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A MODIFIED NEWTON METHOD FOR A MATRIX POLYNOMIAL EQUATION ARISING IN STOCHASTIC PROBLEM*

SANG-HYUP SEO†, JONG-HYEON SEO‡, AND HYUN-MIN KIM†

Abstract. The Newton iteration is considered for a matrix polynomial equation which arises in stochastic problem. In this paper, it is shown that the elementwise minimal nonnegative solution of the matrix polynomial equation can be obtained using Newton's method if the equation satisfies the sufficient condition, and the convergence rate of the iteration is quadratic if the solution is simple. Moreover, it is shown that the convergence rate is at least linear if the solution is non-simple, but a modified Newton method whose iteration number is less than the pure Newton iteration number can be applied. Finally, numerical experiments are given to compare the effectiveness of the modified Newton method and the standard Newton method.

Key words. Matrix polynomial equation, Elementwise positive solution, Elementwise nonnegative solution, M-matrix, Newton's method, Convergence rate, Acceleration of a method.

AMS subject classifications. 65H10.

1. Introduction.

We consider a matrix polynomial equation(MPE) with n-degree defined by

$$P(X) = \sum_{k=0}^{n} A_k X^k = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0, \tag{1.1}$$

where the coefficient matrices A_k's are $m \times m$ matrices. Then, the unknown matrix X must be an $m \times m$ matrix.

The MPE (1.1) often occurs in the theory of differential equations, system theory, network theory, stochastic theory, quasi-birth-and-death and other areas [1–4, 7, 14, 21–23].

Davis [5, 6], and Higham and Kim [15, 16] studied the Newton method for a quadratic matrix equation. Guo and Laub [11] considered a nonsymmetric algebraic Riccati equation, and they proposed iteration algorithms which converge to the minimal positive solution. In [8], Guo provided a sufficient condition for the existence of nonnegative solutions of nonsymmetric algebraic Riccati equations. Kim [20] showed that the minimal positive solutions also can be found by the Newton method with the zero initial matrices in some different types of quadratic equations. Hautphenne, Latouche, and Remiche [12] studied the Newton method for the Markovian binary tree.

Seo and Kim [26, 27] studied the Newton iteration for a quadratic matrix equation and a matrix polynomial equation. Specially, in [26], they provided a relaxed Newton method whose convergence is faster than the pure one. Guo and Lancaster [10] analyzed and provided a modification about Newton's method for algebraic Riccati equations. They showed that the modification of Newton's method is better than the pure one if the minimal nonnegative solution is non-simple.

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Assumption 1.1. For the MPE (1.1), the following hold:

1) The coefficient matrices A_k's are nonnegative except A_1.
2) $-A_1$ is a nonsingular M-matrix.
3) A_0, A_1, and $\sum_{k=2}^{n} A_k$ are irreducible.

The goal of this paper is to propose a modified Newton method of the MPE (1.1) which satisfies Assumption 1.1. This MPE is useful for stochastic theory, quasi-birth-and-death area, and so on. The modified Newton method is better than the pure Newton's method if the elementwise minimal positive solution is non-simple. The idea of the modified Newton method is from the modification of Newton's method for algebraic Riccati equtions of [10]. In [10], Guo and Lancaster showed that $\|Y_{i+1} - S\| < c\varepsilon$ for the modified iteration Y_{i+1}, the solution S, a constant $c > 0$, and small $\varepsilon > 0$. On the other hand, we show that the modified Newton iteration Y_{i+1} for the MPE is closer to the solution S than the pure Newton iteration X_{i+1}.

We start with some basic definitions.

Definition 1.2. Let a matrix $A \in \mathbb{R}^{m\times m}$. A is an *Z-matrix* if all its off-diagonal elements are nonpositive.

It is clear that any Z-matrix A can be written as $sI - B$ with $B \geq 0$ and $s \in \mathbb{R}$. Then M-matrix can be defined as follows.

Definition 1.3. A matrix $A \in \mathbb{R}^{m\times m}$ is an *M-matrix* if $A = rI - B$ for some nonnegative matrix B with $r \geq \rho(B)$ where ρ is the spectral radius; it is a singular M-matrix if $r = \rho(B)$ and a nonsingular M-matrix if $r > \rho(B)$.

The following result is well known and can be found in [9] and [25] for example.

Theorem 1.4. *For a Z-matrix A, the following are equivalent:*

1. *A is a nonsingular M-matrix.*
2. *A^{-1} is nonnegative.*
3. *$Av > 0$ for some vector $v > 0$.*
4. *All eigenvalues of A have positive real parts.*

Definition 1.5. A positive solution S_1 of the matrix equation $P(X) = 0$ is the *elementwise minimal positive solution* and a positive solution S_2 of $P(X) = 0$ is the *elementwise maximal positive solution* if, for any positive solution S of $P(X)$,

$$S_1 \leq S \leq S_2. \tag{1.2}$$

Similarly, if nonnegative solutions S_1 and S_2 satisfy (1.2) for any nonnegative solution S, S_1 is called the *elementwise minimal nonnegative solution* and S_2 is called the *elementwise maximal nonnegative solution*.

Definition 1.6. [18, Definitions 4.2.1 and 4.2.9] The *Kronecker product* of $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{C}^{p\times q}$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}.$$

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The vec operator vec : $\mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ is defined by

$$\mathrm{vec}(A) = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_n^T \end{bmatrix}^T,$$

where $\mathbf{a}_i^T = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni} \end{bmatrix}$.

LEMMA 1.7. [18, Lemma 4.3.1] *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$ be given and let $X \in \mathbb{C}^{n\times p}$ be unknown. The matrix equation*

$$AXB = C \tag{1.3}$$

is equivalent to the system of qm equations in np unknowns given by

$$(B^T \otimes A)\mathrm{vec}(X) = \mathrm{vec}(C), \tag{1.4}$$

that is, $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$.

