6. (Conjecture 5.2 of [12]) *Let $A, B, C \in \Omega_n$. Then*

$$h(AB) + h(BC) - h(ABC) \leq h(B).$$

Note that (3.7) is a special case of Conjecture 3.6 by letting B be the identity matrix. Although the Sylvester's law of h function is true and Conjecture 3.6 still remains mysterious to us, it is not true if we replace Ω_n by ω_n in Conjecture 3.6. Here is an example.

Example 3.7. Let

$$A = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$B = A^t$ and $C = A$. Then

$$AB = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad BC = \frac{3}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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and

$$ABC = \frac{3}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, $h(A) = h(B) = h(AB) = h(BC) = \frac{3}{5}$ and $h(ABC) = \frac{9}{25}$, and hence,

$$\frac{21}{25} = h(AB) + h(BC) - h(ABC) > h(B) = \frac{3}{5}.$$

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