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ON THE INTERVAL GENERALIZED COUPLED MATRIX EQUATIONS*

MARZIEH DEHGHANI-MADISEH†

Abstract. In this work, the interval generalized coupled matrix equations

$$\sum_{j=1}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} = \mathbf{C}_i, \qquad i = 1, \dots, p+q,$$

are studied in which $\mathbf{A}_{ij}$, $\mathbf{B}_{ik}$ and $\mathbf{C}_i$ are known real interval matrices, while X_j and Y_k are the unknown matrices for $j = 1, \dots, p$, $k = 1, \dots, q$ and $i = 1, \dots, p+q$. This paper discusses the so-called AE-solution sets for this system. In these types of solution sets, the elements of the involved interval matrices are quantified and all occurrences of the universal quantifier $\forall$ (if any) precede the occurrences of the existential quantifier $\exists$. The AE-solution sets are characterized and some sufficient conditions under which these types of solution sets are bounded are given. Also some approaches are proposed which include a numerical technique and an algebraic approach for enclosing some types of the AE-solution sets.

Key words. Interval arithmetic, Generalized coupled matrix equations, AE-solution set.

AMS subject classifications. 65G30, 15A24.

1. Introduction.

Consider the generalized coupled matrix equations

$$\sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} = C_i, \qquad i = 1, \dots, p+q, \tag{1.1}$$

where $A_{ij} \in \mathbb{R}^{m\times m}$, $B_{ik} \in \mathbb{R}^{n\times n}$ and $C_i \in \mathbb{R}^{m\times n}$ are known matrices and $X_j, Y_k \in \mathbb{R}^{m\times n}$ are unknown matrices for $i = 1, \dots, p+q$, $j = 1, \dots, p$ and $k = 1, \dots, q$. These types of systems have nice applications in various branches of science and engineering. For example, Sylvester and Lyapunov matrix equations that are special cases of (1.1) appear frequently in a variety of subjects such as vibration theory [3, 6, 40], image restoration [2], control theory [1, 4], model reduction and so on, see [14, 19]. Therefore, in the literature these types of problems have been widely studied [4, 5, 10, 11, 12, 13, 36, 37, 38, 39, 41, 42, 43, 44].

Even though the system of matrix equations of the form (1.1) are studied in the literature, less attention has been paid to the form of uncertainties that may occur in the elements of A_{ij}, B_{ik} and C_i, for $i = 1, \dots, p+q$, $j = 1, \dots, p$ and $k = 1, \dots, q$. These uncertainties that usually arise from rounding errors and measurement errors, can be described by intervals, and hence, we will have the interval generalized coupled matrix equations

$$\sum_{j=1}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} = \mathbf{C}_i, \qquad i = 1, \dots, p+q, \tag{1.2}$$

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or equivalently,

$$\begin{cases} \sum_{j=1}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} = \mathbf{C}_1, \\ \vdots \\ \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{(p+q)k} = \mathbf{C}_{p+q}, \end{cases}$$

therein the boldface letters stand for the interval matrices. The interval generalized coupled matrix equations (1.2) can be transformed to the interval linear system

$$\mathbf{P} z = \mathbf{f}, \tag{1.3}$$

where

$$\mathbf{P} = \begin{pmatrix} I_n \otimes \mathbf{A}_{11} & \cdots & I_n \otimes \mathbf{A}_{1p} & \mathbf{B}_{11}^\top \otimes I_m & \cdots & \mathbf{B}_{1q}^\top \otimes I_m \\ \vdots & & \vdots & \vdots & & \vdots \\ I_n \otimes \mathbf{A}_{(p+q)1} & \cdots & I_n \otimes \mathbf{A}_{(p+q)p} & \mathbf{B}_{(p+q)1}^\top \otimes I_m & \cdots & \mathbf{B}_{(p+q)q}^\top \otimes I_m \end{pmatrix},$$

$z = (\mathrm{vec}(X_1)^\top, \ldots, \mathrm{vec}(X_p)^\top, \mathrm{vec}(Y_1)^\top, \ldots, \mathrm{vec}(Y_q)^\top)^\top$ and $\mathbf{f} = (\mathrm{vec}(\mathbf{C}_1)^\top, \ldots, \mathrm{vec}(\mathbf{C}_{p+q})^\top)^\top$ in which I_n stands for the identity matrix of order n and $\otimes$ denotes the Kronecker product and for $X \in \mathbb{R}^{m\times n}$, $\mathrm{vec}(X)$ is obtained by stacking the columns of X to a large vector. A common approach when considering the interval system (1.2) is to first transform it to the interval system (1.3) and then using a technique for enclosing the solution set of that interval linear system. But it is to be noted that the interval system (1.3) is a large linear system even for small integers m, n, p or q and so in computational point of view, is not efficient. For instance some methods for handling system (1.3) need to compute an approximate inverse of $\mathrm{mid}(\mathbf{P}) \in \mathbb{R}^{mn(p+q)\times mn(p+q)}$ ($\mathrm{mid}(\mathbf{P})$ denotes the midpoint of interval matrix $\mathbf{P}$). It is obvious that computing such an approximate inverse is too costly. On the other hand, the elements of the transformed system (1.3) have some dependencies and in fact, it should be treated as a parametric linear system. So, considering it as an interval system causes some overestimation in the solution set. These reasons motivate us to propose some methods that work with the original formulation (1.2) instead of (1.3).

Because computing the exact solution of an interval linear system is NP-hard [25], in general, so providing some approximations for the solution set is considered by most researchers. Up to now, only a few techniques for determining the solution set of some special cases of the interval system (1.2) have been implemented. Some techniques for enclosing the united solution set of the interval Sylvester matrix equation $\mathbf{A}X + X\mathbf{B} = \mathbf{C}$ have been proposed in [26]. Shashikhin [33, 34] studied the interval matrix equation $\mathbf{A}X + X\mathbf{B} = \mathbf{C}$ using its correspondence by the interval linear system of equations

$$\big((I_n \otimes \mathbf{A}) + (\mathbf{B}^\top \otimes I_m)\big)x = \mathbf{c}, \qquad x = \mathrm{vec}(X), \quad \mathbf{c} = \mathrm{vec}(\mathbf{C}).$$

Hashemi and Dehghan [15] studied the interval linear system with multiple right-hand sides $\mathbf{A}X = \mathbf{B}$ and used an interval Gaussian elimination technique to find an enclosure for the united solution set of $\mathbf{A}X = \mathbf{B}$. Also, they [16] gave some analytical characterizations of the AE-solution set to the interval Lyapunov matrix equation

$$\mathbf{A}X + X\mathbf{A}^\top = \mathbf{F},$$

and proposed a modification of the Krawczyk operator to obtain an outer estimation for the united solution set of this equation. The authors in [7] characterized the generalized AE-solution set to the interval generalized Sylvester matrix equation

$$\sum_{i=1}^{p} \mathbf{A}_i X_i + \sum_{j=1}^{q} Y_j \mathbf{B}_j = \mathbf{C},$$

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and developed some algebraic approaches and numerical techniques for obtaining inner and outer estimations for some special cases of the AE-solution set of this equation. Rivaz et al. [24] considered the interval system of matrix equations

$$\begin{cases} \mathbf{A}_{11}X + Y\mathbf{A}_{12} = \mathbf{C}_1, \\ \mathbf{A}_{21}X + Y\mathbf{A}_{22} = \mathbf{C}_2, \end{cases}$$

and defined its united solution set and studied some conditions under which the united solution set is bounded. Also they presented a direct method and an iterative method for solving this interval system. Dehghani-Madiseh and Hladík [9] studied the interval generalized Sylvester matrix equation

$$\mathbf{A}X\mathbf{B} + \mathbf{C}X\mathbf{D} = \mathbf{F},$$

and presented a modified variant of the Krawczyk operator and some iterative techniques for enclosing its solution set reducing significantly computational complexity, compared to the Kronecker product form of the mentioned interval system. In this paper, we want to consider a more general case that includes many interval matrix equations as its special cases, such as the generalized (coupled) Lyapunov and Sylvester matrix equations. We consider the interval generalized coupled matrix equations

$$\sum_{j=1}^{p} \mathbf{A}_{ij}X_j + \sum_{k=1}^{q} Y_k\mathbf{B}_{ik} = \mathbf{C}_i, \qquad i = 1, \ldots, p+q,$$

and define the concept of the AE-solution sets and characterize them. Also we give some sufficient conditions for boundedness of the AE-solution set. Then some approaches for enclosing some types of the AE-solution set will be proposed.

In this paper, $\mathbb{IR}$, $\mathbb{IR}^n$ and $\mathbb{IR}^{m\times n}$ denote the set of proper intervals, n-dimensional interval vectors and m-by-n interval matrices, respectively. Ordinary letters stand for real values and boldface letters denote interval quantities. For interval $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ define the *midpoint* $\text{mid}(\mathbf{x}) \equiv \hat{\mathbf{x}} := \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$, the *radius* $\text{rad}(\mathbf{x}) \equiv \check{\mathbf{x}} := \frac{\overline{\mathbf{x}}-\underline{\mathbf{x}}}{2}$ and the *absolute value* $\text{mag}(\mathbf{x}) \equiv |\mathbf{x}| := max\{|x| : x \in \mathbf{x}\}$. Kaucher [20] extended the set of proper intervals $\mathbb{IR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \leq \overline{\mathbf{x}}, \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$ by the set $\overline{\mathbb{IR}} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \geq \overline{\mathbf{x}}, \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$ of improper intervals, resulting in a more flexible set of generalized intervals $\mathbb{KR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$. In this new set, "*dual*" is an important operator that is defined as $\text{dual}(\mathbf{x}) = [\overline{\mathbf{x}}, \underline{\mathbf{x}}]$. The set of generalized intervals $\mathbb{KR}$ has better algebraic properties than the set of classical intervals $\mathbb{IR}$. For example, the addition in $\mathbb{KR}$ is a group and the opposite of an interval number $\mathbf{x}$ is $-\text{dual}(\mathbf{x})$, i.e., $\mathbf{x} + (-\text{dual}(\mathbf{x})) = [0, 0]$. Also, the multiplication in $\mathbb{KR}$ restricted to zero free intervals, is a group and the inverse of such interval $\mathbf{x}$ is $\frac{1}{\text{dual}(\mathbf{x})}$, i.e., $\frac{\mathbf{x}}{\text{dual}(\mathbf{x})} = [1, 1]$, see [20]. The concept of midpoint, radius, absolute value and dual for interval vectors and matrices are defined componentwise. The symbol $\square$ stands for the interval hull of a bounded set, i.e., if $\mathcal{A} \subseteq \mathbb{R}^{m\times n}$ is a bounded set of the real matrices then $\inf \mathcal{A}$ and $\sup \mathcal{A}$ exist and the interval hull of $\mathcal{A}$ defined by

$$\square\mathcal{A} = [\inf \mathcal{A}, \sup \mathcal{A}] = \bigcap\{\mathbf{X} : \mathbf{X} \in \mathbb{IR}^{m\times n}, \mathbf{X} \supseteq \mathcal{A}\},$$

is the tightest interval matrix enclosing $\mathcal{A}$.

The rest of the paper is organized as follows: In Section 2, we define and characterize the generalized AE-solution set to the interval generalized coupled matrix equations (1.2) and also some sufficient conditions for boundedness of the AE-solution set will be given. In Section 3, we propose some approaches for enclosing the AE-solution set $\Xi_{\exists\exists\gamma}$. Finally, the paper is completed by a short conclusion in Section 4.

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2. The AE-solution set. In this section, first we define the concept of the AE-solution sets for the interval generalized coupled matrix equations (1.2) and then characterize these types of solution sets.

2.1. Definition of the AE-solution set. Consider the interval generalized coupled matrix equations (1.2). Each element of this interval system can correspond to different types of uncertainty. Shary [31] for the first time introduced the concept of the AE-solution set for an interval linear system in order to specify distribution of the uncertainty type with respect to the interval elements of the system. Here, we want to define the AE-solution set for the interval system (1.2) using a similar convention. We focus on the generalized solution sets of the interval system (1.2) in which all occurrences of the universal quantifier $\forall$ precede the occurrences of the existential quantifier $\exists$ (AE-form). Here "A" ("E") stands for *all* (*exist*) and we say an interval parameter has A-uncertainty (E-uncertainty) when it appears with the universal quantifier "$\forall$" (existential quantifier "$\exists$").