DEFINITION 1.8. Let a matrix function $F : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ be given, and let a matrix equation

$$F(X) = 0 \tag{1.5}$$

be given. Then, a solution $S \in \mathbb{C}^{m\times n}$ of (1.5) is called *simple* if the Fréchet derivative of F at S is nonsingular.

To reach our goal, in Section 2, we study the minimal nonnegative solution S of (1.1) and the Fréchet derivative of P in (1.1) at S, and we show the convergence of the Newton iteration for (1.1). We give an analysis about Newton's method for the non-simple minimal nonnegative solution S, in Section 3. In Section 4, we propose a modified Newton method which is better for finding the minimal nonnegative solution S. Finally, we give some numerical experiments, in Section 5.

For convenience, the notation $||\cdot||$ is used instead of the Frobenius norm $||\cdot||_F$ and $\mathbb{N}_0$ is used as $\mathbb{N}\cup\{0\}$ because the Frobenius norm and $\mathbb{N}_0$ are used very frequently in this paper.

2. Convergence of Newton's method for an MPE. In this section, we introduce a sufficient condition of the existence of the minimal nonnegative solution of the MPE (1.1) with Assumption 1.1, and give some analysis for Newton's method.

THEOREM 2.1. [24, Theorem 2.1] *Let the MPE (1.1) with 1) and 2) in Assumption 1.1 be given. Then, there exists the minimal nonnegative solution if*

$$B = -\sum_{k=0}^{n} A_k \text{ is a nonsingular or singular irreducible } M\text{-matrix.} \tag{2.6}$$

The Fréchet derivative of the matrix polynomial equation (1.1) at X in the direction H is given by

$$P'_X(H) = \sum_{k=1}^{n}\sum_{l=0}^{k-1} A_k X^l H X^{k-l-1}. \tag{2.7}$$

The second Fréchet derivative of the quadratic matrix equation (1.1) at X is given by

$$P''_X(K,H) = \sum_{k=2}^{n}\sum_{l=0}^{k-2}\sum_{j=0}^{l} A_k \left(X^l H X^j K X^{n-l-j-2} + X^l K X^j H X^{n-l-j-2}\right). \tag{2.8}$$

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For the equation (1.1), each step of the Newton iteration with given X_0 can be written as

$$X_{i+1} = X_i - P_{X_i}'^{-1}(P(X_i)) \tag{2.9}$$

if P'_{X_i} is invertible for all $i \in \mathbb{N}_0$.

(2.9) can be separated into two parts as

$$\begin{cases} P'_{X_i}(H_i) = -P(X_i), \\ X_{i+1} = X_i + H_i, \end{cases} \qquad i = 1, 2, \dots \tag{2.10}$$

The general approach for solving (2.10) is to solve the $m^2 \times m^2$ linear system derived by Lemma 1.7 such as

$$\mathcal{P}'_{X_i}\mathrm{vec}(H_i) = \mathrm{vec}(-P(X_i)),$$

where

$$\mathcal{P}'_X = \sum_{k=1}^{n}\sum_{l=0}^{k-1}(X^{k-l-1})^T \otimes A_k X^l. \tag{2.11}$$

Theorem 2.2. *Suppose that the MPE (1.1) satisfies Assumption 1.1 and (2.6). Then, the Newton sequence $\{X_i\}$ with $X_0 = 0$ is well defined, is monotone nondecreasing, and converges to the elementwise minimal positive solution S. Furthermore, $-\mathcal{P}'_{X_i}$ is a nonsingular irreducible M-matrix for $i \in \mathbb{N}$, and $-\mathcal{P}'_S$ is an irreducible M-matrix.*

Proof. According to the proof of [24, Theorem 2.1], the elementwise minimal nonnegative solution S of (1.1) is the limit of the monotone nondecreasing sequence $\{X_i^G\}_{i=0}^{\infty}$ which is defined by

$$\begin{cases} X_{i+1}^G = G(X_i^G), \\ X_0^G = 0, \end{cases}$$

where

$$G(X) = -A_1^{-1}\left(\sum_{k=2}^{n} A_k X^k + A_0\right).$$

Since $X_1^G = -A_1^{-1}A_0 > 0$, the solution S is also positive. Thus, $S \in \{Y \in \mathbb{R}^{m\times m} | Y > 0, \;\; F(Y) \le 0\}$. From [26, Theorem 2.9], the Newton sequence $\{X_i\}$ with $X_0 = 0$ is well-defined, monotone nondecreasing, and converges to the elementwise minimal positive solution S. Moreover,

$$-\mathcal{P}'_{X_i} = -\sum_{k=1}^{n}\sum_{l=0}^{k-1}(X_i^{k-l-1})^T \otimes A_k X_i^l$$

is a nonsingular M-matrix for each $i \in \mathbb{N}_0$.

Now, it is sufficient to show that $-\mathcal{P}'_{X_i}$ is irreducible for $i \in \mathbb{N}$. Since $X_1 = -A_1^{-1}A_0 > 0$ and $A_p \ge 0$ for all $p \ge 2$,

$$\sum_{k=2}^{n}\sum_{l=0}^{k-1}(X_i^{k-l-1})^T \otimes A_k X_i^l \ge \sum_{k=2}^{n}(X_i^{k-1})^T \otimes A_k \ge 0.$$

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From that $X_i > 0$, we obtain that $\sum_{k=2}^{n}(X_i^{k-1})^T \otimes A_k$ is irreducible if and only if $\sum_{k=2}^{n} \mathbf{1}_{m\times m} \otimes A_k = \mathbf{1}_{m\times m} \otimes (\sum_{k=2}^{n} A_k)$ is irreducible. Since $\sum_{k=2}^{n} A_k$ is irreducible,

$$\sum_{k=2}^{n}\sum_{l=0}^{k-1}(X_i^{k-l-1})^T \otimes A_k X_i^l$$

is an irreducible nonnegative matrix. Therefore,

$$-\mathcal{P}'_{X_i} = -I_m \otimes A_1 - \sum_{k=2}^{n}\sum_{l=0}^{k-1}(X_i^{k-l-1})^T \otimes A_k X_i^l$$

is also irreducible. □

The next theorem follows directly from Theorem 2.2 and the well known local quadratic convergence of Newton's method.

