

2019

## A NOTE ON VARIANTS OF ZERO FORCING

Lon Mitchell

*University of South Florida*, [lonmitchell@mail.usf.edu](mailto:lonmitchell@mail.usf.edu)

Follow this and additional works at: <https://repository.uwyo.edu/ela>



Part of the [Discrete Mathematics and Combinatorics Commons](#)

---

### Recommended Citation

Mitchell, Lon. (2019), "A NOTE ON VARIANTS OF ZERO FORCING", *Electronic Journal of Linear Algebra*, Volume 35, pp. 468-472.

DOI: <https://doi.org/10.13001/1081-3810.3822>

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in *Electronic Journal of Linear Algebra* by an authorized editor of Wyoming Scholars Repository. For more information, please contact [scholcom@uwyo.edu](mailto:scholcom@uwyo.edu).



## A NOTE ON VARIANTS OF ZERO FORCING\*

LON MITCHELL<sup>†</sup>

**Abstract.** A small improvement is made to the zero-forcing variants defined by Butler, Grout, and Hall (2015) for matrices with a given number of negative eigenvalues, resulting in a better value for the Barioli-Fallat tree and one negative eigenvalue.

**Key words.** Zero forcing, Barioli-Fallat tree.

**AMS subject classifications.** 05C50.

**1. Introduction.** Each  $n$ -by- $n$  Hermitian matrix  $M = (m_{ij})$  has an associated simple graph  $G = (V, E)$  with vertex set  $V = \{1, \dots, n\}$  and edge set  $E = \{ij \mid i < j, m_{ij} \neq 0\}$ . Conversely, starting with a simple graph  $G$ , we can consider the set  $S(G)$  of all Hermitian matrices whose graph is  $G$ . The maximum nullity among the matrices in  $S(G)$  is an invariant of  $G$ , denoted  $M(G)$ .

Zero forcing is a combinatorial game played on the graph  $G$  that gives an upper bound, the zero-forcing number  $Z(G)$ , for  $M(G)$  [1]. A modified game gives an upper bound,  $Z_+(G)$ , for the maximum nullity of positive semidefinite matrices in  $S(G)$  [2]. Butler, Grout, and Hall [4] defined variants of these games and related numbers  $Z_q(G)$  that interpolate between  $Z_+(G) = Z_0(G)$  and  $Z(G) = Z_n(G)$ . They noted that for most trees  $T$  on ten or fewer vertices,  $Z_q$  usually gives a tight bound for the maximum nullity of matrices in  $S(T)$  having exactly  $q$  negative eigenvalues,  $M_q(G)$ . One notable exception is the Barioli-Fallat tree  $BF$  [3], shown in Figure 1, which has  $M_1(BF) = 2$  but  $Z_1(BF) = 3$ .

In this note, we will introduce an improvement to Butler, Grout, and Hall's original definition for  $Z_q$ , resulting in new values  $\dot{Z}_q(G)$  that are tighter bounds in many cases, including providing a tight bound for  $M_1$  of the Barioli-Fallat tree.

**2. Definitions.** The *change rule*, given a graph with vertices that are either “filled” or “unfilled”, is to fill a vertex  $v$  that is the unique unfilled neighbor of a filled vertex  $u$  ( $v$  is said to be *forced* by  $u$ ). This rule is derived from the following idea: If we consider a vector in the null space of a matrix  $M \in S(G)$  that has zero entries corresponding to the filled vertices and  $v$  and  $u$  are as above, then matrix multiplication implies the entry corresponding to  $v$  must also be zero.

**DEFINITION 2.1.** ( $Z_q$ -Forcing Game) All the vertices of the graph  $G$  are initially unfilled. There are two players, known as Player (who has tokens) and Opponent. Player will repeatedly apply one of the following three options until all vertices are filled:

1. For one token, Player can fill any vertex.
2. At no cost, Player can apply the change rule on the entire graph  $G$ .
3. Let the vertices currently filled be denoted by  $B$ , and let  $W_1, \dots, W_k$  be the vertex sets of the connected components of  $G - B$ . Player selects at least  $q + 1$  of the  $W_i$  and announces the selection

---

\*Received by the editors on June 6, 2018. Accepted for publication on August 5, 2019. Handling Editor: Leslie Hogben.

<sup>†</sup>University of South Florida St. Petersburg, USA (lonmitchell@mail.usf.edu).

to Opponent. Opponent then selects a nonempty subset of these components, say  $\{W_{i_1}, \dots, W_{i_l}\}$ , and announces it to Player. At no cost, Player can apply the change rule on  $G[B \cup W_{i_1} \cup \dots \cup W_{i_l}]$ .