For describing these uncertainties, we define the m-by-m matrices $\alpha_{ij} = ((\alpha_{ij})_{st})$, the n-by-n matrices $\beta_{ik} = ((\beta_{ik})_{st})$ and the m-by-n matrices $\gamma_i = ((\gamma_i)_{st})$, for $i = 1, \ldots, p+q$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$ as follows:

$$(\alpha_{ij})_{st} := \begin{cases} \forall, & \text{if } (\mathbf{A}_{ij})_{st} \text{ has A-uncertainty,} \\ \exists, & \text{if } (\mathbf{A}_{ij})_{st} \text{ has E-uncertainty,} \end{cases}$$
$$(\beta_{ik})_{st} := \begin{cases} \forall, & \text{if } (\mathbf{B}_{ik})_{st} \text{ has A-uncertainty,} \\ \exists, & \text{if } (\mathbf{B}_{ik})_{st} \text{ has E-uncertainty,} \end{cases}$$
$$(\gamma_i)_{st} := \begin{cases} \forall, & \text{if } (\mathbf{C}_i)_{st} \text{ has A-uncertainty,} \\ \exists, & \text{if } (\mathbf{C}_i)_{st} \text{ has E-uncertainty.} \end{cases}$$

Using the above definitions, we define interval matrices $\mathbf{A}_{ij}^{\forall}$, $\mathbf{A}_{ij}^{\exists}$, $\mathbf{B}_{ik}^{\forall}$, $\mathbf{B}_{ik}^{\exists}$, $\mathbf{C}_i^{\forall}$ and $\mathbf{C}_i^{\exists}$ for $i = 1, \ldots, p+q$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$ as follows:

$$(\mathbf{A}_{ij}^{\forall})_{st} = \begin{cases} (\mathbf{A}_{ij})_{st}, & \text{if } (\alpha_{ij})_{st} = \forall, \\ 0, & \text{otherwise,} \end{cases} \qquad (\mathbf{A}_{ij}^{\exists})_{st} = \begin{cases} (\mathbf{A}_{ij})_{st}, & \text{if } (\alpha_{ij})_{st} = \exists, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathbf{B}_{ik}^{\forall})_{st} = \begin{cases} (\mathbf{B}_{ik})_{st}, & \text{if } (\beta_{ik})_{st} = \forall, \\ 0, & \text{otherwise,} \end{cases} \qquad (\mathbf{B}_{ik}^{\exists})_{st} = \begin{cases} (\mathbf{B}_{ik})_{st}, & \text{if } (\beta_{ik})_{st} = \exists, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathbf{C}_i^{\forall})_{st} = \begin{cases} (\mathbf{C}_i)_{st}, & \text{if } (\gamma_i)_{st} = \forall, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbf{C}_{ij}^{\exists} = \begin{cases} (\mathbf{C}_i)_{st}, & \text{if } (\gamma_i)_{st} = \exists, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that by these definitions $\mathbf{A}_{ij} = \mathbf{A}_{ij}^{\forall} + \mathbf{A}_{ij}^{\exists}$, $\mathbf{B}_{ik} = \mathbf{B}_{ik}^{\forall} + \mathbf{B}_{ik}^{\exists}$ and $\mathbf{C}_i = \mathbf{C}_i^{\forall} + \mathbf{C}_i^{\exists}$, for $i = 1, \ldots, p+q$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$. The above mentioned interpretation enables us to formulate what we mean by a solution of the interval generalized coupled matrix equations (1.2).

Also for specifying the uncertainty type distribution corresponding to the elements of the interval system (1.2), we introduce the matrix groups α, β and γ as

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Volume 34, pp. 695-717, December 2018.
http://repository.uwyo.edu/ela

$$\alpha = (\alpha_{11}, \ldots, \alpha_{1p}, \ldots, \alpha_{(p+q)1}, \ldots, \alpha_{(p+q)p}),$$

$$\beta = (\beta_{11}, \ldots, \beta_{1q}, \ldots, \beta_{(p+q)1}, \ldots, \beta_{(p+q)q}), \qquad \gamma = (\gamma_1, \ldots, \gamma_{p+q}).$$

DEFINITION 2.1. We define the *AE-solution set of type* $\alpha\beta\gamma$ to the interval generalized coupled matrix equations (1.2) as the set

$$\begin{aligned}\Xi_{\alpha\beta\gamma} = \Big\{ & (X_1, \ldots, X_p, Y_1, \ldots, Y_q) : \\ & (\forall A'_{11} \in \mathbf{A}^{\forall}_{11}) \ldots (\forall A'_{(p+q)p} \in \mathbf{A}^{\forall}_{(p+q)p})(\forall B'_{11} \in \mathbf{B}^{\forall}_{11}) \ldots (\forall B'_{(p+q)q} \in \mathbf{B}^{\forall}_{(p+q)q}) \\ & (\forall C'_1 \in \mathbf{C}^{\forall}_1) \ldots (\forall C'_{p+q} \in \mathbf{C}^{\forall}_{p+q}) \\ & (\exists A''_{11} \in \mathbf{A}^{\exists}_{11}) \ldots (\exists A''_{(p+q)p} \in \mathbf{A}^{\exists}_{(p+q)p})(\exists B''_{11} \in \mathbf{B}^{\exists}_{11}) \ldots (\exists B''_{(p+q)q} \in \mathbf{B}^{\exists}_{(p+q)q}) \\ & (\exists C''_1 \in \mathbf{C}^{\exists}_1) \ldots (\exists C''_{p+q} \in \mathbf{C}^{\exists}_{p+q}) \\ & \Big(\sum_{j=1}^{p} (A'_{ij} + A''_{ij}) X_j + \sum_{k=1}^{q} Y_k (B'_{ik} + B''_{ik}) = C'_i + C''_i \Big), \quad i = 1, \ldots, p+q \Big\}.\end{aligned} \tag{2.4}$$

where the matrix group $(X_1, \ldots, X_p, Y_1, \ldots, Y_q)$ is such that $X_j, Y_k \in \mathbb{R}^{m \times n}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

As said in [32], generalized solution sets to systems of equations naturally come into decision making theory and existence in operation research with considerable applications [27]. These solution sets can be practically interpreted as solutions of some games or multi-step decision-making processor under interval uncertainty and solutions to some minimax operation research problems [28, 30].

For an interval system of equations, different types of generalized solution sets can be considered. But among all of them, the united solution set is the widest and has numerous applications [16, 21]. Before defining the united solution set for our problem, note that by $\alpha = \forall$ ($\alpha = \exists$) we mean for $i = 1, \ldots, p+q$ and $j = 1, \ldots, p$, $\alpha_{ij} = \forall$ ($\alpha_{ij} = \exists$), i.e., every element of the interval matrix $\mathbf{A}_{ij}$ has interval A-uncertainty (E-uncertainty). Notations $\beta = \forall$, $\beta = \exists$, $\gamma = \forall$ and $\gamma = \exists$ are defined similarly.

• **The united solution set:** The united solution set to the interval generalized coupled matrix equations (1.2) is formed by solutions of all point generalized coupled matrix equations of the form (1.1), with $A_{ij} \in \mathbf{A}_{ij}$, $B_{ik} \in \mathbf{B}_{ik}$ and $C_i \in \mathbf{C}_i$, for $i = 1, \ldots, p+q$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$, i.e., the set

$$\begin{aligned}\Xi_{\exists\exists\exists} = \Big\{ & (X_1, \ldots, X_p, Y_1, \ldots, Y_q) : \\ & (\exists A_{11} \in \mathbf{A}_{11}) \ldots (\exists A_{(p+q)p} \in \mathbf{A}_{(p+q)p})(\exists B_{11} \in \mathbf{B}_{11}) \ldots (\exists B_{(p+q)q} \in \mathbf{B}_{(p+q)q}) \\ & (\exists C_1 \in \mathbf{C}_1) \ldots (\exists C_{p+q} \in \mathbf{C}_{p+q}) \Big(\sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} = C_i \Big), \quad i = 1, \ldots, p+q \Big\}.\end{aligned} \tag{2.5}$$

2.2. Characterization of the AE-solution sets. In this subsection, we give some properties and characterizations for the AE-solution sets to the interval generalized coupled matrix equations (1.2).

Electronic Journal of Linear Algebra, ISSN 1081-3810
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http://repository.uwyo.edu/ela

THEOREM 2.2.

$$\Xi_{\alpha\beta\gamma} = \bigcap_{A'\in\mathbf{A}^{\forall}} \bigcap_{B'\in\mathbf{B}^{\forall}} \bigcap_{C'\in\mathbf{C}^{\forall}} \bigcup_{A''\in\mathbf{A}^{\exists}} \bigcup_{B''\in\mathbf{B}^{\exists}} \bigcup_{C''\in\mathbf{C}^{\exists}} \Bigg\{(X_1,\ldots,X_p,Y_1,\ldots,Y_q): \\ \left(\sum_{j=1}^{p}(A'_{ij}+A''_{ij})X_j+\sum_{k=1}^{q}Y_k(B'_{ik}+B''_{ik})=C'_i+C''_i\right),\quad i=1,\ldots,p+q\Bigg\},$$

where the above intersection and union symbols mean

$$\bigcap_{A'\in\mathbf{A}^{\forall}} \equiv \bigcap_{A'_{11}\in\mathbf{A}^{\forall}_{11}} \cdots \bigcap_{A'_{(p+q)p}\in\mathbf{A}^{\forall}_{(p+q)p}}, \qquad \bigcup_{A''\in\mathbf{A}^{\exists}} \equiv \bigcup_{A''_{11}\in\mathbf{A}^{\exists}_{11}} \cdots \bigcup_{A''_{(p+q)p}\in\mathbf{A}^{\exists}_{(p+q)p}},$$

$$\bigcap_{B'\in\mathbf{B}^{\forall}} \equiv \bigcap_{B'_{11}\in\mathbf{B}^{\forall}_{11}} \cdots \bigcap_{B'_{(p+q)q}\in\mathbf{B}^{\forall}_{(p+q)q}}, \qquad \bigcup_{B''\in\mathbf{B}^{\exists}} \equiv \bigcup_{B''_{11}\in\mathbf{B}^{\exists}_{11}} \cdots \bigcup_{B''_{(p+q)q}\in\mathbf{B}^{\exists}_{(p+q)q}},$$

$$\bigcap_{C'\in\mathbf{C}^{\forall}} \equiv \bigcap_{C'_{1}\in\mathbf{C}^{\forall}_{1}} \cdots \bigcap_{C'_{p+q}\in\mathbf{C}^{\forall}_{p+q}}, \qquad \bigcup_{C''\in\mathbf{C}^{\exists}} \equiv \bigcup_{C''_{1}\in\mathbf{C}^{\exists}_{1}} \cdots \bigcup_{C''_{p+q}\in\mathbf{C}^{\exists}_{p+q}}.$$

Proof. According to definition of the intersection of the sets and using the above symbols, the solution set $\Xi_{\alpha\beta\gamma}$ in Definition 2.1 can be written as

$$\Xi_{\alpha\beta\gamma} = \bigcap_{A'\in\mathbf{A}^{\forall}} \bigcap_{B'\in\mathbf{B}^{\forall}} \bigcap_{C'\in\mathbf{C}^{\forall}} \Bigg\{(X_1,\ldots,X_p,Y_1,\ldots,Y_q): \\ (\exists A''_{11}\in\mathbf{A}^{\exists}_{11})\ldots(\exists A''_{(p+q)p}\in\mathbf{A}^{\exists}_{(p+q)p})(\exists B''_{11}\in\mathbf{B}^{\exists}_{11})\ldots(\exists B''_{(p+q)q}\in\mathbf{B}^{\exists}_{(p+q)q}) \\ (\exists C''_{1}\in\mathbf{C}^{\exists}_{1})\ldots(\exists C''_{p+q}\in\mathbf{C}^{\exists}_{p+q}) \\ \left(\sum_{j=1}^{p}(A'_{ij}+A''_{ij})X_j+\sum_{k=1}^{q}Y_k(B'_{ik}+B''_{ik})=C'_i+C''_i\right),\quad i=1,\ldots,p+q\Bigg\}.$$