Theorem 2.3. *If the matrix $-\mathcal{P}'_S$ in Theorem 2.2 is nonsingular, then for $X_0 = 0$, the Newton sequence $\{X_i\}$ converges to S, quadratically.*

3. Analysis for the singular M-matrix $-\mathcal{P}'_S$. According to Theorem 2.3, the Newton iteration (2.9) converges quadratically if $-\mathcal{P}'_S$ is nonsingular. In this section, we will see the convergence rate of (2.9) when $-\mathcal{P}'_S$ is a singular M-matrix. If P'_S is not invertible, then P'_S has a null space $\mathcal{N} = \mathrm{Ker}(P'_S)$ and closed range $\mathcal{M} = \mathrm{Im}(P'_S)$. Suppose that the direct sum $\mathcal{N} \oplus \mathcal{M} = \mathbb{R}^{m\times m}$. Then, we can define $\mathbf{P}_\mathcal{N}$ to be the projection onto $\mathcal{N}$ parallel to $\mathcal{M}$ and $\mathbf{P}_\mathcal{M} = I - \mathbf{P}_\mathcal{N}$. For a nonzero matrix $N_0 \in \mathcal{N}$, define the map $\mathcal{B}_{N_0} : \mathcal{N} \to \mathcal{N}$ given by

$$\mathcal{B}_{N_0}(N) = \mathbf{P}_\mathcal{N} P''_S(N_0, N). \tag{3.12}$$

In fact, $\mathcal{B}_{N_0}$ is a linear map. The main result of this section is an application of the following theorem which shows the local convergence and the convergence rate of Newton's method under some conditions.

Theorem 3.1. (cf. [10, Theorem 1.5], [19, Theorem 1.1]) *Let $\mathcal{B}_{N_0}$ in (3.12) be invertible for some nonzero $N_0 \in \mathcal{N}$, $\mathcal{N} = \mathrm{span}\{N_0\} \oplus \mathcal{N}_1$ for some subspace $\mathcal{N}_1$, and let*

$$W(\rho,\theta,\eta) = \left\{ X \;\middle|\; \begin{array}{l} 0 < \|X - S\| < \rho, \|\mathbf{P}_\mathcal{M}(X-S)\| \le \theta\|\mathbf{P}_\mathcal{N}(X-S)\|, \\ \|(\mathbf{P}_\mathcal{N} - \mathbf{P}_0)(X-S)\| \le \eta\|\mathbf{P}_\mathcal{N}(X-S)\| \end{array} \right\}, \tag{3.13}$$

where $\mathbf{P}_0$ is the projection onto $\mathrm{span}\{N_0\}$ parallel to $\mathcal{N}_1 \oplus \mathcal{M}$. If $X_0 \in W(\rho_0,\theta_0,\eta_0)$ for ρ_0,θ_0,η_0 sufficiently small, then the Newton sequence $\{X_i\}$ is well defined and $\|P'^{-1}_{X_i}\| \le c\|X_i - S\|^{-1}$ for all $i \ge 1$ and some constant $c > 0$. Moreover,

$$\lim_{i\to\infty}\frac{\|X_{i+1}-S\|}{\|X_i - S\|} = \frac{1}{2}, \qquad \lim_{i\to\infty}\frac{\|\mathbf{P}_\mathcal{M}(X_i-S)\|}{\|\mathbf{P}_\mathcal{N}(X_i-S)\|^2} = 0.$$

To analyze convergence of Newton's method when $-\mathcal{P}'_S$ is singular, we will show that (1.1) satisfies the conditions of Theorem 3.1. From this point, for convenience, we let $\tilde{X}_i = X_i - S$.

Lemma 3.2. *Suppose the matrix polynomial equation (1.1) satisfies Assumption 1.1. If the matrix $-\mathcal{P}'_S$ is a singular M-matrix, then 0 is a simple eigenvalue of $-\mathcal{P}'_S$, $\mathcal{N} \oplus \mathcal{M} = \mathbb{R}^{m\times m}$, $\mathcal{N}$ is one-dimensional and the map $\mathcal{B}_{N_0}$ is invertible for some nonzero $N_0 \in \mathcal{N}$.*

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Proof. Since S is positive and A_k's are irreducible, $-\mathcal{P}'_S$ is irreducible. Then, by Perron-Frobenius Theorem [17, Theorem 8.4.4], 0 is a simple eigenvalue of $\mathcal{P}'_S$ with a positive eigenvector. Thus, we can find n^2 linearly independent vectors $\chi_1, \chi_2, \ldots, \chi_{n^2}$ such that $\chi_1 > 0$ and

$$\mathcal{X}^{-1}\mathcal{P}'_S\mathcal{X} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D} \end{bmatrix}, \text{ where } \mathcal{X} = \left[\begin{array}{c|c|c|c} \chi_1 & \chi_2 & \cdots & \chi_{n^2} \end{array}\right] \tag{3.14}$$