We can try to understand this game as follows: Consider a matrix  $M \in S(G)$  expressed as

$$(2.1) \quad M = \begin{bmatrix} M_1 & 0 & \cdots & 0 & B_1^T \\ 0 & M_2 & \cdots & 0 & B_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_k & B_k^T \\ B_1 & B_2 & \cdots & B_k & C \end{bmatrix},$$

where each  $M_i$  corresponds to the vertices of  $W_i$  and  $C$  corresponds to the filled vertices  $B$  at a given stage in the zero-forcing process. Let  $X$  be the subspace of null vectors of  $M$  whose support is contained in  $G - B$ . Let

$$x = [x_1 \quad x_2 \quad \cdots \quad x_k \quad 0]^*$$

be a column vector in  $X$  whose support is the union of all supports occurring in  $X$  (where each  $x_i$  corresponds to the vertices of  $W_i$ ). If we are able to conclude that  $\sum_{i \in S} B_i x_i = 0$  for some subset  $S$  of  $\{1, \dots, k\}$ , then we can apply the change rule on the subgraph induced by  $B$  and the  $W_i$  with  $i \in S$ , since showing that an entry of  $x$  must be zero is equivalent to showing that entry is zero in every vector in  $X$ .

Assuming a fixed number of negative eigenvalues allows this sort of argument, as follows: An *isotropic* subspace for  $M$  is a subspace of vectors  $y$  such that  $y^* M y = 0$ .

**THEOREM 2.2.** ([4]) *Let  $A$  be a Hermitian matrix and  $R$  an isotropic subspace for  $A$  of dimension more than  $\min\{p, q\}$ , where  $p$  and  $q$  are the number (counting multiplicities) of positive and negative eigenvalues, respectively. Then  $R$  contains a nonzero vector in the null space of  $A$ .*

Suppose then we know that  $M$  has exactly  $q$  negative eigenvalues and  $p \geq q$ . Take any  $q + 1$  of the  $W_i$ , say  $W_1, \dots, W_{q+1}$ . Since  $Mx = 0$ , for each  $j$  we have  $M_j x_j = 0$ . If  $q = 0$ , then the row span of  $B_j$  is in the row span of  $M_j$  for each  $j$  (often called the row and column inclusion property of positive semidefinite matrices), meaning  $B_j x_j = 0$  for each  $j$ . In particular, if  $M$  is positive semidefinite, then we can apply the change rule to  $B$  and any component. If  $q > 0$  and  $B_j x_j \neq 0$  for each  $j$  with  $1 \leq j \leq q + 1$ , if we extend each  $x_j$  to an  $n$ -by-1 column vector  $y_j$  whose entries corresponding to  $W_j$  taken from  $x_j$  and which has zeros elsewhere, we find the  $y_j$  are linearly independent and span an isotropic subspace for  $M$ . By Theorem 2.2, there is some nontrivial linear combination of the  $y_j$  that is a null vector:  $M \sum_{i=1}^{q+1} b_i y_i = \sum_{i=1}^{q+1} b_i B_i y_i = 0$ . Since an entry of  $y_j$  corresponding to  $W_j$  is zero if and only if the corresponding entry of  $x_j$  is zero, we can apply the change rule to the subgraph of  $G$  induced by  $B$  and those  $W_j$  whose corresponding coefficient  $b_j$  is nonzero in the above sum.

The design of the game to include Opponent reflects that, in general,  $q$ -zero forcing can not be applied to all matrices at once (as it can with  $q = 0$  or  $q = n$ ), but must be considered on a matrix-by-matrix basis. Each matrix  $M \in S(G)$  with exactly  $q$  negative eigenvalues gives a strategy for Opponent: If Player selects  $W_1, \dots, W_{q+1}$  and  $B_j x_j = 0$  for some  $j$ , then Opponent returns  $W_j$ ; otherwise, Opponent returns those  $W_i$  such that the corresponding  $b_i$  is nonzero. Butler, Grout, and Hall showed that this strategy forces Player to spend at least as many tokens as the nullity of  $M$  [4]. In particular,  $M$  cannot have nullity more than  $Z_q(G)$ , so that  $Z_q(G) \geq M_q(G)$ .

**3. Improvement.** In this section, we will combine two key elements of the game to suggest an improvement. We will let  $\dot{Z}_q$  denote the resulting parameter. First, note that the change rule can be applied to  $B$  and any number of components  $W_i$  (as determined by Opponent) and the implementation of the change rule may involve as few as one filled vertex (once we have the null vector, applying the change rule amounts to studying the result of multiplying the null vector by a single row of  $M$ ). Thus, the only way the  $q$  value appears in the game is in deciding how many components  $W_i$  we must have in order to convince Opponent to look for a vector in the null space that allows application of the change rule. Second, the strategies used by Opponent to show that  $Z_q(G) \geq M_q(G)$  come from actual matrices  $M \in S(G)$ . Thus, in altering the game to restrict the play of Opponent, as long as we still allow Opponent to act based on any  $M \in S(G)$ , we can still conclude  $\dot{Z}_q(G) \geq M_q(G)$ .