Now, by definition of the union of the sets, we have

$$\Xi_{\alpha\beta\gamma} = \bigcap_{A'\in\mathbf{A}^{\forall}} \bigcap_{B'\in\mathbf{B}^{\forall}} \bigcap_{C'\in\mathbf{C}^{\forall}} \bigcup_{A''_{11}\in\mathbf{A}^{\exists}_{11}} \cdots \bigcup_{A''_{(p+q)p}\in\mathbf{A}^{\exists}_{(p+q)p}} \bigcup_{B''_{11}\in\mathbf{B}^{\exists}_{11}} \cdots \bigcup_{B''_{(p+q)q}\in\mathbf{B}^{\exists}_{(p+q)q}} \bigcup_{C''_{1}\in\mathbf{C}^{\exists}_{1}} \cdots \bigcup_{C''_{p+q}\in\mathbf{C}^{\exists}_{p+q}} \\ \Bigg\{(X_1,\ldots,X_p,Y_1,\ldots,Y_q): \\ \left(\sum_{j=1}^{p}(A'_{ij}+A''_{ij})X_j+\sum_{k=1}^{q}Y_k(B'_{ik}+B''_{ik})=C'_i+C''_i\right),\quad i=1,\ldots,p+q\Bigg\},$$

and finally, by substitution,

$$\bigcup_{A''\in\mathbf{A}^{\exists}} \bigcup_{B''\in\mathbf{B}^{\exists}} \bigcup_{C''\in\mathbf{C}^{\exists}} = \bigcup_{A''_{11}\in\mathbf{A}^{\exists}_{11}} \cdots \bigcup_{A''_{(p+q)p}\in\mathbf{A}^{\exists}_{(p+q)p}} \bigcup_{B''_{11}\in\mathbf{B}^{\exists}_{11}} \cdots \bigcup_{B''_{(p+q)q}\in\mathbf{B}^{\exists}_{(p+q)q}} \bigcup_{C''_{1}\in\mathbf{C}^{\exists}_{1}} \cdots \bigcup_{C''_{p+q}\in\mathbf{C}^{\exists}_{p+q}},$$

the proof is completed. ∎

COROLLARY 2.3. *For the united solution set (2.5) of the interval generalized coupled matrix equations (1.2), we have*

$$\Xi_{\alpha\beta\gamma} = \bigcup_{A_{11}\in\mathbf{A}_{11}} \cdots \bigcup_{A_{(p+q)p}\in\mathbf{A}_{(p+q)p}} \bigcup_{B_{11}\in\mathbf{B}_{11}} \cdots \bigcup_{B_{(p+q)q}\in\mathbf{B}_{(p+q)q}} \bigcup_{C_{1}\in\mathbf{C}_{1}} \cdots \bigcup_{C_{p+q}\in\mathbf{C}_{p+q}} \\ \Bigg\{(X_1,\ldots,X_p,Y_1,\ldots,Y_q): \left(\sum_{j=1}^{p}A_{ij}X_j+\sum_{k=1}^{q}Y_kB_{ik}=C_i\right),\quad i=1,\ldots,p+q\Bigg\}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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It is worth noting that for the involved interval matrices in the interval generalized coupled matrix equations (1.2), we treat their elements as independent intervals so we have the following relation

$$\sum_{j=1}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} - \mathbf{C}_i = \square \Big\{ \sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} - C_i : A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i \in \mathbf{C}_i \Big\} \tag{2.6}$$

for $i = 1, \dots, p+q$.

Lemma 2.4. [22] *Let* $\mathbf{A} \in \mathbb{IR}^{m\times n}$ *and* $\Sigma, \Sigma' \subseteq \mathbb{R}^{m\times n}$. *Then*

(i) $\Sigma \subseteq \mathbf{A} \Longrightarrow \square\Sigma \subseteq \mathbf{A}$, (ii) $\Sigma \subseteq \Sigma' \Longrightarrow \square\Sigma \subseteq \square\Sigma'$.

Lemma 2.5. [22] *For interval matrices* $\mathbf{A}$, $\mathbf{B}$ *and real point matrix* X, *we have*

i. $mid(\mathbf{A} \pm \mathbf{B}) = mid(\mathbf{A}) \pm mid(\mathbf{B})$, $rad(\mathbf{A} \pm \mathbf{B}) = rad(\mathbf{A}) + rad(\mathbf{B})$,
ii. $mid(\mathbf{A}X) = mid(\mathbf{A})X$, $mid(X\mathbf{A}) = X mid(\mathbf{A})$,
iii. $rad(\mathbf{A}X) = rad(\mathbf{A})|X|$, $rad(X\mathbf{A}) = |X| rad(\mathbf{A})$.

Theorem 2.6. $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\alpha\beta\gamma}$ *if and only if*

$$\begin{aligned} &\Big\{ \sum_{j=1}^{p} A'_{ij} X_j + \sum_{k=1}^{q} Y_k B'_{ik} - C'_i : A'_{ij} \in \mathbf{A}^{\forall}_{ij}, B'_{ik} \in \mathbf{B}^{\forall}_{ik}, C'_i \in \mathbf{C}^{\forall}_i \Big\} \\ &\subseteq \Big\{ C''_i - \sum_{j=1}^{p} A''_{ij} X_j - \sum_{k=1}^{q} Y_k B''_{ik} : A''_{ij} \in \mathbf{A}^{\exists}_{ij}, B''_{ik} \in \mathbf{B}^{\exists}_{ik}, C''_i \in \mathbf{C}^{\exists}_i \Big\}, \qquad i = 1, \dots, p+q. \end{aligned} \tag{2.7}$$

Proof. Suppose $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\alpha\beta\gamma}$. So, using (2.4) for all $A'_{ij} \in \mathbf{A}^{\forall}_{ij}, B'_{ik} \in \mathbf{B}^{\forall}_{ik}, C'_i \in \mathbf{C}^{\forall}_i$, $i = 1, \dots, p+q$, $j = 1, \dots, p$ and $k = 1, \dots, q$, there exist $A''_{ij} \in \mathbf{A}^{\exists}_{ij}, B''_{ik} \in \mathbf{B}^{\exists}_{ik}, C''_i \in \mathbf{C}^{\exists}_i$ such that

$$\sum_{j=1}^{p} A'_{ij} X_j + \sum_{k=1}^{q} Y_k B'_{ik} - C'_i = C''_i - \sum_{j=1}^{p} A''_{ij} X_j - \sum_{k=1}^{q} Y_k B''_{ik}, \qquad i = 1, \dots, p+q. \tag{2.8}$$

Now, let $\mathcal{T}_i$ belongs to the left-side of the inclusion relation (2.7) for $i = 1, \dots, p+q$. So, $\mathcal{T}_i = \sum_{j=1}^{p} A'_{ij} X_j + \sum_{k=1}^{q} Y_k B'_{ik} - C'_i$ for appropriate matrices $A'_{ij} \in \mathbf{A}^{\forall}_{ij}, B'_{ik} \in \mathbf{B}^{\forall}_{ik}, C'_i \in \mathbf{C}^{\forall}_i$, $j = 1, \dots, p$ and $k = 1, \dots, q$. Using (2.8) there exist $A''_{ij} \in \mathbf{A}^{\exists}_{ij}, B''_{ik} \in \mathbf{B}^{\exists}_{ik}, C''_i \in \mathbf{C}^{\exists}_i$, for $j = 1, \dots, p$ and $k = 1, \dots, q$, such that $\mathcal{T}_i = C''_i - \sum_{j=1}^{p} A''_{ij} X_j - \sum_{k=1}^{q} Y_k B''_{ik}$, for $i = 1, \dots, p+q$, which yields (2.7).

Conversely, let (2.7) hold. According to (2.7) for all $A'_{ij} \in \mathbf{A}^{\forall}_{ij}, B'_{ik} \in \mathbf{B}^{\forall}_{ik}, C'_i \in \mathbf{C}^{\forall}_i$, $j = 1, \dots, p$ and $k = 1, \dots, q$, $\sum_{j=1}^{p} A'_{ij} X_j + \sum_{k=1}^{q} Y_k B'_{ik} - C'_i$ belongs to the right-hand side of (2.7) for $i = 1, \dots, p+q$. This means that for $i = 1, \dots, p+q$, there exist appropriate matrices $A''_{ij} \in \mathbf{A}^{\exists}_{ij}, B''_{ik} \in \mathbf{B}^{\exists}_{ik}, C''_i \in \mathbf{C}^{\exists}_i$, for $j = 1, \dots, p$ and $k = 1, \dots, q$ such that $\sum_{j=1}^{p} A'_{ij} X_j + \sum_{k=1}^{q} Y_k B'_{ik} - C'_i = C''_i - \sum_{j=1}^{p} A''_{ij} X_j - \sum_{k=1}^{q} Y_k B''_{ik}$. By (2.4), we conclude that $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\alpha\beta\gamma}$, and the proof is completed. $\square$

Now, let us consider some special cases of the AE-solution set $\Xi_{\alpha\beta\gamma}$. These cases include some important solution sets such as united solution sets, controllable solution sets and tolerable solution sets for the interval linear systems.

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http://repository.uwyo.edu/ela

THEOREM 2.7. *If*

$$\left\{\begin{array}{c} \sum_{j=1}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} - \mathbf{C}_1^{\forall} \subseteq \mathbf{C}_1^{\exists}, \\ \vdots \\ \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{(p+q)k} - \mathbf{C}_{p+q}^{\forall} \subseteq \mathbf{C}_{p+q}^{\exists}, \end{array}\right. \tag{2.9}$$

then $(X_1, \ldots, X_p, Y_1, \ldots, Y_q) \in \Xi_{\forall\forall\gamma}$.

Proof. Let (2.9) hold. By (2.6) and the assumption of the theorem, we can write

$$\begin{aligned} &\left\{ \sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} - C_i' : A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i' \in \mathbf{C}_i^{\forall} \right\} \\ &\subseteq \square \left\{ \sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} - C_i' : A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i' \in \mathbf{C}_i^{\forall} \right\} \\ &= \sum_{j=1}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} - \mathbf{C}_i^{\forall} \subseteq \mathbf{C}_i^{\exists} = \{C_i'' : C_i'' \in \mathbf{C}_i^{\exists}\}, \qquad i = 1, \ldots, p+q. \end{aligned}$$

Thus, by Theorem 2.6, we conclude that $(X_1, \ldots, X_p, Y_1, \ldots, Y_q) \in \Xi_{\forall\forall\gamma}$. ∎

THEOREM 2.8. *If* $(X_1, \ldots, X_p, Y_1, \ldots, Y_q) \in \Xi_{\exists\exists\gamma}$, *then*

$$\left\{\begin{array}{c} \mathbf{C}_1^{\forall} \subseteq \sum_{j=1}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} - \mathbf{C}_1^{\exists}, \\ \vdots \\ \mathbf{C}_{p+q}^{\forall} \subseteq \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{(p+q)k} - \mathbf{C}_{p+q}^{\exists}, \end{array}\right.$$

Proof. Let $(X_1, \ldots, X_p, Y_1, \ldots, Y_q) \in \Xi_{\exists\exists\gamma}$. So by Theorem 2.6, we can write

$$\{-C_i' : C_i' \in \mathbf{C}_i^{\forall}\} \subseteq \left\{ C_i'' - \sum_{j=1}^{p} A_{ij} X_j - \sum_{k=1}^{q} Y_k B_{ik} : A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i'' \in \mathbf{C}_i^{\exists} \right\}, \quad i = 1, \ldots, p+q.$$

