Electronic Journal of Linear Algebra

Volume 35 *Volume 35* Article 23

2019

Alpha Adjacency: A generalization of adjacency matrices

Matt Hudelson
Washington State University, mhudelson@wsu.edu

Judi McDonald
Washington State University, Jmcdonald1@wsu.edu

Enzo Wendler
Washington State University, enzo.wendler@wsu.edu

Recommended Citation

Hudelson, Matt; McDonald, Judi; and Wendler, Enzo. (2019), "Alpha Adjacency: A generalization of adjacency matrices", *Electronic Journal of Linear Algebra*, Volume 35, pp. 365-375.
DOI: https://doi.org/10.13001/1081-3810.3828

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

α-ADJACENCY: A GENERALIZATION OF ADJACENCY MATRICES*

M. HUDELSON†, J. MCDONALD†, AND E. WENDLER†

Abstract. B. Shader and W. So introduced the idea of the skew adjacency matrix. Their idea was to give an orientation δ to a simple undirected graph G from which a skew adjacency matrix $S(G^\delta)$ is created. The α-adjacency matrix extends this idea to an arbitrary field $\mathbb{F}$. To study the underlying undirected graph, the average α-characteristic polynomial can be created by averaging the characteristic polynomials over all the possible orientations. In particular, a Harary-Sachs theorem for the average α-characteristic polynomial is derived and used to determine a few features of the graph from the average α-characteristic polynomial.

Key words. Graph spectra, Adjacency matrices.

AMS subject classifications. 05C31, 05C50.

1. Introduction. Let G be a simple graph with a vertex set $V = \{1, 2, \dots, n\}$. The (standard) adjacency matrix A is defined by $a_{ij} = 1$ if i is adjacent to j, (i.e., if ij is an edge) and $a_{ij} = 0$ if i is not adjacent to j. The (standard) spectrum of a graph is the spectrum of its adjacency matrix. While the adjacency matrix does depend on the labeling of the vertices, the spectrum does not.

As the spectrum of a graph is uniquely determined by the characteristic polynomial of its adjacency matrix, we will often only focus on the characteristic polynomial. A useful result in calculating the characteristic polynomial is the Harary-Sachs Theorem.

We let $\mathcal{U}_k$ denote the collection of edges and cycles no two on which share a vertex that cover exactly k vertices. We will say $\overrightarrow{U}$ is a *routing* of $U \in \mathcal{U}_k$ and denote this $\overrightarrow{U} \sim U$ if $\overrightarrow{U}$ can be obtained by directing the cycles in U.

THEOREM 1.1 ([2], Theorem 1.3). *The characteristic polynomial for (the standard adjacency matrix) of a undirected graph G is given by*

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1.1}$$

in which the coefficients are

$$a_k = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} = \sum_{U \in \mathcal{U}_k} (-1)^{\#U} 2^{c(U)}, \tag{1.2}$$

where $\#U$ denotes the number of connected components in U and $c(U)$ denotes the number of cycles (not including edges) in U.

This can be generalized to a weighted directed graph, in which the ij entry in the adjacency matrix is the weight of the arc from i to j or 0 if there is no arc. The Harary-Sachs theorem can be extended in this case.

*Received by the editors on June 19, 2018. Accepted for publication on July 18, 2019. Handling Editor: Bryan L. Shader. Corresponding Author: Judi McDonald.

†Department of Mathematics and Statistics, Washington State University, Pullman, WA, 99164, USA (mhudelson@wsu.edu, jmcdonald1@wsu.edu, enzo.wendler@wsu.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

THEOREM 1.2 ([2], Theorem 1.3 with equation (1.35)'). *The characteristic polynomial of a weighted digraph $D(A)$ is given by*

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1.3}$$

in which the coefficients are given by

$$a_k = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \Pi_{\overrightarrow{U}}(A), \tag{1.4}$$

where $\overrightarrow{\mathcal{U}_k}$ is the collection of all sets of vertex disjoint dicycles on exactly k vertices, $\#(\overrightarrow{U})$ is the number of cycles in $\overrightarrow{U}$ and $\Pi_{\overrightarrow{U}}(A)$ is the product of the weights of the edges in U.