**DEFINITION 3.1.** ( $\dot{Z}_q$ -Forcing Game) All the vertices of the graph  $G$  are initially unfilled. There are two players, known as Player (who has tokens) and Opponent. Before the game begins, for each induced subgraph  $H$  of  $G$ , Opponent must specify a value  $e(H)$  subject to the following conditions:

- $0 \leq e(H) \leq q = e(G)$ ;
- if  $H_1, \dots, H_k$  are the connected components of  $H - B$  for some subgraph  $H$  and set of vertices  $B$ , then  $\sum_{i=1}^k e(H_i) \leq e(H)$ .

Next, Player will repeatedly apply one of the following three options until all vertices are filled:

1. For one token, Player can fill any vertex.
2. At no cost, Player can apply the change rule on the entire graph  $G$ .
3. Let the vertices currently filled be denoted by  $B$ , and let  $W_1, \dots, W_k$  be the vertex sets of the connected components of  $G - B$ . Player selects  $m$  of the  $W_i$  and a subset  $B'$  of  $B$  such that, if  $H$  is the subgraph induced by  $B'$  and the selected  $W_i$ , then  $m > e(H)$ . Player then announces the selection to Opponent. Opponent selects a nonempty subset of the components, say  $\{W_{i_1}, \dots, W_{i_l}\}$ , and announces it to Player. At no cost, Player can apply the change rule on  $G[B' \cup W_{i_1} \cup \dots \cup W_{i_l}]$ .

To see how this new definition takes advantage of the key elements discussed above, consider a matrix  $M$  as in [Equation \(2.1\)](#). First, if  $M$  has exactly  $q$  negative eigenvalues, using Cauchy eigenvalue interlacing, we know that the  $M_j$  combined have at most  $q$  negative eigenvalues. Initially, that does not provide any benefit. However, once some vertices of, for example,  $W_1$  are filled, we can see if we can apply the change rule using just vertices of  $W_1$ . If  $M_1$  has fewer than  $q$  negative eigenvalues, then we need not provide as many connected components of  $W_1$  to Opponent to convince Opponent that a null vector for those components exists. This comes from the fact that if a vector  $x$  is an isotropic vector for a matrix  $A$  whose support is some subset of vertices  $S$ , then  $x$  is also an isotropic vector for any principal submatrix of  $A$  corresponding to a superset of  $S$ . Thus, we may be able to tailor our choices of components and filled vertices based on the  $e$  distribution specified by Opponent.

Second, Opponent can still get a strategy for the game from each matrix  $M \in S(G)$ : Choose  $e(H)$  to be the number of negative eigenvalues of the principal submatrix of  $M$  corresponding to  $H$ ; play exactly as before when Player specifies certain components (but now Player is a little craftier in knowing when a null vector will exist). As in the proof by Butler, Grout, and Hall [\[4\]](#), Opponent's play using the matrix strategy can still protect a null vector supported on the unfilled vertices that has largest support, forcing Player to pay a token for each dimension of the null space (and perhaps more).

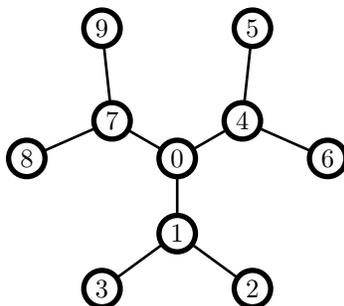


FIGURE 1. The Barioli-Fallat tree.

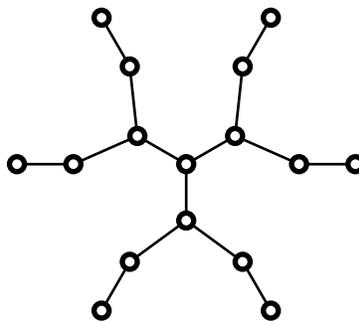


FIGURE 2. The extended Barioli-Fallat tree.