Part (ii) of Lemma 2.4 and (2.6) yield

$$\begin{aligned} -\mathbf{C}_i^{\forall} = \square\{-C_i' : C_i' \in \mathbf{C}_i^{\forall}\} &\subseteq \square \left\{ C_i'' - \sum_{j=1}^{p} A_{ij} X_j - \sum_{k=1}^{q} Y_k B_{ik} : A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i'' \in \mathbf{C}_i^{\exists} \right\} \\ &= \mathbf{C}_i^{\exists} - \sum_{j=1}^{p} \mathbf{A}_{ij} X_j - \sum_{k=1}^{q} Y_k \mathbf{B}_{ik}, \qquad i = 1, \ldots, p+q. \end{aligned}$$

Now using this property that for two interval quantities $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \subseteq \mathbf{b} \Leftrightarrow -\mathbf{a} \subseteq -\mathbf{b}$, we obtain

$$\left\{\begin{array}{c} \mathbf{C}_1^{\forall} \subseteq \sum_{j=1}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} - \mathbf{C}_1^{\exists}, \\ \vdots \\ \mathbf{C}_{p+q}^{\forall} \subseteq \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{(p+q)k} - \mathbf{C}_{p+q}^{\exists}. \end{array}\right. \qquad ∎$$

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THEOREM 2.9. *If*

$$\Big|\sum_{j=1}^{p}\hat{\mathbf{A}}_{ij}X_j+\sum_{k=1}^{q}Y_k\hat{\mathbf{B}}_{ik}-\hat{\mathbf{C}}_i\Big|\le \check{\mathbf{C}}_i^{\exists}-\check{\mathbf{C}}_i^{\forall}-\sum_{j=1}^{p}\check{\mathbf{A}}_{ij}|X_j|-\sum_{k=1}^{q}|Y_k|\check{\mathbf{B}}_{ik},\qquad i=1,\ldots,p+q, \tag{2.10}$$

then $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)\in\Xi_{\forall\forall\gamma}$.

Proof. By Lemma 2.5, the system of inequalities (2.10) is equivalent to

$$\begin{aligned}&\left|\operatorname{mid}\left(\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}_i^{\forall}\right)-\operatorname{mid}(\mathbf{C}_i^{\exists})\right|\\ &\le \operatorname{rad}(\mathbf{C}_i^{\exists})-\operatorname{rad}\left(\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}_i^{\forall}\right),\qquad i=1,\ldots,p+q.\end{aligned} \tag{2.11}$$

On the other hand, for two interval matrices $\mathbf{A},\mathbf{B}\in\mathbb{IR}^{m\times n}$, we have

$$\mathbf{A}\subseteq\mathbf{B}\Longleftrightarrow|\operatorname{mid}(\mathbf{B})-\operatorname{mid}(\mathbf{A})|\le\operatorname{rad}(\mathbf{B})-\operatorname{rad}(\mathbf{A}), \tag{2.12}$$

see [22]. Now, using (2.12) the system of inequalities (2.11) is equivalent to

$$\begin{cases}\sum_{j=1}^{p}\mathbf{A}_{1j}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{1k}-\mathbf{C}_1^{\forall}\subseteq\mathbf{C}_1^{\exists},\\ \qquad\vdots\\ \sum_{j=1}^{p}\mathbf{A}_{(p+q)j}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{(p+q)k}-\mathbf{C}_{p+q}^{\forall}\subseteq\mathbf{C}_{p+q}^{\exists},\end{cases}$$

that Theorem 2.7 implies $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)\in\Xi_{\forall\forall\gamma}$. ∎

THEOREM 2.10. *If* $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)\in\Xi_{\exists\exists\gamma}$, *then*

$$\Big|\sum_{j=1}^{p}\hat{\mathbf{A}}_{ij}X_j+\sum_{k=1}^{q}Y_k\hat{\mathbf{B}}_{ik}-\hat{\mathbf{C}}_i\Big|\le\sum_{j=1}^{p}\check{\mathbf{A}}_{ij}|X_j|+\sum_{k=1}^{q}|Y_k|\check{\mathbf{B}}_{ik}+\check{\mathbf{C}}_i^{\exists}-\check{\mathbf{C}}_i^{\forall},\qquad i=1,\ldots,p+q. \tag{2.13}$$

Proof. Let $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)\in\Xi_{\exists\exists\gamma}$. So, by Theorem 2.8, we can write

$$\begin{cases}\mathbf{C}_1^{\forall}\subseteq\sum_{j=1}^{p}\mathbf{A}_{1j}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{1k}-\mathbf{C}_1^{\exists},\\ \qquad\vdots\\ \mathbf{C}_{p+q}^{\forall}\subseteq\sum_{j=1}^{p}\mathbf{A}_{(p+q)j}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{(p+q)k}-\mathbf{C}_{p+q}^{\exists}.\end{cases}$$

The above system of inequalities and (2.12) yield

$$\begin{aligned}&\left|\operatorname{mid}\left(\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}_i^{\exists}\right)-\operatorname{mid}(\mathbf{C}_i^{\forall})\right|\\ &\le\operatorname{rad}\left(\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}_i^{\exists}\right)-\operatorname{rad}(\mathbf{C}_i^{\forall}),\qquad i=1,\ldots,p+q,\end{aligned}$$

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that by Lemma 2.5 is equivalent to

$$\Big|\sum_{j=1}^{p}\hat{\mathbf{A}}_{ij}X_j+\sum_{k=1}^{q}Y_k\hat{\mathbf{B}}_{ik}-\hat{\mathbf{C}}_i\Big|\le\sum_{j=1}^{p}\breve{\mathbf{A}}_{ij}|X_j|+\sum_{k=1}^{q}|Y_k|\breve{\mathbf{B}}_{ik}+\breve{\mathbf{C}}_i^{\exists}-\breve{\mathbf{C}}_i^{\forall},\qquad i=1,\dots,p+q.\qquad\square$$

Corollary 2.11. *If the matrix group $(X_1,\dots,X_p,Y_1,\dots,Y_q)$ belongs to the united solution set $\Xi_{\exists\exists\exists}$ of the interval system (1.2), then*

$$0\in\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}_i,\qquad i=1,\dots,p+q,$$

and also

$$\Big|\sum_{j=1}^{p}\hat{\mathbf{A}}_{ij}X_j+\sum_{k=1}^{q}Y_k\hat{\mathbf{B}}_{ik}-\hat{\mathbf{C}}_i\Big|\le\sum_{j=1}^{p}\breve{\mathbf{A}}_{ij}|X_j|+\sum_{k=1}^{q}|Y_k|\breve{\mathbf{B}}_{ik}+\breve{\mathbf{C}}_i,\qquad i=1,\dots,p+q.$$

Proof. It is enough to put $\mathbf{C}_i^{\forall}=0$, for $i=1,\dots,p+q$, in Theorems 2.8 and 2.10. $\square$

2.3. A sufficient condition for boundedness of $\Xi_{\alpha\beta\gamma}$. One of the goals of this paper is obtaining an enclosure for the solution set $\Xi_{\alpha\beta\gamma}$. But this enclosure is achievable only if $\Xi_{\alpha\beta\gamma}$ is bounded. In this subsection, we present a sufficient condition for boundedness of the solution set $\Xi_{\alpha\beta\gamma}$ to the interval generalized coupled matrix equations (1.2).

Theorem 2.12. *For all m-by-n interval matrices $\mathbf{C}_1,\dots,\mathbf{C}_{p+q}$, the AE-solution set $\Xi_{\alpha\beta\gamma}$ to the interval generalized coupled matrix equations (1.2) is bounded if the system of inequalities*

$$\Big|\sum_{j=1}^{p}\hat{\mathbf{A}}_{ij}X_j+\sum_{k=1}^{q}Y_k\hat{\mathbf{B}}_{ik}\Big|\le\sum_{j=1}^{p}\breve{\mathbf{A}}_{ij}|X_j|+\sum_{k=1}^{q}|Y_k|\breve{\mathbf{B}}_{ik},\qquad i=1,\dots,p+q,\tag{2.14}$$

has only the trivial solution $(X_1,\dots,X_p,Y_1,\dots,Y_q)=(0,\dots,0)$.

Proof. As we said previously, the united solution set $\Xi_{\exists\exists\exists}$ is the widest solution set for the interval system (1.2), i.e., $\Xi_{\alpha\beta\gamma}\subseteq\Xi_{\exists\exists\exists}$. So it is enough to show that $\Xi_{\exists\exists\exists}$ is bounded. Using Theorem 2.10, if $(X_1,\dots,X_p,Y_1,\dots,Y_q)$ belongs to the united solution set of the interval generalized coupled matrix equations

$$\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{k=1}^{q}Y_k\mathbf{B}_{ik}=0,\qquad i=1,\dots,p+q,\tag{2.15}$$

then $(X_1,\dots,X_p,Y_1,\dots,Y_q)$ solves (2.14). But the system of inequalities (2.14) has only the trivial solution $(X_1,\dots,X_p,Y_1,\dots,Y_q)=(0,\dots,0)$, so the united solution set of the interval system (2.15) is the singleton set $\{(0,\dots,0)\}$, i.e., for all $A_{ij}\in\mathbf{A}_{ij}$ and $B_{ik}\in\mathbf{B}_{ik}$, $i=1,\dots,p+q$, $j=1,\dots,p$ and $k=1,\dots,q$, the generalized coupled matrix equations

$$\sum_{j=1}^{p}A_{ij}X_j+\sum_{k=1}^{q}Y_kB_{ik}=0,\qquad i=1,\dots,p+q,$$

has only the trivial solution $(0,\dots,0)$. Hence, its equivalent system $Pz=0$ in which

$$P=\begin{pmatrix} I_n\otimes A_{11} & \cdots & I_n\otimes A_{1p} & B_{11}^{\top}\otimes I_m & \cdots & B_{1q}^{\top}\otimes I_m\\ \vdots & & \vdots & \vdots & & \vdots\\ I_n\otimes A_{(p+q)1} & \cdots & I_n\otimes A_{(p+q)p} & B_{(p+q)1}^{\top}\otimes I_m & \cdots & B_{(p+q)q}^{\top}\otimes I_m\end{pmatrix},$$

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and $z = (\mathrm{vec}(X_1)^\top, \dots, \mathrm{vec}(X_p)^\top, \mathrm{vec}(Y_1)^\top, \dots, \mathrm{vec}(Y_q)^\top)^\top$ for all $A_{ij} \in \mathbf{A}_{ij}$ and $B_{ik} \in \mathbf{B}_{ik}$, $i = 1, \dots, p+q$, $j = 1, \dots, p$, $k = 1, \dots, q$, has the unique solution $z = 0 \in \mathbb{R}^{mn(p+q)}$. This implies the non-singularity of P for all $A_{ij} \in \mathbf{A}_{ij}$ and $B_{ik} \in \mathbf{B}_{ik}$. Thus, the following set is bounded for every interval vector $\mathbf{f} \in \mathbb{R}^{mn(p+q)}$

$$\{z \in \mathbb{R}^{mn(p+q)} : Pz = f,\ A_{ij} \in \mathbf{A}_{ij},\ B_{ik} \in \mathbf{B}_{ik},\ f \in \mathbf{f}\}.$$

Bringing the above set back to its equivalent set (using vectorization operator), i.e., the set

$$\Bigg\{(X_1, \dots, X_p, Y_1, \dots, Y_q) : \Bigg(\sum_{j=1}^{p} A_{ij} X_j + \sum_{k=1}^{q} Y_k B_{ik} = C_i,\ i = 1, \dots, p+q\Bigg) \\ A_{ij} \in \mathbf{A}_{ij}, B_{ik} \in \mathbf{B}_{ik}, C_i \in \mathbf{C}_i\Bigg\},$$

in which $\mathbf{C}_i$, $i = 1, \dots, p+q$, is organized such that $\mathbf{f} = (\mathrm{vec}(\mathbf{C}_1)^\top, \dots, \mathrm{vec}(\mathbf{C}_{p+q})^\top)^\top$, yields the boundedness of the united solution set to the interval generalized coupled matrix equations (1.2) for all interval matrices $\mathbf{C}_1, \dots, \mathbf{C}_{p+q} \in \mathbb{IR}^{m\times n}$. ◻

3. Some approaches for enclosing the AE-solution set of type $\exists\exists\gamma$.

In this section, we consider the problem of outer estimation of the AE-solution set of type $\exists\exists\gamma$ to the interval generalized coupled matrix equations (1.2). An interval matrix group $(\mathbf{X}_1, \dots, \mathbf{X}_p, \mathbf{Y}_1, \dots, \mathbf{Y}_q)$ is an outer estimation for the solution set $\Xi_{\alpha\beta\gamma}$ if

$$\Xi_{\alpha\beta\gamma} \subseteq (\mathbf{X}_1, \dots, \mathbf{X}_p, \mathbf{Y}_j, \dots, \mathbf{Y}_q).$$

3.1. An iterative technique.

Here, we want to propose an iterative method based on the Gauss-Seidel iteration for enclosing the truncated solution set $\Xi_{\exists\exists\gamma}$ with any desired interval matrix group. The Gauss-Seidel iteration which is a well-known method for solving the linear systems has been used by some authors for enclosing the solution set of the interval and parametric systems, for example see [7, 8, 17, 23, 32, 35].