and $\mathcal{D}$ is an $(n^2-1)\times(n^2-1)$ nonsingular matrix. By the same way, we also have a positive vector ψ such that $\psi^T\mathcal{P}'_S = 0$. Now, $P'_S(N) = 0$ if and only if $\mathcal{P}'_S\mathrm{vec}(N) = 0$. From (3.14), $\mathcal{P}'_S\mathrm{vec}(N) = 0$ if and only if $\mathrm{vec}(N) \in \mathrm{span}(\chi_1)$, in which case we write $N = a\mathrm{vec}^{-1}(\chi_1)$ for some nonzero $a \in \mathbb{R}$. Thus, $\mathcal{N} = \mathrm{span}(\mathrm{vec}^{-1}(\chi_1))$. Similarly, $\mathcal{M} = \mathrm{span}(\mathrm{vec}^{-1}(\chi_2), \ldots, \mathrm{vec}^{-1}(\chi_{n^2}))$. Therefore, $\mathcal{N}$ is one-dimensional and $\mathbb{R}^{m\times m} = \mathcal{N} \oplus \mathcal{M}$.

To prove the map $\mathcal{B}_{N_0}$ is invertible for a nonzero matrix $N_0 \in \mathcal{N}$, we only need to show that

$$\mathbf{P}_{\mathcal{N}}\left(P''_S(N_0, N)\right) \neq 0,$$

for all nonzero $N \in \mathcal{N}$ because $\mathcal{B}_{N_0}$ is linear and $\mathcal{N}$ is one-dimensional. Since $\mathrm{vec}^{-1}(\chi_1) > 0$ and $S > 0$, we have

$$\begin{aligned} P''_S(N_0, N) &= \sum_{k=2}^{n}\sum_{l=0}^{k-2}\sum_{j=0}^{l} A_k\left(S^l N_0 S^j N S^{n-l-j-2} + S^l N S^j N_0 S^{n-l-j-2}\right) \\ &= 2ab\sum_{k=2}^{n}\sum_{l=0}^{k-2}\sum_{j=0}^{l} A_k\left(S^l \mathrm{vec}^{-1}(\chi_1) S^j \mathrm{vec}^{-1}(\chi_1) S^{n-l-j-2}\right) \neq 0, \end{aligned}$$

where $N = a\mathrm{vec}^{-1}(\chi_1)$ and $N_0 = b\mathrm{vec}^{-1}(\chi_1)$. Moreover, $P''_S(N_0, N)$ is either positive or negative.

On the other hand,

$$\mathrm{vec}\left(P''_S(N_0, N)\right) = k_1\chi_1 + k_2\chi_2 + \cdots + k_{n^2}\chi_{n^2}$$

for some real numbers $k_1, k_2, \ldots, k_{n^2}$. By Fundamental theorem of linear algebra in [28] and Lemma 6.3.10 in [17], we have

$$\psi^T\mathrm{vec}\left(P''_S(N_0, N)\right) = k_1\psi^T\chi_1. \tag{3.15}$$

Since $P''_S(N_0, N)$ is either positive or negative and ψ is positive, the left side of (3.15) is also either positive or negative. So, k_1 cannot be zero. Therefore,

$$\mathbf{P}_{\mathcal{N}}\left(P''_S(N_0, N)\right) = k_1\mathrm{vec}^{-1}(\chi_1) \neq 0. \qquad \square$$

Lemma 3.3. *Let S be the minimal positive solution of (1.1) with Assumption 1.1, and let $\{X_i\}_{i=0}^{\infty}$ be a Newton sequence in (2.9). Then,*

$$\|P(X_i)\| \leq a\|\tilde{X}_i\|^2 + b\|\tilde{X}_i\|\|\tilde{X}_{i-1}\| + c\|\tilde{X}_{i-1}\|^2$$

for some positive real numbers a, b, c.

Proof. From Taylor's Theorem and putting $S = X_{i-1} - \tilde{X}_{i-1}$, we have

$$P(X_i) = P(S) + P'_S(X_i - S) + O(\|X_i - S\|^2), \tag{3.16}$$

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and

$$0 = P(S) = P(X_{i-1} - \tilde{X}_{i-1}) = P(X_{i-1}) - P'_{X_{i-1}}(\tilde{X}_{i-1}) + O(\|\tilde{X}_{i-1}\|^2),$$

which is equivalent to

$$-P(X_{i-1}) + P'_{X_{i-1}}(\tilde{X}_{i-1}) = O(\|\tilde{X}_{i-1}\|^2). \tag{3.17}$$

From (2.10), we have

$$0 = P'_{X_{i-1}}(X_i - X_{i-1}) + P(X_{i-1}), \tag{3.18}$$

and clearly,

$$X_i - X_{i-1} = \tilde{X}_i - \tilde{X}_{i-1}. \tag{3.19}$$

If we subtract (3.18) from (3.16) and substitute (3.17), we obtain

$$\begin{aligned} P(X_i) &= P(S) + P'_S(\tilde{X}_i) - P(X_{i-1}) - P'_{X_{i-1}}(\tilde{X}_i - \tilde{X}_{i-1}) + O(\|\tilde{X}_i\|^2) \\ &= P'_S(\tilde{X}_i) - P'_{X_{i-1}}(\tilde{X}_i) + O(\|\tilde{X}_i\|^2) + O(\|\tilde{X}_{i-1}\|^2). \end{aligned}$$