EXAMPLE 3.2. Consider the Barioli-Fallat tree  $BF$  as in Figure 1. Removing the center vertex leaves three isomorphic subtrees. Regardless of how Opponent assigns  $e$  values, two of the three can be considered to have no negative eigenvalues. Without loss of generality, let  $e(G[4, 5, 6]) = 1$ . We have  $\dot{Z}_1(BF) > 1$  since starting with one filled vertex must leave 5 or 6 (or both) unfilled regardless of how the game is played. On the other hand, if we start with, for example, 3 and 5 filled, we can force 1 and 4 using the global rule. Next, since  $e(G[1, 2]) = 0$ , we can use positive semidefinite zero forcing to fill 2. Then, the global rule can be applied in turn to 0, 6, and 7. Finally, since  $e(G[7, 8, 9]) = 0$ , we can again use positive semidefinite zero forcing to fill 8 and 9 in turn. Thus,  $\dot{Z}_1(BF) = 2$ .

EXAMPLE 3.3. The star  $K_{1,m}$  provides an example where there is no difference between  $\dot{Z}_q$  and  $Z_q$ , since Opponent can define  $e$  to equal  $q$  for any connected subgraph.

REMARK 3.4. Butler, Grout, and Hall [4] also discussed the extended Barioli-Fallat tree,  $EBF$ , as shown in Figure 2. Adding information about the diagonal entries (looped and unlooped vertices) gives an improvement on the traditional zero forcing number,  $\dot{Z}(G)$ . They defined  $\dot{Z}_q(G)$  in a corresponding fashion and found  $EBF$  to be problematic for  $\dot{Z}_1$  just as  $BF$  is for  $Z_1$ . The same technique of Example 3.2 works to show  $\dot{Z}_1(EBF) = 2$ . It would be interesting to know if  $\dot{Z}_1$  correctly gives the corresponding maximum nullity for all trees.

REMARK 3.5. Butler, Grout, and Hall [4] showed that, for the tree  $T$  in Figure 3,  $Z_0(T) = 1$ ,  $Z_1(T) = 2$ , and  $Z_2(T) = 3$ , while  $Z_2$  of the disjoint union  $T \cup T$  is 5. For  $Z_q(G \cup H)$ , they proposed instead considering  $Z_s(G)$  and  $Z_t(H)$  where  $s + t = q$ . The new game incorporates this idea, so that, in general,

$$\dot{Z}_q(G \cup H) = \max_{s+t=q} (\dot{Z}_s(G) + \dot{Z}_t(H)).$$

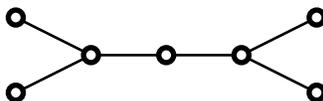


FIGURE 3. The tree  $T$  in Remark 3.5.

**4. Further remarks.** Butler, Grout, and Hall [4] contend that “on all but one tree with 10 or fewer vertices,  $Z_q(G)$  is a tight bound for the maximum nullity of a matrix associated with the tree that has  $q$  negative eigenvalues”. Of course, this is not true for large values of  $q$  relative to  $n$ . For example, for a path  $P_n$  on  $n$  vertices,  $Z_n(P_n) = Z(P_n) = 1$  but the nullity of an  $n$ -by- $n$  matrix with exactly  $n$  negative eigenvalues is zero. However, their remark does seem to be true if we consider matrices with *at most*  $q$  negative eigenvalues, and in fact they point out their main result is still true in this case (using the same proof): The nullity of  $A \in S(G)$  when  $A$  has at most  $q$  negative eigenvalues is at most  $Z_q(G)$ .

For the purposes of applying the change rule in the new game, principal submatrices that have no negative eigenvalues are helpful. But row inclusion also holds for negative semidefinite matrices, so it should be just as helpful to find submatrices that have no positive eigenvalues. The current definition, which relies just on negative eigenvalues could also be improved to consider the minimum of  $p$  and  $q$  as in Theorem 2.2. Then the  $\dot{Z}_m$ -forcing game could be applied where  $m = \min\{p, q\}$ . As above, high values of  $m$  relative to  $n$  could be ignored. For example,  $m \geq n/2$  would force nullity zero! We could incorporate this, for example, by defining  $\bar{Z}_m(G) = \min\{\dot{Z}_m(G), n - 2m\}$  and  $\bar{Z}(G) = \max_m \bar{Z}_m(G)$ . From the definitions and because  $Z_q(G) \leq Z(G)$  for each  $q$ ,  $\bar{Z}(G) \leq Z(G)$ . It would be interesting to know if graphs exist where  $\bar{Z}(G)$  is strictly less than  $Z(G)$ .

#### REFERENCES

- [1] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.
- [2] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433:401–411, 2010.
- [3] F. Barioli and S.M. Fallat. On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices. *Electronic Journal of Linear Algebra*, 11:41–50, 2004.
- [4] S. Butler, J. Grout, and H.T. Hall. Using variants of zero forcing to bound the inertia set of a graph. *Electronic Journal of Linear Algebra*, 30:1–18, 2015.