For arbitrary interval matrix group $(\mathbf{X}_1, \dots, \mathbf{X}_p, \mathbf{Y}_1, \dots, \mathbf{Y}_q)$ where $\mathbf{X}_j, \mathbf{Y}_k \in \mathbb{IR}^{m\times n}$ for $j = 1, \dots, p$ and $k = 1, \dots, q$, we are interested in good enclosures for the truncated solution set

$$\Xi_{\exists\exists\gamma} \cap (\mathbf{X}_1, \dots, \mathbf{X}_p, \mathbf{Y}_1, \dots, \mathbf{Y}_q).$$

Let $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\exists\exists\gamma}$, thus Theorem 2.8 yields

$$\mathbf{C}_i^{\forall} \subseteq \sum_{j=1}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} - \mathbf{C}_i^{\exists}, \qquad i = 1, \dots, p+q. \tag{3.16}$$

For $r = 1, \dots, p$, (3.16) is equivalent to

$$-\mathrm{dual}(\mathbf{A}_{ir} X_r) \subseteq \sum_{\substack{j=1\\ j\neq r}}^{p} \mathbf{A}_{ij} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{ik} - (\mathbf{C}_i^{\exists} + \mathrm{dual}(\mathbf{C}_i^{\forall})), \qquad i = 1, \dots, p+q,$$

and by putting $\mathbf{C}_i^* = \mathbf{C}_i^{\exists} + \mathrm{dual}(\mathbf{C}_i^{\forall})$, we obtain

$$\mathrm{dual}(\mathbf{A}_{ir}) X_r \subseteq \mathbf{C}_i^* - \sum_{\substack{j=1\\ j\neq r}}^{p} \mathbf{A}_{ij} X_j - \sum_{k=1}^{q} Y_k \mathbf{B}_{ik}, \qquad i = 1, \dots, p+q. \tag{3.17}$$

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If we write inclusion (3.17) componentwise, for $s=1,\ldots,m$ and $t=1,\ldots,n$, we will have

$$\sum_{\ell=1}^{m}\mathrm{dual}((\mathbf{A}_{ir})_{s\ell})X_{r\ell t}\subseteq \mathbf{C}^*_{ist}-\sum_{\substack{j=1\\ j\neq r}}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}X_{j\ell t}-\sum_{k=1}^{q}\sum_{\ell=1}^{n}Y_{ks\ell}(\mathbf{B}_{ik})_{\ell t},\qquad i=1,\ldots,p+q. \tag{3.18}$$

By adding term $-\sum_{\substack{\ell=1\\ \ell\neq s}}^{m}(\mathbf{A}_{ir})_{s\ell}X_{r\ell t}$ to both sides of inclusion (3.18) and then dividing both sides of the resulting inclusion by $(\mathbf{A}_{ir})_{ss}$, we obtain

$$\begin{aligned}X_{rst}&\subseteq\Big(\mathbf{C}^*_{ist}-\sum_{\substack{j=1\\ j\neq r}}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}X_{j\ell t}-\sum_{k=1}^{q}\sum_{\ell=1}^{n}Y_{ks\ell}(\mathbf{B}_{ik})_{\ell t}-\sum_{\substack{\ell=1\\ \ell\neq s}}^{m}(\mathbf{A}_{ir})_{s\ell}X_{r\ell t}\Big)/(\mathbf{A}_{ir})_{ss}\\ &\subseteq\Big(\mathbf{C}^*_{ist}-\sum_{\substack{j=1\\ j\neq r}}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}\mathbf{X}_{j\ell t}-\sum_{k=1}^{q}\sum_{\ell=1}^{n}\mathbf{Y}_{ks\ell}(\mathbf{B}_{ik})_{\ell t}-\sum_{\substack{\ell=1\\ \ell\neq s}}^{m}(\mathbf{A}_{ir})_{s\ell}\mathbf{X}_{r\ell t}\Big)/(\mathbf{A}_{ir})_{ss}\\ &=:\boldsymbol{\mathcal{X}}^i_{rst},\qquad i=1,\ldots,p+q.\end{aligned}$$

Then we set $\boldsymbol{\mathcal{X}}_{rst}$ as follows

$$\boldsymbol{\mathcal{X}}_{rst}=\bigcap_{i=1}^{p+q}\boldsymbol{\mathcal{X}}^i_{rst}, \tag{3.19}$$

for $s=1,\ldots,m$ and $t=1,\ldots,n$. The above inclusion for X_{rst} is provided that $(\mathbf{A}_{ir})_{ss}$ is invertible and the interval matrix group $(\mathbf{X}_1,\ldots,\mathbf{X}_p,\mathbf{Y}_1,\ldots,\mathbf{Y}_q)$ contains $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)$. Note that $\mathbf{X}_{rst}$ denotes the (s,t)-th component of $\mathbf{X}_r$, i.e., $\mathbf{X}_{rst}\equiv(\mathbf{X}_r)_{st}$.

In a similar manner, for $\nu=1,\ldots,q$, (3.16) is equivalent to

$$-\mathrm{dual}(Y_\nu\mathbf{B}_{i\nu})\subseteq\sum_{j=1}^{p}\mathbf{A}_{ij}X_j+\sum_{\substack{k=1\\ k\neq\nu}}^{q}Y_k\mathbf{B}_{ik}-\mathbf{C}^*_i,\qquad i=1,\ldots,p+q,$$

which yields

$$Y_\nu\mathrm{dual}(\mathbf{B}_{i\nu})\subseteq\mathbf{C}^*_i-\sum_{j=1}^{p}\mathbf{A}_{ij}X_j-\sum_{\substack{k=1\\ k\neq\nu}}^{q}Y_k\mathbf{B}_{ik},\qquad i=1,\ldots,p+q. \tag{3.20}$$

Writing (3.20) componentwise, for $s=1,\ldots,m$ and $t=1,\ldots,n$, we obtain

$$\sum_{\ell=1}^{m}Y_{\nu s\ell}\mathrm{dual}((\mathbf{B}_{i\nu})_{\ell t})\subseteq\mathbf{C}^*_{ist}-\sum_{j=1}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}X_{j\ell t}-\sum_{\substack{k=1\\ k\neq\nu}}^{q}\sum_{\ell=1}^{n}Y_{ks\ell}(\mathbf{B}_{ik})_{\ell t},\qquad i=1,\ldots,p+q. \tag{3.21}$$

Adding term $-\sum_{\substack{\ell=1\\ \ell\neq t}}^{n}Y_{\nu s\ell}(\mathbf{B}_{i\nu})_{\ell t}$ to both sides of inclusion (3.21) and then dividing both sides of the resulting

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Volume 34, pp. 695-717, December 2018.
http://repository.uwyo.edu/ela

inclusion by $(\mathbf{B}_{i\nu})_{tt}$, yields

$$\begin{aligned} Y_{\nu st} &\subseteq \Big(\mathbf{C}^*_{ist} - \sum_{j=1}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}X_{j\ell t} - \sum_{\substack{k=1\\k\neq\nu}}^{q}\sum_{\ell=1}^{n}Y_{ks\ell}(\mathbf{B}_{ik})_{\ell t} - \sum_{\substack{\ell=1\\\ell\neq t}}^{n}Y_{\nu s\ell}(\mathbf{B}_{i\nu})_{\ell t}\Big)/(\mathbf{B}_{i\nu})_{tt} \\ &\subseteq \Big(\mathbf{C}^*_{ist} - \sum_{j=1}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}\mathbf{X}_{j\ell t} - \sum_{\substack{k=1\\k\neq\nu}}^{q}\sum_{\ell=1}^{n}\mathbf{Y}_{ks\ell}(\mathbf{B}_{ik})_{\ell t} - \sum_{\substack{\ell=1\\\ell\neq t}}^{n}\mathbf{Y}_{\nu s\ell}(\mathbf{B}_{i\nu})_{\ell t}\Big)/(\mathbf{B}_{i\nu})_{tt} \\ &\qquad =: \boldsymbol{\mathcal{Y}}^i_{\nu st}, \qquad i = 1,\ldots,p+q, \end{aligned}$$

and we define $\boldsymbol{\mathcal{Y}}_{\nu st}$ as follows

$$\boldsymbol{\mathcal{Y}}_{\nu st} = \bigcap_{i=1}^{p+q}\boldsymbol{\mathcal{Y}}^i_{\nu st}, \tag{3.22}$$

for $s = 1,\ldots,m$ and $t = 1,\ldots,n$. Applying (3.19) and (3.22) for $r = 1,\ldots,p$ and $\nu = 1,\ldots,q$, yields a new enclosure matrix group $(\boldsymbol{\mathcal{X}}_1,\ldots,\boldsymbol{\mathcal{X}}_p,\boldsymbol{\mathcal{Y}}_1,\ldots,\boldsymbol{\mathcal{Y}}_q)$ for $(X_1,\ldots,X_p,Y_1,\ldots,Y_q)$ and since this holds for all members of $\Xi_{\exists\exists\gamma}$, we can write

$$\Xi_{\exists\exists\gamma}\cap(\mathbf{X}_1,\ldots,\mathbf{X}_p,\mathbf{Y}_1,\ldots,\mathbf{Y}_q)\subseteq(\boldsymbol{\mathcal{X}}_1,\ldots,\boldsymbol{\mathcal{X}}_p,\boldsymbol{\mathcal{Y}}_1,\ldots,\boldsymbol{\mathcal{Y}}_q)\cap(\mathbf{X}_1,\ldots,\mathbf{X}_p,\mathbf{Y}_1,\ldots,\mathbf{Y}_q).$$

Now, similar to the strategy in Gauss-Seidel method, we can obtain an improved enclosure $(\tilde{\mathbf{X}}_1,\ldots,\tilde{\mathbf{X}}_p,$ $\tilde{\mathbf{Y}}_1,\ldots,\tilde{\mathbf{Y}}_q)$ as follows

$$\begin{aligned} \tilde{\mathbf{X}}^i_{rst} = \Big(&\mathbf{C}^*_{ist} - \sum_{k=1}^{q}\sum_{\ell=1}^{n}\mathbf{Y}_{ks\ell}(\mathbf{B}_{ik})_{\ell t} - \sum_{j=1}^{r-1}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}\tilde{\mathbf{X}}_{j\ell t} - \sum_{j=r+1}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}\mathbf{X}_{j\ell t} \\ &-\sum_{\ell=1}^{s-1}(\mathbf{A}_{ir})_{s\ell}\tilde{\mathbf{X}}_{r\ell t} - \sum_{\ell=s+1}^{m}(\mathbf{A}_{ir})_{s\ell}\mathbf{X}_{r\ell t}\Big)/(\mathbf{A}_{ir})_{ss}, \qquad i = 1,\ldots,p+q, \end{aligned} \tag{3.23}$$

and then define $\tilde{\mathbf{X}}_{rst}$ for $r = 1,\ldots,p$, as

$$\tilde{\mathbf{X}}_{rst} = (\bigcap_{i=1}^{p+q}\tilde{\mathbf{X}}^i_{rst})\cap\mathbf{X}_{rst}. \tag{3.24}$$