Putting $S = X_{i-1} - \tilde{X}_{i-1}$ in the previous equality,

$$\begin{aligned} P(X_i) &= \sum_{k=1}^{n}\sum_{l=0}^{k-1} A_k (X_{i-1} - \tilde{X}_{i-1})^l \tilde{X}_i (X_{i-1} - \tilde{X}_{i-1})^{k-l-1} \\ &\qquad\qquad - P'_{X_{i-1}}(\tilde{X}_i) + O(\|\tilde{X}_i\|^2) + O(\|\tilde{X}_{i-1}\|^2) \\ &= \sum_{k=1}^{n}\sum_{l=0}^{k-1} A_k X_{i-1}^l \tilde{X}_i X_{i-1}^{k-l-1} + O(\|\tilde{X}_i\|\|\tilde{X}_{i-1}\|) \\ &\qquad\qquad - P'_{X_{i-1}}(\tilde{X}_i) + O(\|\tilde{X}_i\|^2) + O(\|\tilde{X}_{i-1}\|^2) \\ &= P'_{X_{i-1}}(\tilde{X}_i) + O(\|\tilde{X}_i\|\|\tilde{X}_{i-1}\|) - P'_{X_{i-1}}(\tilde{X}_i) + O(\|\tilde{X}_i\|^2) + O(\|\tilde{X}_{i-1}\|^2) \\ &= O(\|\tilde{X}_{i-1}\|^2) + O(\|\tilde{X}_i\|\|\tilde{X}_{i-1}\|) + O(\|\tilde{X}_i\|^2). \end{aligned}$$

Since $\|\cdot\|$ is a multiplicative matrix norm on $\mathbb{R}^{m\times m}$, we have required result. ∎

LEMMA 3.4. *For any fixed $\theta > 0$, let*

$$\mathcal{Q} = \{i \in \mathbb{N} \cup \{0\} | \|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| > \theta \|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|\}$$

where $\{X_i\}$ is the Newton sequence in Theorem 2.2. Then, there exist an integer i_0 and a constant $c > 0$ such that $\|\tilde{X}_i\| \le c\|\tilde{X}_{i-1}\|^2$ for all $i \ge i_0$ in $\mathcal{Q}$.

Proof. Using Taylor's Theorem and the fact that $P'_S\left(\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\right) = 0$,

$$P(X_i) = P(S) + P'_S(\tilde{X}_i) + O(\|\tilde{X}_i\|^2) = P'_S\left(\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\right) + O(\|\tilde{X}_i\|^2). \tag{3.20}$$

Since $P'_S|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is invertible, $\left\|P'_S\left(\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\right)\right\| \ge c_1 \|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\|$ for some constant $c_1 > 0$. For $i \in \mathcal{Q}$, we have

$$\|\tilde{X}_i\| \le \left\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\right\| + \left\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\right\| \le (\theta^{-1} + 1)\left\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\right\|.$$

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Thus, by (3.20),

$$\|P(X_i)\| \geq c_1\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| - c_2\|\tilde{X}_i\|^2 \geq c_1(\theta^{-1}+1)^{-1}\|\tilde{X}_i\| - c_2\|\tilde{X}_i\|^2. \tag{3.21}$$

On the other hand, from Lemma 3.3, we have

$$\|P(X_i)\| \leq c_3\|\tilde{X}_i\|^2 + c_4\|\tilde{X}_{i-1}\|\|\tilde{X}_i\| + c_5\|\tilde{X}_{i-1}\|^2.$$

From (3.21) and the fact that $X_i \neq S$ for any i, we have

$$c_1(\theta^{-1}+1)^{-1} - c_2\|\tilde{X}_i\| \leq c_3\|\tilde{X}_i\| + c_4\|\tilde{X}_{i-1}\| + c_5\frac{\|\tilde{X}_{i-1}\|^2}{\|\tilde{X}_i\|}.$$

Since $\tilde{X}_i$ converges to 0 by Theorem 2.2, we can find an i_0 such that $\|\tilde{X}_i\| \leq c\|\tilde{X}_{i-1}\|^2$ for all $i \geq i_0$. ◻

Corollary 3.5. *Assume that, for given $\theta > 0$, $\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| > \theta\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|$ for all i large enough. Then $X_i \to S$ quadratically.*

When P'_S is singular practically the Newton sequence converges linearly, according to the corollary we conclude that *the error will generally be dominated by its $\mathcal{N}$ component* [10]. From Lemmas 3.2 and 3.4 we have the following main theorem.

Theorem 3.6. *If $-\mathcal{P}'_S$ is a singular M-matrix and the convergence rate of the Newton sequence $\{X_i\}$ in Theorem 2.2 is not quadratic, then $\|P'^{-1}_{X_i}\| \leq c\|\tilde{X}_i\|^{-1}$ for all $i \geq 1$ and some constant $c > 0$. Moreover,*

$$\lim_{i\to\infty}\frac{\|\tilde{X}_{i+1}\|}{\|\tilde{X}_i\|} = \frac{1}{2}, \qquad \lim_{i\to\infty}\frac{\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\|}{\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|^2} = 0.$$

4. A modified Newton method.

Under the conditions in Theorem 3.6, the convergence rate of the Newton sequence is 1/2. Furthermore, $\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\|/\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|^2$ converges to 0, i.e., $\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_{i_0})\| < \varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i_0})\|$ holds for sufficiently small $\varepsilon > 0$ and large integer i_0 to make $\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i_0})\| < 1$. Intuitivley, we understand that $\mathbf{P}_{\mathcal{M}}(\tilde{X}_{i_0})$ is almost terminated, and $\{\tilde{X}_i\}_{i=i_0}^{\infty}$ is located near a one-dimensional subspace $\mathcal{N}$. Then, we will give a modified Newton method which has faster convergence than the pure one.

Lemma 4.1. *Let $\{X_i\}$ be the Newton sequence in Theorem 2.2, and let the derivative P'_S be singular. Suppose that there exists $i_0 \in \mathbb{N}_0$ such that $i \geq i_0$ implies that*

$$\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| < \varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|, \tag{4.22}$$

for $0 < \varepsilon \leq 1$. Then, $\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)$ is negative for $i \geq i_0$.