2. α-Characteristic polynomials and spectra. The notion of skew spectra was introduced by B. Shader and W. So in [5] and was used to distinguish co-spectral graphs in [1]. The idea is to introduce an orientation $\delta : E \to \{-1, 1\}$ to a simple undirected graph $G = (V, E)$. We replace each edge with two directed arcs, one with weight 1 and the other with weight -1, and we denote the new digraph by $G^\delta = (V, E^\delta)$. The skew adjacency matrix $S(G^\delta)$ is a $\{-1, 0, 1\}$ matrix which is the weighted adjacency matrix of G^δ.

The weight $\Pi_{\overrightarrow{U}}$ of a routing $\overrightarrow{U}$ in a skew symmetric graph could be either ± 1 depending on the orientation δ. Thus, for a given graph G, there may be multiple skew spectra depending on the orientation. In [1] it is shown that the skew adjacency matrices S of a graph G are all cospectral if and only if G contains no even cycles.

Similar to skew adjacency, we will use an orientation $\delta : E \to \{-1, 1\}$ to define the α-adjacency matrix $H_\alpha(G^\delta) = [h_{ij}]$. Let $\mathbb{F}$ represent an arbitrary field, let α be an indeterminate or a non-zero field element, and define the α-adjacency matrix by

$$h_{ij} = \begin{cases} \alpha^{\delta(ij)} & \text{if } (ij) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We will often use a directed graph to denote the orientation of an oriented graph by only showing the edges ij such that $\delta(ij) = 1$. Figure 1 shows the construction of an α-oriented graph from a simple graph G. Figure 2 shows the simplified drawing of the oriented graph and the α-adjacency matrix of the graph in Figure 1.

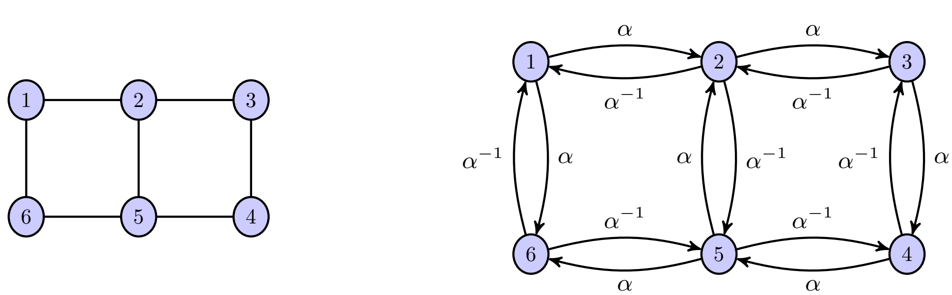
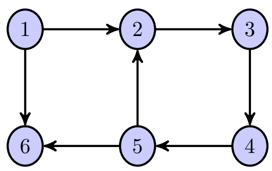


FIGURE 1. *Example of an α-orientation of G.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela



$$H_\alpha(G^\delta) = \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 & \alpha \\ \alpha^{-1} & 0 & \alpha & 0 & \alpha^{-1} & 0 \\ 0 & \alpha^{-1} & 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha^{-1} & 0 & \alpha \\ \alpha^{-1} & 0 & 0 & 0 & \alpha^{-1} & 0 \end{bmatrix}$$

FIGURE 2. *Simplified drawing of the orientation in Figure 1 and the α-δ-adjacency matrix.*

Often it is convenient to use the field of complex numbers and let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$. Notice that $H_1(G^\delta) = A(G)$ and so when $\alpha = 1$ we recover the standard adjacency matrix. Note that this is independent of the choice of the orientation δ. We can also recover the skew-adjacency matrix S^δ by setting $\alpha = i$ and scaling the resulting matrix by $-i$, that is, $-iH_i(G^\delta) = S(G^\delta)$. Also notice that if $|\alpha| = 1$, then $\alpha^{-1} = \bar{\alpha}$, and thus, these α-adjacency matrices are Hermitian matrices.