And

$$\begin{aligned} \tilde{\mathbf{Y}}^i_{\nu st} = \Big(&\mathbf{C}^*_{ist} - \sum_{j=1}^{p}\sum_{\ell=1}^{m}(\mathbf{A}_{ij})_{s\ell}\tilde{\mathbf{X}}_{j\ell t} - \sum_{k=1}^{\nu-1}\sum_{\ell=1}^{n}\tilde{\mathbf{Y}}_{ks\ell}(\mathbf{B}_{ik})_{\ell t} - \sum_{k=\nu+1}^{q}\sum_{\ell=1}^{n}\mathbf{Y}_{ks\ell}(\mathbf{B}_{ik})_{\ell t} \\ &-\sum_{\ell=1}^{t-1}\tilde{\mathbf{Y}}_{\nu s\ell}(\mathbf{B}_{i\nu})_{\ell t} - \sum_{\ell=t+1}^{n}\mathbf{Y}_{\nu s\ell}(\mathbf{B}_{i\nu})_{\ell t}\Big)/(\mathbf{B}_{i\nu})_{tt}, \qquad i = 1,\ldots,p+q, \end{aligned} \tag{3.25}$$

and then define $\tilde{\mathbf{Y}}_{\nu st}$ for $\nu = 1,\ldots,q$, as

$$\tilde{\mathbf{Y}}_{\nu st} = (\bigcap_{i=1}^{p+q}\tilde{\mathbf{Y}}_{\nu st})\cap\mathbf{Y}_{\nu st}. \tag{3.26}$$

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It is to be noted that for fixed indices r, s and t, first the values of $\tilde{\mathbf{X}}^i_{rst}$ for $i = 1, \ldots, p+q$ are computed in parallel by (3.23) and then $\tilde{\mathbf{X}}_{rst}$ is computed by (3.24). The values of $\tilde{\mathbf{Y}}^i_{\nu st}$ for $i = 1, \ldots, p+q$ and $\tilde{\mathbf{Y}}_{\nu st}$ are computed in a similar manner, respectively by (3.25) and (3.26). Also, first interval matrices $\tilde{\mathbf{X}}_1, \ldots, \tilde{\mathbf{X}}_p$ must be constructed column–by–column by (3.23) and (3.24) and then interval matrices $\tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_q$ are constructed row–by–row using (3.25) and (3.26). Argument leading to the construction of Equations (3.24) and (3.26) yields the following theorem.

Theorem 3.1. *Consider the interval generalized coupled matrix equations* (1.2). *For the given interval matrix group* $(\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$ *in which* $\mathbf{X}_j, \mathbf{Y}_k \in \mathbb{IR}^{m\times n}$, *for* $j = 1, \ldots, p$ *and* $k = 1, \ldots, q$, *if the interval matrix group* $(\tilde{\mathbf{X}}_1, \ldots, \tilde{\mathbf{X}}_p, \tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_q)$ *is constructed by Equations* (3.23)–(3.26) *then*

$$\Xi_{\exists\exists\gamma} \cap (\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q) \subseteq (\tilde{\mathbf{X}}_1, \ldots, \tilde{\mathbf{X}}_p, \tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_q) \subseteq (\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q).$$

If $(\tilde{\mathbf{X}}_1, \ldots, \tilde{\mathbf{X}}_p, \tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_q) \subset (\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$ then iterating the above procedure can provide a further improved enclosure. Algorithm 1 describes the computational scheme of the proposed modified interval Gauss-Seidel method (MIGS). Since we want to enclose the truncated solution set $\Xi_{\exists\exists\gamma}$ with the initial interval matrix group $(\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$, then $(\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$ can be chosen arbitrarily. However, in practice we are interested in finding an initial interval matrix group which is an enclosure for the solution set $\Xi_{\exists\exists\gamma}$. For instance, it can be the solution obtained by another method for enclosing the solution set $\Xi_{\exists\exists\gamma}$ or, in some cases, it is chosen as an interval matrix group containing zero with a large radius. Similar to the convention in [7], distance between two interval matrix groups $(\mathbf{A}_1, \ldots, \mathbf{A}_s)$ and $(\mathbf{B}_1, \ldots, \mathbf{B}_s)$ is considered as the maximum distances between their components, i.e.,

$$\text{distance}\big((\mathbf{A}_1, \ldots, \mathbf{A}_s), (\mathbf{B}_1, \ldots, \mathbf{B}_s)\big) = \max_{1\le z\le s}\{\text{distance}(\mathbf{A}_z, \mathbf{B}_z)\},$$

in which distance$(\mathbf{A}_z, \mathbf{B}_z)$ is considered as any arbitrary interval metric between two interval matrices, see [22].

Theorem 3.2. *Algorithm 1 requires* $\mathcal{O}(mn(p+q)^2(mp+nq))$ *arithmetic operations.*

Proof. Computing $\tilde{\mathbf{X}}^i_{rst}$ in line 8 of Algorithm 1 by (3.23) needs the following amount of operations

$$\begin{aligned} 2(n-1)(q-1) + 2(m-1)(r-2) + 2(m-1)(p-r-1) + 2(s-2) + 2(m-s-1) + 6 \\ = 2(mp+nq) - 2p - 2q - 4m - 2n + 8, \end{aligned}$$

so it requires $\mathcal{O}(mp+nq)$ arithmetic operations. But this value should be computed in four loops $t = 1, \ldots, n$, $r = 1, \ldots, p$, $s = 1, \ldots, m$ and $i = 1, \ldots, p+q$. So, if we don't consider the probability of executing the commands "*break*" in lines 10 and 15 then we will need at most $\mathcal{O}(mnp(p+q)(mp+nq))$ operations. Similarly, computing $\tilde{\mathbf{Y}}^i_{\nu st}$ in line 24 using relation (3.25) needs

$$\begin{aligned} 2(m-1)(p-1) + 2(n-1)(\nu-2) + 2(n-1)(q-\nu-1) + 2(t-2) + 2(n-t-1) + 6 \\ = 2(mp+nq) - 2p - 2q - 2m - 4n + 8, \end{aligned}$$

operations. Thus, it costs $\mathcal{O}(mp+nq)$. Because it is computed in four loops $s = 1, \ldots, m$, $\nu = 1, \ldots, q$, $t = 1, \ldots, n$ and $i = 1, \ldots, p+q$, so again ignoring the commands "*break*" in lines 26 and 31 results in at most $\mathcal{O}(mnq(p+q)(mp+nq))$ arithmetic operations. Now, since time complexity of the remaining parts of the algorithm is negligible, so totally Algorithm 1 requires $\mathcal{O}(mn(p+q)^2(mp+nq))$ arithmetic operations. $\square$

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Algorithm 1 MIGS method for providing an enclosure $(\tilde{\mathbf{X}}_1,\dots,\tilde{\mathbf{X}}_p,\tilde{\mathbf{Y}}_1,\dots,\tilde{\mathbf{Y}}_q)$ for $\Xi_{\exists\exists\gamma} \cap (\mathbf{X}_1,\dots,\mathbf{X}_p,\mathbf{Y}_1,\dots,\mathbf{Y}_q)$ to the interval system $\sum_{j=1}^{p} \mathbf{A}_{ij}X_j + \sum_{k=1}^{q} Y_k\mathbf{B}_{ik} = \mathbf{C}_i, \quad i=1,\dots,p+q.$

1: (m,n)=size$(\mathbf{X}_1)$;
2: dis=∞;
3: **while** dis $>=$ *tol* **do**
4:     **for** t=1:n **do**
5:         **for** r=1:p **do**
6:             **for** s=1:m **do**
7:                 **for** i=1:p+q **do**
8:                     Compute $\tilde{\mathbf{X}}^i_{rst}$ from (3.23); $\triangleright$ {provided that $0 \notin (\mathbf{A}_{ir})_{ss}$}
9:                     **if** ($\tilde{\mathbf{X}}^i_{rst}$ is an improper interval) **then**
10:                         break (end algorithm), disp "$\Xi_{\exists\exists\gamma}$ does not intersect the interval matrix group."
11:                     **end if**
12:                 **end for**
13:                 $\tilde{\mathbf{X}}_{rst} = (\bigcap_{i=1}^{p+q} \tilde{\mathbf{X}}^i_{rst}) \cap \mathbf{X}_{rst}$;
14:                 **if** ($\tilde{\mathbf{X}}_{rst}$ is an empty set) **then**
15:                     break (end algorithm), disp "$\Xi_{\exists\exists\gamma}$ does not intersect the interval matrix group."
16:                 **end if**
17:             **end for**
18:         **end for**
19:     **end for**
20:     **for** s=1:m **do**
21:         **for** ν=1:q **do**
22:             **for** t=1:n **do**
23:                 **for** i=1:p+q **do**
24:                     Compute $\tilde{\mathbf{Y}}^i_{\nu st}$ from (3.25); $\triangleright$ {provided that $0 \notin (\mathbf{B}_{i\nu})_{tt}$}
25:                     **if** ($\tilde{\mathbf{Y}}^i_{\nu st}$ is an improper interval) **then**
26:                         break (end algorithm), disp "$\Xi_{\exists\exists\gamma}$ does not intersect the interval matrix group."
27:                     **end if**
28:                 **end for**
29:                 $\tilde{\mathbf{Y}}_{\nu st} = (\bigcap_{i=1}^{p+q} \tilde{\mathbf{Y}}_{\nu st}) \cap \mathbf{Y}_{\nu st}$;
30:                 **if** ($\tilde{\mathbf{Y}}_{\nu st}$ is an empty set) **then**
31:                     break (end algorithm), disp "$\Xi_{\exists\exists\gamma}$ does not intersect the interval matrix group."
32:                 **end if**
33:             **end for**
34:         **end for**
35:     **end for**
36:     dis=distance $\big((\tilde{\mathbf{X}}_1,\dots,\tilde{\mathbf{X}}_p,\tilde{\mathbf{Y}}_1,\dots,\tilde{\mathbf{Y}}_q),(\mathbf{X}_1,\dots,\mathbf{X}_p,\mathbf{Y}_1,\dots,\mathbf{Y}_q)\big)$;
37:     $(\mathbf{X}_1,\dots,\mathbf{X}_p,\mathbf{Y}_1,\dots,\mathbf{Y}_q) := (\tilde{\mathbf{X}}_1,\dots,\tilde{\mathbf{X}}_p,\tilde{\mathbf{Y}}_1,\dots,\tilde{\mathbf{Y}}_q)$;
38: **Go to Line 3**

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Example 3.3. Consider the interval generalized coupled matrix equations

$$\begin{cases} \mathbf{A}X + Y\mathbf{C} = \mathbf{E}, \\ \mathbf{B}X + Y\mathbf{D} = \mathbf{F}, \end{cases}$$

in which

$$\mathbf{A} = \begin{pmatrix} [1,3] & [2,2] \\ [-2,-1] & [5,7] \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} [0.1,0.3] & [2,3.5] \\ [-0.2,0.2] & [0.1,0.5] \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} [2,4] & [-1,2] \\ [0,1] & [6,8] \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} [0.1,1.3] & [0,0.5] \\ [-0.4,-0.3] & [0.1,2] \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} [3,4] & [-4,-2] \\ [1,2] & [5,8] \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} [3,7] & [1,3] \\ [8,10] & [4,8] \end{pmatrix}.$$

The MIGS method with the following initial point

$$(\mathbf{X}^0, \mathbf{Y}^0) = \left(\begin{pmatrix} [-1000000, 1000] & [-1000000, 1] \\ [-1000000, 2] & [-1000000, 3] \end{pmatrix}, \begin{pmatrix} [-1000000, 3] & [-1000000, 3] \\ [-1000000, 4] & [-1000000, 4] \end{pmatrix} \right),$$

encloses $\Xi_{\exists\exists\exists} \cap (\mathbf{X}^0, \mathbf{Y}^0)$ by the following interval matrix group

$$(\mathbf{X}, \mathbf{Y}) = \left(\begin{pmatrix} [-47.9001, 228.9174] & [-20.5, 1] \\ [-19.7287, 2] & [-10.2, 3] \end{pmatrix}, \begin{pmatrix} [-100.5309, 3] & [-80.7847, 3] \\ [-7.7201, 4] & [-8.2161, 4] \end{pmatrix} \right).$$

Now, we want to present some numerical examples in higher dimensions to compare the results obtained by the proposed iterative method (MIGS) in this paper and applying interval Gauss-Seidel iteration method (IGS) on the transformed system (1.3). We compare these methods by relative sums of radii (RSR) with respect to the enclosure obtained by MIGS method, i.e., for the enclosure $(\mathbf{X}'_1, \dots, \mathbf{X}'_p, \mathbf{Y}'_1, \dots, \mathbf{Y}'_q)$ obtained by MIGS method and enclosure $(\mathbf{X}_1, \dots, \mathbf{X}_p, \mathbf{Y}_1, \dots, \mathbf{Y}_q)$ obtained by IGS method, we display

$$\mathtt{RSR} = \frac{\sum_{i,j,s,t} \Big(\mathrm{rad}((\mathbf{X}_i)_{st}) + \mathrm{rad}((\mathbf{Y}_j)_{st}) \Big)}{\sum_{i,j,s,t} \Big(\mathrm{rad}((\mathbf{X}'_i)_{st}) + \mathrm{rad}((\mathbf{Y}'_j)_{st}) \Big)}.$$

In the examples below, T_M and T_I show the execution times of the MIGS and IGS methods, respectively. For a fixed dimension n, we run our code for a collection of examples. All computational times are in seconds and we utilize some functions of Matlab to produce the input data.