Proof. From the proof of Lemma 3.2, $V_1 := \mathrm{vec}^{-1}(\chi_1)$ is a positive basis for $\mathcal{N}$. Let $c_{i,1}$ be a scalar such that $\mathbf{P}_{\mathcal{N}}(\tilde{X}_i) = c_{i,1}V_1$. Suppose that $c_{i,1} \geq 0$. Then, $\mathbf{P}_{\mathcal{N}}(\tilde{X}_i) \geq 0$. Since $\mathbf{P}_{\mathcal{N}}(\tilde{X}_i) + \mathbf{P}_{\mathcal{M}}(\tilde{X}_i) = \tilde{X}_i \leq 0$, we obtain that $0 \leq \mathbf{P}_{\mathcal{N}}(\tilde{X}_i) \leq -\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)$. Thus,

$$\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| \leq \|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| < \varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|. \tag{4.23}$$

It means that $\varepsilon > 1$, i.e., it contradicts the hypothesis. Therefore, $c_{i,1} < 0$ and $\mathbf{P}_{\mathcal{N}}(\tilde{X}_i) < 0$. ◻

Consider a polynomial $f(x) = px^3 + 2px^2 + (9p+1)x - 1$ for $p > 0$. Since $f(0) = -1$ and $f(1) = 12p$, f has a root in the interval $(0,1)$. From that $f'(x) = 3px^2 + 4px + 9p + 1 > 0$ for all $x > 0$, f is monotone increasing in $(0,1)$, i.e., the root t of f in $(0,1)$ is unique. Hence, for $x \in (0,t)$, it holds that

$$\frac{9x + 2x^2 + x^3}{1-x} < \frac{1}{p}. \tag{4.24}$$

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The previous inequality is useful to prove the following theorem.

THEOREM 4.2. *Let* $Y_{i+1} = X_i - 2P'^{-1}_{X_i}(P(X_i))$, $p = \|\mathbf{P}_{\mathcal{N}}\|$, *and let* $\varepsilon \in (0,t)$ *be given where* t *is the real root of* $f(x) = px^3 + 2px^2 + (9p+1)x - 1$ *in* $(0,1)$. *Suppose that* $i \geq i_0$ *implies that*

$$\left| \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} - 2 \right| < \varepsilon, \qquad \|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| < \varepsilon \|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| \tag{4.25}$$

for some $i_0 \in \mathbb{N}_0$. *Then, for* $i \geq i_0$,

$$\|Y_{i+1} - S\| < \|\tilde{X}_{i+1}\|. \tag{4.26}$$

Proof. From the definition of Y_{i+1}, we get that

$$\begin{aligned} Y_{i+1} - S &= X_i - 2P'^{-1}_{X_i}(P(X_i)) - S \\ &= 2\left(X_i - P'^{-1}_{X_i}(P(X_i))\right) - X_i - S \\ &= 2X_{i+1} - X_i - 2S + S \\ &= 2\tilde{X}_{i+1} - \tilde{X}_i. \end{aligned}$$

So, we will show that $\|2\tilde{X}_{i+1} - \tilde{X}_i\| < \|\tilde{X}_{i+1}\|$. From the hypothesis, we obtain that

$$(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| < \|\tilde{X}_i\| < (1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|. \tag{4.27}$$

It yields two following inequalities,

$$\begin{aligned} 2 - \varepsilon < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < \frac{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|}, \\ \frac{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|} < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < 2 + \varepsilon. \end{aligned} \tag{4.28}$$

From the previous two inequalities, we get the inequality

$$\frac{(2-\varepsilon)(1-\varepsilon)}{1+\varepsilon} < \frac{\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|} = \frac{c_{i,1}}{c_{i+1,1}} < \frac{(2+\varepsilon)(1+\varepsilon)}{1-\varepsilon}, \tag{4.29}$$

where $c_{i,1}$ and $c_{i+1,1}$ are scalars which are in the proof of Lemma 4.1, i.e., they are negative. It is obtained that

$$\frac{-5\varepsilon + \varepsilon^2}{1+\varepsilon}(-c_{i+1,1}) < 2c_{i+1,1} - c_{i,1} < \frac{5\varepsilon + \varepsilon^2}{1-\varepsilon}(-c_{i+1,1}). \tag{4.30}$$

Since $\dfrac{5\varepsilon + \varepsilon^2}{1-\varepsilon} - \dfrac{5\varepsilon - \varepsilon^2}{1+\varepsilon}$ is positive for $0 < \varepsilon < 1$,

$$|2c_{i+1,1} - c_{i,1}| < \frac{5\varepsilon + \varepsilon^2}{1-\varepsilon}|c_{i+1,1}|. \tag{4.31}$$

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Therefore, we get the following inequality,

$$
\begin{aligned}
\|2\tilde{X}_{i+1}-\tilde{X}_i\| &= \|2\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})-\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)+2\mathbf{P}_{\mathcal{M}}(\tilde{X}_{i+1})-\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| \\
&\le \|2\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})-\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|+\|2\mathbf{P}_{\mathcal{M}}(\tilde{X}_{i+1})\|+\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| \\
&< \|2\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})-\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|+2\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|+\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| \\
&= |2c_{i+1,1}-c_{i,1}|\|V_1\|+2\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|+\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| \qquad (4.32)\\
&< \frac{5\varepsilon+\varepsilon^2}{1-\varepsilon}|c_{i+1,1}|\|V_1\|+2\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|+\frac{(2+\varepsilon)(1+\varepsilon)}{1-\varepsilon}\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\| \\
&= \frac{5\varepsilon+\varepsilon^2}{1-\varepsilon}\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|+2\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|+\frac{(2+\varepsilon)(1+\varepsilon)}{1-\varepsilon}\varepsilon\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\| \\
&= \frac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon}\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\| \\
&\le \frac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon}p\|\tilde{X}_{i+1}\|.
\end{aligned}
$$

Since $\varepsilon \in (0,t)$, $\dfrac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon} < \dfrac{1}{p}$ holds,

$$\|Y_{i+1}-S\| = \|2\tilde{X}_{i+1}-\tilde{X}_i\| < \|\tilde{X}_{i+1}\|. \qquad \square \tag{4.33}$$

The theoretical result in Theorem 4.2 suggests a modified Newton method, as in [10] for algebraic Riccati equations. The main ideas of the algorithm are that we choose X_{i+1} as the next step of X_i if $\|P(Y_{i+1})\| \ge \eta$ for given tolerance η and the iteration is terminated if $\|P(X_{i+1})\| < \eta$ or $\|P(Y_{i+1})\| < \eta$.