For a routing, $\overrightarrow{U}$, let $\delta(\overrightarrow{U}) = \sum_{ij \in \overrightarrow{U}} \delta(ij)$. Then we have that

$$\Pi_{\overrightarrow{U}}(H^\delta_\alpha) = \Pi_{(ij) \in \overrightarrow{U}} \alpha^{\delta(ij)} = \alpha^{\delta(\overrightarrow{U})}. \tag{2.5}$$

This is different than is typically used for skew spectra where a product is used instead of a sum.

Given a cycle C in G, there are two routings of C, $\overrightarrow{C}$ and $\overleftarrow{C}$. Note that $\delta(\overrightarrow{C}) = -\delta(\overleftarrow{C})$. We will use the notation $\delta(C) = |\delta(\overrightarrow{C})|$.

Our first result is to extend the Harary-Sachs theorem for α-adjacency matrices.

THEOREM 2.1. *Given a graph G and an orientation δ on G, the characteristic polynomial of $H^\delta_\alpha(G)$ is given by*

$$p_H(x) = x^n + h_1 x^{n-1} + \cdots + h_{n-1} x + h_n \tag{2.6}$$

in which the coefficients are given by

$$h_k = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \left(\frac{\alpha^{\delta(\overrightarrow{U})} + \alpha^{-\delta(\overrightarrow{U})}}{2} \right) \tag{2.7}$$

$$= \sum_{U \in \mathcal{U}_k} (-1)^{\#U} \prod_{\substack{C \in U \\ |C| \geq 3}} (\alpha^{\delta(C)} + \alpha^{-\delta(C)}), \tag{2.8}$$

where $\#U$ is the number of components in U.

Proof. From (1.4) and (2.5) we obtain

$$h_k = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \alpha^{\delta(\overrightarrow{U})}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

Further notice that $\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k$ if and only if $\overleftarrow{U} \in \overrightarrow{\mathcal{U}}_k$, where $\overleftarrow{U}$ is obtained by reversing the directions of all the cycles in $\overrightarrow{U}$. Also note that $\#\overrightarrow{U} = \#\overleftarrow{U}$. Thus,

$$\begin{aligned}\sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \alpha^{\delta(\overrightarrow{U})} &= \frac{1}{2} \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} \left((-1)^{\#\overrightarrow{U}} \alpha^{\delta(\overrightarrow{U})} + (-1)^{\#\overleftarrow{U}} \alpha^{\delta(\overleftarrow{U})} \right) \\ &= \frac{1}{2} \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \left(\alpha^{\delta(\overrightarrow{U})} + \alpha^{\delta(\overleftarrow{U})} \right) \\ &= \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \left(\frac{\alpha^{\delta(\overrightarrow{U})} + \alpha^{-\delta(\overrightarrow{U})}}{2} \right).\end{aligned}$$

This shows (2.7). Next, we note that to sum over all $\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k$ we can first fix a $U \in \mathcal{U}_k$ and sum over all routings of U and then sum over all $U \in \mathcal{U}_k$. Thus,

$$h_k = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}}_k} (-1)^{\#\overrightarrow{U}} \alpha^{\delta(\overrightarrow{U})} \tag{2.9}$$

$$= \sum_{U \in \mathcal{U}_k} (-1)^{\#U} \sum_{\overrightarrow{U} \sim U} \alpha^{\delta(\overrightarrow{U})}. \tag{2.10}$$

Focusing on a fixed $U \in \mathcal{U}_k$, let $U = C_1 \cup C_2 \cdots \cup C_p$, where $C_1, C_2, \ldots, C_p$ are the connected components of U. Note $\overrightarrow{U} \sim U$ if and only if $\overrightarrow{U} = \overrightarrow{C_1} \cup \overrightarrow{C_2} \cdots \cup \overrightarrow{C_p}$, where $\overrightarrow{C_i} \sim C_i$. Further note that $\delta(\overrightarrow{U}) = \sum_{i=1}^{p} \delta(\overrightarrow{C_i})$.