Example 3.4. Let us consider the interval generalized coupled matrix equations

$$\begin{cases} \mathbf{A}X + Y\mathbf{C} = \mathbf{E}, \\ \mathbf{B}X + Y\mathbf{D} = \mathbf{F}, \end{cases}$$

in which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ and initial enclosures $\mathbf{X}$ and $\mathbf{Y}$ are obtained by the following Matlab's functions

```
Al=gallery('parter',n); Au=Al+alpha*rand(n,n); A=infsup(Al,Au);
Bl=ones(n,n); Bu=B1+alpha*rand(n,n); B=infsup(Bl,Bu);
Cl=gallery('lehmer',n); Cu=C1+alpha*ones(n,n); C=infsup(Cl,Cu);
Dl=ones(n,n);Du=Dl+alpha*gallery('lehmer',n); D=infsup(Dl,Du);
E=ones(n,n);F=E;
X=infsup(-100000*ones(n,n),10*ones(n,n));Y=X;
```

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Table 1
Results obtained for solving Example 3.4.

α	10^{-2}			10^{-3}			10^{-4}		
	RSR	T_M	T_I	RSR	T_M	T_I	RSR	T_M	T_I
n									
5	851.6038	1.4674	0.6680	859.2323	1.4557	0.2959	859.9939	1.3834	0.2658
10	384.1710	5.9364	1.2574	387.1505	6.0341	1.3346	387.4475	6.0445	1.3041
15	245.1936	13.9684	7.4800	247.0178	13.8966	7.5788	247.1997	13.9381	7.3881
20	178.4395	24.7795	31.7804	179.7575	24.7881	32.7978	179.8889	24.2119	32.4483
25	141.1575	39.3013	109.8477	142.1823	38.3899	109.6584	142.2844	38.3111	110.8069
30	115.6686	55.5269	312.0254	116.5130	55.2683	311.6932	116.5971	55.1674	311.4249
35	98.2702	76.8896	774.2434	98.9862	76.3107	774.0558	99.0575	76.1981	772.6186
40	85.6007	100.0065	1.6957e+03	86.2209	100.4416	1.7104e+03	86.2828	100.7314	1.7041e+03
45	75.6175	127.2180	3.4681e+03	76.1658	126.1649	3.4519e+03	76.2204	127.9629	3.4320e+03
50	67.9999	156.1719	6.4504e+03	68.4894	156.4499	6.4825e+03	68.5382	156.8516	6.4724e+03

The numerical results for enclosing the united solution set $\Xi_{\exists\exists\exists}$ are reported in Table 1 for various dimensions n and parameters `alpha`.

From the results presented in Table 1, we see that the enclosures obtained by MIGS are tighter than those obtained by IGS. This is because transforming the interval system (1.2) to the interval system (1.3) ignores the dependencies between elements and so causes overestimation in the results. Also, MIGS performs faster than IGS except for small dimensions.

Example 3.5. Consider the same interval generalized coupled matrix equations as in previous example therein $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ and initial enclosures $\mathbf{X}$ and $\mathbf{Y}$ are obtained by the following Matlab's functions

```
Al=abs(gallery('orthog',n,1)); Au=Al+alpha*ones(n,n); A=infsup(Al,Au);
Bl=abs(gallery('orthog',n,2)); Bu=Bl+alpha*ones(n,n); B=infsup(Bl,Bu);
Cl=abs(gallery('orthog',n,3)); Cu=Cl+alpha*ones(n,n); C=infsup(Cl,Cu);
Dl=abs(gallery('orthog',n,4)); Du=Dl+alpha*ones(n,n); D=infsup(Dl,Du);
E=gallery('parter',n);F=gallery('lehmer',n);
X=infsup(-1000000*ones(n,n),ones(n,n));Y=X;
```

The obtained results by executing IGS and MIGS methods for enclosing the solution set $\Xi_{\exists\exists\exists}$ can be seen in Table 2.

As one can see, Table 2 shows that MIGS yields tighter enclosures than IGS in all cases. Also, IGS is more time consuming than MIGS except for small values of n.

Example 3.6. Consider the interval generalized coupled matrix equations

$$\begin{cases} \mathbf{A}X + Y\mathbf{C} = \mathbf{E}, \\ \mathbf{B}X + Y\mathbf{D} = \mathbf{F}, \end{cases}$$

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TABLE 2
Results obtained for solving Example 3.5.

α	10^{-2}			10^{-3}			10^{-4}		
	RSR	T_M	T_I	RSR	T_M	T_I	RSR	T_M	T_I
n									
5	426.6819	0.8712	0.2212	432.8059	0.9034	0.2555	433.4216	0.9321	0.2486
10	137.1097	3.8122	1.1529	139.3320	3.8319	1.1899	139.5556	3.8129	1.2064
15	51.5231	9.1718	7.2429	52.4240	9.0672	7.1714	52.5148	9.1900	7.1738
20	42.5616	18.2889	33.9876	43.3121	16.1030	32.5841	43.3878	16.3844	32.3749
25	27.1383	25.4731	108.6837	27.6229	25.2769	109.8197	27.6717	25.6964	109.1824
30	24.3860	37.0612	306.0820	24.8345	36.6803	311.4656	24.8797	36.5491	308.3196
35	13.5287	50.7129	754.9611	13.7818	50.1552	758.0909	13.8074	50.8408	761.2317
40	11.9473	66.7771	1.6768e+03	12.1683	67.0212	1.6687e+03	12.1906	66.1123	1.6866e+03
45	12.9174	83.4457	3.3901e+03	13.1599	84.1187	3.4102e+03	13.1844	86.6640	3.4533e+03
50	9.8018	106.0270	6.3883e+03	9.9864	103.7530	6.4386e+03	10.0051	106.3511	6.4127e+03

in which **A**, **B**, **C**, **D**, **E**, **F** and initial enclosures **X** and **Y** are

```
Al=ones(n,n); Au=Al+alpha*ones(n,n); A=infsup(Al,Au);
Bl=gallery('parter',n); Bu=B1+alpha*ones(n,n); B=infsup(Bl,Bu);
Cl=rand(n,n); Cu=C1+alpha*ones(n,n); C=infsup(Cl,Cu);
Dl=gallery('lehmer',n);Du=Dl+alpha*ones(n,n); D=infsup(Dl,Du);
E=ones(n,n);F=E;
X=infsup(-100000*ones(n,n),10*ones(n,n));Y=X;
```

Table 3 shows the results obtained by MIGS and IGS methods to find enclosure to the solution set $\Xi_{\exists\exists\exists}$ of the above system, for various dimensions n and parameters `alpha`.

The reported numbers in Table 3 show that in all cases, the enclosures obtained by MIGS method are tighter than those obtained by IGS method. For small dimensions, IGS performs faster than MIGS while from the dimension of $n = 20$ onwards MIGS is much faster.

TABLE 3
Results obtained for solving Example 3.6.

α	10^{-2}			10^{-3}			10^{-4}		
	RSR	T_M	T_I	RSR	T_M	T_I	RSR	T_M	T_I
n									
5	763.2762	1.3970	0.2789	771.9315	1.3681	0.2903	772.7978	1.4125	0.2647
10	329.4500	5.7849	1.2444	333.3682	5.9083	1.3095	333.7606	5.7996	1.1899
15	209.6711	13.7301	7.2319	212.1919	13.6152	7.1780	212.4445	13.6998	7.2710
20	153.2019	24.2242	30.9281	155.0463	24.7118	31.9140	155.2311	24.2849	32.0589
25	120.3922	38.9477	108.0699	121.8390	38.3314	107.2100	121.9839	38.3494	107.2441
30	99.6439	55.4460	308.5164	100.8459	56.1523	309.8872	100.9663	55.1510	309.4986
35	84.5898	75.5929	770.4143	85.6080	75.6881	771.9300	85.7100	72.4440	773.2570
40	73.5375	93.1517	1.6854e+03	74.4218	94.5540	1.7056e+03	74.5104	95.6662	1.7042e+03
45	65.0159	126.8136	3.5788e+03	65.7969	123.6490	3.4899e+03	65.8752	123.2872	3.4713e+03
50	58.2705	154.1580	6.5421e+03	58.9700	152.8976	6.5650e+03	59.0401	154.5989	6.5466e+03

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 695-717, December 2018.
http://repository.uwyo.edu/ela

3.2. An algebraic approach. Here we want to enclose the AE-solution set $\Xi_{\exists\exists\gamma}$ to the interval generalized coupled matrix equations (1.2) by an algebraic approach. The AE-solution set $\Xi_{\exists\exists\gamma}$ has the important united solution set $\Xi_{\exists\exists\exists}$ as its special case.

Definition 3.7. [29] An interval quantity is said to be *algebraic solution to an interval system of equations* if substitution of it into the system and execution of all interval arithmetic operations results in a valid equality.