Algorithm 4.3. The Modified Newton Method for the Matrix Polynomial Equations for the given tolerance η.

1. $X_0 \leftarrow 0$;
2. Calculate H such that $\mathcal{P}'_{X_0}\mathrm{vec}(H) = -\mathrm{vec}(P(X_0))$;
3. $X_0 \leftarrow X_0 + 2H$;
4. If $\|P(X_0)\| < \eta$, then go to step 7;
5. $X_0 \leftarrow X_0 - H$;
6. If $\|P(X_0)\| \ge \eta$, then go to step 2;
7. $S \leftarrow X_0$.

5. Numerical experiments. In this section, we compare the effectiveness of the modified Newton method and the pure one for the MPE. The experiments are made with MATLAB R2016a. The tolerance of the algorithm of Newton's method is given by $\eta = m \times 10^{-16}$ and we will stop the iteration if $\|P(X_{i+1})\| < \eta$.

Example 5.1. This example is given to check Theorem 4.2, theoretically. So, we give the example whose solution is easy to be found, and the calculations of the iterations in the experiments are computed up to 100 digits with `vpa` function. Let an MPE (1.1) with degree $n = 6$ be given with the following coefficients

$$\begin{cases} A_k = a_k W, & \text{for } k = 0,2,3,4,5, \\ A_1 = a_1 W - I_m, & \\ A_6 = W, & \end{cases} \tag{5.34}$$

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where

$$W = \frac{1}{6200(m-1)}(\mathbf{1}_{m\times m} - I_m) \quad \text{and} \quad \begin{cases} a_0 = 4096, & a_1 = 56, \\ a_2 = 384, & a_3 = 1312, \\ a_4 = 321, & a_5 = 30. \end{cases} \tag{5.35}$$

Then, A_k's satisfy Assumption 1.1 and (2.6). Hence, the MPE has the minimal positive solution S.

Let $m = 3$. Then, the minimal nonnegative solution

$$S = \begin{bmatrix} \frac{2r+1}{3} & \frac{1-r}{3} & \frac{1-r}{3} \\ \frac{1-r}{3} & \frac{2r+1}{3} & \frac{1-r}{3} \\ \frac{1-r}{3} & \frac{1-r}{3} & \frac{2r+1}{3} \end{bmatrix},$$

where $r \approx -0.3287191$ which is the nearest real root to 0 of the equation $x^6 + 30x^5 + 321x^4 + 1312x^3 + 384x^2 + 12456x + 4096 = 0$. Furthermore, $-\mathcal{P}'_S$ is a singular M-matrix. Therefore, we can calculate $\mathbf{P}_\mathcal{N}$ and $\mathbf{P}_\mathcal{M}$, easily. In this example, in fact, $-\mathcal{P}'_S$ is symmetric, so, $\mathbf{P}_\mathcal{N}$ and $\mathbf{P}_\mathcal{M}$ are orthogonal projections, i.e., $p = \|\mathbf{P}_\mathcal{N}\| = 1$. Thus, $t \approx 0.097985683$ is the real root of $x^3 + 2x^2 + 10x - 1$.

If we calculate the Newton iteration $\{X_i\}_{i=1}^\infty$ in (2.9) with $X_0 = 0$ and $\{Y_i\}_{i=1}^\infty$ in Theorem 4.2, then we obtain Figures 1, 2, 3 as results of $\left|\frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} - 2\right|$, $\|\mathbf{P}_\mathcal{N}(\tilde{X}_i)\|$, $\|\mathbf{P}_\mathcal{M}(\tilde{X}_i)\|$, $\|X_i - S\|$, and $\|Y_i - S\|$. Figure 1 shows that $\left|\frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} - 2\right| < t$ if $i \geq 2$. From Figure 2, we get that $\|\mathbf{P}_\mathcal{M}(\tilde{X}_i)\| < t\|\mathbf{P}_\mathcal{N}(\tilde{X}_i)\|$ when $i \geq 1$. Finally, we see that $\|Y_i - S\| < \|X_i - S\|$ for $i \geq 2$ through Figure 3.

This experiment shows that, for given i, Y_i is closer to the solution than X_i if it satisfies the conditions of Theorem 4.2. Furthermore, we obtain that Y_i's for $i \geq 13$ are closer to the solution than any X_i's. It means that we do not need to compute X_i for $i \geq 13$ since Y_{13} is thought to be a "numerical" minimal nonnegative solution which is close sufficiently to the "mathematical" minimal nonnegative solution.

Example 5.2. (cf. [13, Example 1]) In this example, we use Algorithm 4.3 for the following MPEs and $m = 8$. Let MPEs

$$Q(X) = W_0 + (W_1 - I_m)X + W_2X^2 = 0, \tag{5.36}$$

$$R(X) = W_3 + (W_4 - I_m)X + W_5X^2 = 0 \tag{5.37}$$

be given, where W_k is a random nonnegative matrix which has null diagonal entries and positive off-diagonal entries such that $W_k\mathbf{1}_m = \frac{1}{3}\mathbf{1}_m$ for $k = 0, 1, 2$, $W_3\mathbf{1}_m = \frac{1}{2}\mathbf{1}_m$, and $W_k\mathbf{1}_m = \frac{1}{4}\mathbf{1}_m$ for $k = 4, 5$. Since $I - \sum_{k=0}^{2} W_k$ and $I - \sum_{k=3}^{5} W_k$ are singular irreducible M-matrices and the coefficients satisfy Assumption 1.1, (5.36) and (5.37) have the minimal positive solutions.