Lemma 3.8. [18] *For any three point matrices A, B and C of compatible sizes, we have*

$$vec(ABC) = (C^\top \otimes A)vec(B).$$

Suppose $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\exists\exists\gamma}$, then by Theorem 2.8, we can write

$$\left\{\begin{array}{l} 0 \in \sum_{j=1}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} - (\mathbf{C}_1^{\exists} + \mathrm{dual}(\mathbf{C}_1^{\forall})), \\ \qquad\qquad\vdots \\ 0 \in \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{(p+q)k} - (\mathbf{C}_{p+q}^{\exists} + \mathrm{dual}(\mathbf{C}_{p+q}^{\forall})). \end{array}\right. \tag{3.27}$$

By putting $\mathbf{C}_i^* = \mathbf{C}_i^{\exists} + \mathrm{dual}(\mathbf{C}_i^{\forall})$ for $i = 1, \dots, p+q$, and adding the point matrix X_j to both sides of the j-th relation of (3.27), $j = 1, \dots, p$, and Y_k to both sides of the $(p+k)$-th relation of (3.27), $k = 1, \dots, q$, we will obtain the following equivalent relation

$$\left\{\begin{array}{l} X_1 \in (\mathbf{A}_{11} + I_m) X_1 + \sum_{j=2}^{p} \mathbf{A}_{1j} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{1k} - \mathbf{C}_1^*, \\ \qquad\qquad\vdots \\ X_p \in (\mathbf{A}_{pp} + I_m) X_p + \sum_{j=1}^{p-1} \mathbf{A}_{pj} X_j + \sum_{k=1}^{q} Y_k \mathbf{B}_{pk} - \mathbf{C}_p^*, \\ Y_1 \in \sum_{j=1}^{p} \mathbf{A}_{(p+1)j} X_j + Y_1(\mathbf{B}_{(p+1)1} + I_n) + \sum_{k=2}^{q} Y_k \mathbf{B}_{(p+1)k} - \mathbf{C}_{p+1}^*, \\ \qquad\qquad\vdots \\ Y_q \in \sum_{j=1}^{p} \mathbf{A}_{(p+q)j} X_j + Y_q(\mathbf{B}_{(p+q)q} + I_n) + \sum_{k=1}^{q-1} Y_k \mathbf{B}_{(p+q)k} - \mathbf{C}_{p+q}^*. \end{array}\right.$$

Using the vectorization operator and Lemma 3.8, we will have

$$\begin{aligned} \mathrm{vec}(X_r) \in (I_n \otimes (\mathbf{A}_{rr} + I_m))\mathrm{vec}(X_r) &+ \sum_{\substack{j=1 \\ j\neq r}}^{p} (I_n \otimes \mathbf{A}_{rj})\mathrm{vec}(X_j) \\ &+ \sum_{k=1}^{q} (\mathbf{B}_{rk}^\top \otimes I_m)\mathrm{vec}(Y_k) - \mathrm{vec}(\mathbf{C}_r^*), \qquad r = 1, \dots, p, \end{aligned}$$

$$\begin{aligned} \mathrm{vec}(Y_\nu) \in \sum_{j=1}^{p} (I_n \otimes \mathbf{A}_{(p+\nu)j})\mathrm{vec}(X_j) &+ ((\mathbf{B}_{(p+\nu)\nu}^\top + I_n) \otimes I_m)\mathrm{vec}(Y_\nu) \\ &+ \sum_{\substack{k=1 \\ k\neq \nu}}^{q} (\mathbf{B}_{(p+\nu)k}^\top \otimes I_m)\mathrm{vec}(Y_k) - \mathrm{vec}(\mathbf{C}_{p+\nu}^*), \qquad \nu = 1, \dots, q. \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 695-717, December 2018.
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The above system implies

$$z \in \mathbf{G}z - \mathbf{c}, \tag{3.28}$$

where $z = (\mathrm{vec}(X_1)^\top, \dots, \mathrm{vec}(X_p)^\top, \mathrm{vec}(Y_1)^\top, \dots, \mathrm{vec}(Y_q)^\top)^\top$ and

$$\mathbf{G} = \mathbf{P} + I, \qquad \mathbf{c} = (\mathrm{vec}(\mathbf{C}_1^*)^\top, \dots, \mathrm{vec}(\mathbf{C}_{p+q}^*)^\top)^\top, \tag{3.29}$$

in which $\mathbf{P}$ is the introduced interval matrix in (1.3) and I is an square block diagonal matrix of order $mn(p+q)$ in which the diagonal elements are matrices $I_n \otimes I_m$.

THEOREM 3.9. [31] *Let an interval matrix $\mathbf{Q} \in \mathbb{IR}^{n\times n}$ be such that the spectral radius $\rho(|\mathbf{Q}|)$ of the matrix made up of the moduli of its entries is less than 1. Then for any vector $\mathbf{d} \in \mathbb{IR}^n$, the algebraic solution to the interval linear system*

$$x = \mathbf{Q}x + \mathbf{d},$$

exists and is unique.

Note that here we assume that the right-hand side interval vector $\mathbf{c}$ is proper. Let us introduce the following notation for the AE-solution set $\Xi_{\exists\exists\gamma}$ to the interval generalized coupled matrix equations (1.2). This notation is needed for theoretical proof of the next theorem.

$$\Xi'_{\exists\exists\gamma} = \left\{ \begin{pmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_p) \\ \mathrm{vec}(Y_1) \\ \vdots \\ \mathrm{vec}(Y_q) \end{pmatrix} : (X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\exists\exists\gamma} \right\}. \tag{3.30}$$

THEOREM 3.10. *Let the AE-solution set $\Xi_{\exists\exists\gamma}$ to the interval generalized coupled matrix equations (1.2) be nonempty and $\rho(|\mathbf{G}|) < 1$, where $\mathbf{G}$ is the interval matrix introduced by (3.29). Then the algebraic solution to the interval linear system*

$$z = \mathbf{G}z - \mathbf{c}, \tag{3.31}$$

where $\mathbf{c} = (vec(\mathbf{C}_1^*)^\top, \dots, vec(\mathbf{C}_{p+q}^*)^\top)^\top$ and $\mathbf{C}_i^* = \mathbf{C}_i^\exists + dual(\mathbf{C}_i^\forall)$ for $i = 1, \dots, p+q$, (which according to Theorem 3.9 *exists and is unique) is an interval vector $\mathbf{z}^*$ which encloses $\Xi'_{\exists\exists\gamma}$ introduced in (3.30), i.e., $\Xi'_{\exists\exists\gamma} \subseteq \mathbf{z}^*$.*

Proof. Suppose $\mathbf{z}^*$ is the unique algebraic solution of the interval linear system (3.31). Let $z \in \Xi'_{\exists\exists\gamma}$, so $z = (\mathrm{vec}(X_1)^\top, \dots, \mathrm{vec}(X_p)^\top, \mathrm{vec}(Y_1)^\top, \dots, \mathrm{vec}(Y_q)^\top)^\top$ in which $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\exists\exists\gamma}$, we must show $z \in \mathbf{z}^*$. Since $(X_1, \dots, X_p, Y_1, \dots, Y_q) \in \Xi_{\exists\exists\gamma}$ by Equation (3.28) we conclude that

$$z \in \mathbf{G}z - \mathbf{c}, \tag{3.32}$$

in which the interval matrix $\mathbf{G}$ and interval vector $\mathbf{c}$ are introduced by (3.29). Now, let us consider the following iteration sequence

$$\begin{cases} \mathbf{z}^{(0)} := z, \\ \mathbf{z}^{(k+1)} := \mathbf{G}\mathbf{z}^{(k)} - \mathbf{c}, \qquad \text{for } k = 0, 1, \dots \end{cases} \tag{3.33}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 695-717, December 2018.
http://repository.uwyo.edu/ela

By induction, we can show all interval vectors generated by (3.33) contain z. As the first step it is obvious that $z \in \mathbf{z}^{(0)}$. Now suppose $z \in \mathbf{z}^{(k)}$, so (3.32) and inclusion monotonicity of the interval arithmetic yield

$$z \in \mathbf{G}z - \mathbf{c} \subseteq \mathbf{G}\mathbf{z}^{(k)} - \mathbf{c} = \mathbf{z}^{(k+1)}.$$

Therefore, $z \in \mathbf{z}^{(k)}$ for all $k = 0, 1, \ldots$

On the other hand, condition $\rho(|\mathbf{G}|) < 1$ implies that the generated sequence by (3.33) is convergent (see [22]). This sequence converges to a fixed point of the mapping

$$\mathbf{z} \longmapsto \mathbf{G}\mathbf{z} - \mathbf{c},$$

which is the unique algebraic solution $\mathbf{z}^*$ to Equation (3.31). But since $z \in \mathbf{z}^k$ for all integer k, it is obvious that

$$z \in \lim_{k\to\infty} \mathbf{z}^{(k)} = \mathbf{z}^*,$$

and the proof is completed. $\square$

Note that Theorem 3.10 offers a way to enclose the AE-solution set $\Xi_{\exists\exists\gamma}$, tacitly. In fact, it is enough to convert the algebraic solution $\mathbf{z}^*$ of (3.31) to a matrix group $(\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$ by dividing its elements such that $\mathbf{X}_j, \mathbf{Y}_k \in \mathbb{IR}^{m\times n}$, for $j = 1, \ldots, p$ and $k = 1, \ldots, q$, and $\mathbf{z}^* = (\mathrm{vec}(\mathbf{X}_1)^\top, \ldots, \mathrm{vec}(\mathbf{X}_p)^\top, \mathrm{vec}(\mathbf{Y}_1)^\top, \ldots, \mathrm{vec}(\mathbf{Y}_q)^\top)^\top$. Then the interval matrix group $(\mathbf{X}_1, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \ldots, \mathbf{Y}_q)$ will be an enclosure for $\Xi_{\exists\exists\gamma}$.

It is worth noting that the technique related to Theorem 3.10 can be implemented with an appropriate preconditioning. By (3.27) and Lemma 3.8, we can write

$$0 \in \mathbf{P}z - \mathbf{c},$$

in which $z = (\mathrm{vec}(X_1)^\top, \ldots, \mathrm{vec}(X_p)^\top, \mathrm{vec}(Y_1)^\top, \ldots, \mathrm{vec}(Y_q)^\top)^\top$ and $\mathbf{P}$ and $\mathbf{c}$ are introduced, respectively, by (1.3) and (3.29). Now, using a preconditioner matrix L typically but not necessarily chosen to be $(P^c)^{-1}$ yields

$$0 \in \mathbf{P}_1 z - \mathbf{c}_1, \qquad \text{where } \mathbf{P}_1 = L\mathbf{P} \text{ and } \mathbf{c}_1 = L\mathbf{c}. \tag{3.34}$$

By (3.34), we conclude $z \in \mathbf{P}_1 z + z - \mathbf{c}_1$ and since z is a thin vector, $z \in (\mathbf{P}_1 + I)z - \mathbf{c}_1 = \mathbf{G}_1 z - \mathbf{c}_1$ in which $\mathbf{G}_1 = \mathbf{P}_1 + I$. Thus, the preconditioned system $z = \mathbf{G}_1 z - \mathbf{c}_1$ can be replaced by (3.31) with more tractable coefficient matrix $\mathbf{G}_1$.

The algebraic solution to the equation (3.31) can be obtained by utilizing the numerical algorithm – the sub-differential Newton method – proposed by Shary [29]. We have to point out that equation (3.31) is not equivalent to equation $\mathbf{P}z = \mathbf{c}$ since equation (3.31) can be written as

$$z = \mathbf{G}z - \mathbf{c} = (\mathbf{P} + I)z - \mathbf{c} = \mathbf{P}z + z - \mathbf{c} \longrightarrow 0 = \mathbf{P}z - \mathbf{c} \longrightarrow \mathbf{P}z = \mathrm{dual}(\mathbf{c}).$$

On the other hand, let $\mathbf{z}_1$ be the algebraic solution of the system (3.31) obtained by the Shary method, i.e., $\mathbf{z}_1 = \mathbf{G}\mathbf{z}_1 - \mathbf{c}$, so

$$\mathbf{z}_1 = \mathbf{G}\mathbf{z}_1 - \mathbf{c} = (\mathbf{P} + I)\mathbf{z}_1 - \mathbf{c} \subseteq \mathbf{P}\mathbf{z}_1 + \mathbf{z}_1 - \mathbf{c}.$$

Thus, the interval arithmetic operations do not allow to conclude that $\mathbf{P}\mathbf{z}_1 = \mathbf{c}$. This means that $\mathbf{z}_1$ will not be necessary to the algebraic solution of the interval system $\mathbf{P}z = \mathbf{c}$. So applying the Shary method on systems $\mathbf{P}z = \mathbf{c}$ and $z = \mathbf{G}z - \mathbf{c}$ do not result in a same solution.

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http://repository.uwyo.edu/ela

4. Conclusion. In this work, we investigated the AE-solution sets for the interval generalized coupled matrix equations (1.2). We then characterized these solution sets and gave a sufficient condition under which these solution sets are bounded. Some approaches, including an iterative technique and an algebraic approach were proposed for enclosing the AE-solution set of type $\exists\exists\gamma$. The numerical tests showed the advantage of the proposed iterative technique MIGS with respect to the implementation of the classical Gauss-Seidel method on the transformed interval system (1.3) in the sense of the quality of the obtained results and also the running times. The presented algebraic approach can be handled by utilizing the numerical algorithm – the sub-differential Newton method – proposed by Shary [29].

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