We run the pure Newton algorithm(PNA) and the modified Newton algorithm(MNA) 300 times with $X_0 = 0$, for (5.36) and (5.37) respectively. All of the 300 experiments are run with different coefficients. Averages of the iteration numbers are given in Table 1.

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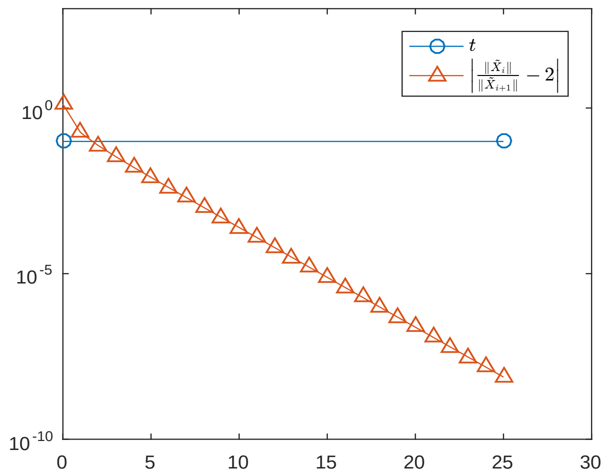

FIGURE 1. *Checking which i satisfies* $\left|\dfrac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|}-2\right| < t$.

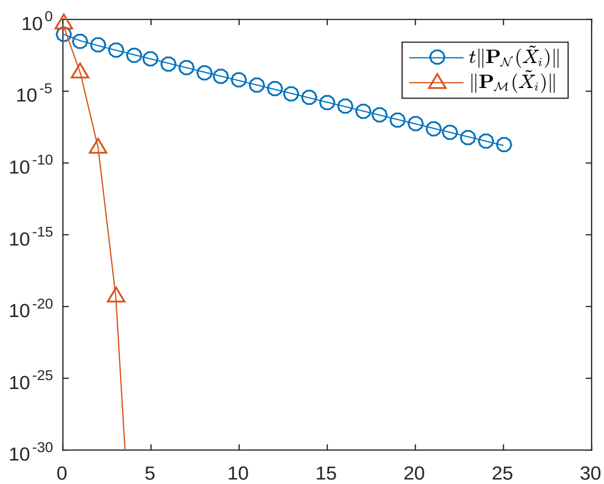

FIGURE 2. *Checking which i satisfies* $\|\mathbf{P}_{\mathcal{M}}(\tilde{X}_i)\| < t\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|$.

	PNA	MNA
$Q(X)=0$	26	10.99
$R(X)=0$	7	7

TABLE 1
Averages of the iteration numbers.

For (5.36), MNA finds the solution faster than PNA. But, for (5.37), the iteration numbers of MNA and PNA are same. Figure 4 shows that the solution S_Q of (5.36) is numerically non-simple and the solution S_R

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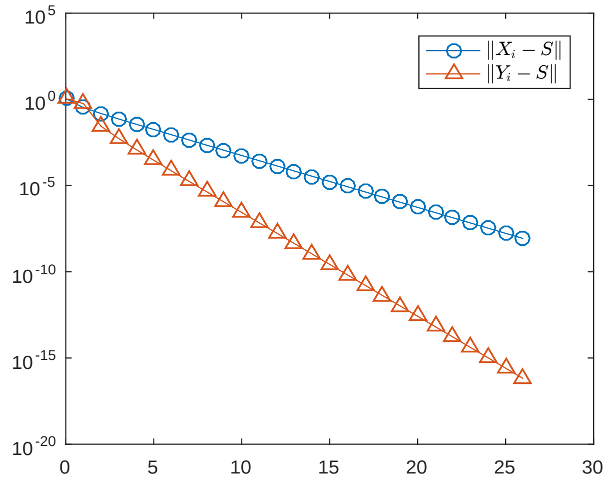

FIGURE 3. *Comparison with* $\|X_i - S\|$ *and* $\|Y_i - S\|$.

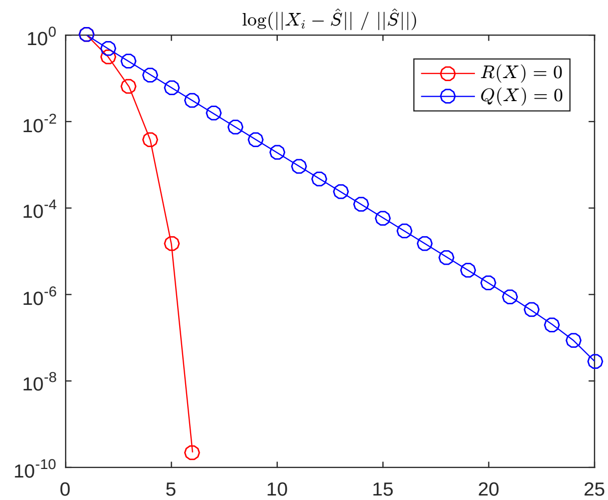

FIGURE 4. *Relative errors of PNA iterations with the numerical solution* $\hat{S}_Q$ *and* $\hat{S}_R$ *from* (5.36) *and* (5.37), *respectively.*

of (5.37) is simple. We can see from Table 1 that MNA is significantly more efficient than PNA for finding non-simple solutions.

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