Thus,

$$\begin{aligned}\sum_{\overrightarrow{U} \sim U} \alpha^{\delta(\overrightarrow{U})} &= \sum_{\overrightarrow{C_1} \sim C_1} \sum_{\overrightarrow{C_2} \sim C_2} \cdots \sum_{\overrightarrow{C_p} \sim C_p} \alpha^{\sum_{i=1}^{p} \delta(\overrightarrow{C_i})} \\ &= \sum_{\overrightarrow{C_1} \sim C_1} \sum_{\overrightarrow{C_2} \sim C_2} \cdots \sum_{\overrightarrow{C_p} \sim C_p} \prod_{i=1}^{p} \alpha^{\delta(\overrightarrow{C_i})} \\ &= \sum_{\overrightarrow{C_1} \sim C_1} \alpha^{\delta(\overrightarrow{C_1})} \sum_{\overrightarrow{C_2} \sim C_2} \alpha^{\delta(\overrightarrow{C_2})} \cdots \sum_{\overrightarrow{C_p} \sim C_p} \alpha^{\delta(\overrightarrow{C_p})} \\ &= \prod_{i=1}^{p} \sum_{\overrightarrow{C_i} \sim C_i} \alpha^{\delta(\overrightarrow{C_i})}.\end{aligned}$$

If C_i is an edge, then there is only one $\overrightarrow{C_i} \sim C_i$ and $\delta(C_i) = 0$, hence $\sum_{\overrightarrow{C_i} \sim C_i} \alpha^{\delta(\overrightarrow{C_i})} = 1$.

If C_i is a cycle of length 3 or more, then there exist two routings of C_i. We will denote these routings as $\overrightarrow{C_i}$ and $\overleftarrow{C_i}$. Since $\delta(\overrightarrow{C_i}) = -\delta(\overleftarrow{C_i})$, we get that $\sum_{\overrightarrow{C_i} \sim C_i} \alpha^{\delta(\overrightarrow{C_i})} = \alpha^{\delta(\overrightarrow{C_i})} + \alpha^{\delta(\overleftarrow{C_i})} = \alpha^{\delta(C_i)} + \alpha^{-\delta(C_i)}$. It follows

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

that

$$\begin{aligned}\sum_{\overrightarrow{U}\sim U}\alpha^{\delta(\overrightarrow{U})} &= \prod_{i=1}^{p}\sum_{\overrightarrow{C_i}\sim C_i}\alpha^{\delta(\overrightarrow{C_i})}\\ &= \prod_{\substack{i=1\\|C_i|\geq 3}}^{p}(\alpha^{\delta(C_i)}+\alpha^{-\delta(C_i)})\\ &= \prod_{\substack{C\in U\\|C|\geq 3}}(\alpha^{\delta(C)}+\alpha^{-\delta(C)}).\end{aligned}$$

Substituting this into (2.10), we have the desired result. ∎

REMARK 2.2. In the special case with $\mathbb{F}=\mathbb{C}$ and $|\alpha|=1$, we have that $\alpha^{\delta(C)}+\alpha^{-\delta(C)}=2\operatorname{Re}(\alpha^{\delta(C)})$.

LEMMA 2.3. *The α-spectrum of a tree is independent of both α and of the orientation δ.*

Proof. Let G be a tree. Note that each $U\in\mathcal{U}_k$ must be a matching. That is, U only contains edges, and thus, there are no cycles $C\in U$. Hence,

$$h_k=\sum_{U\in\mathcal{U}_k}(-1)^{\#U},$$

which is independent of both δ and α. ∎

PROPOSITION 2.4. *Let $\mathbb{F}$ be a field of characteristic 0 and G be a graph with at least one odd cycle. For any orientation δ of G, the α-δ-characteristic polynomial depends on α.*

Proof. Let s be the length of the shortest odd cycle in G. Since the only routes in $\mathcal{U}_s$ must be cycles of length s, the coefficient h_s in the α-characteristic polynomial is given by

$$h_s=\sum_{U\in\mathcal{U}_s}-(\alpha^{\delta(U)}+\alpha^{-\delta(U)}).$$

Further, $\delta(U)$ must be an odd number. Thus, if $\alpha=1$, we have $h_s=\sum_{U\in\mathcal{U}_s}-2<0$, and if $\alpha=-1$, then $h_s=\sum_{U\in\mathcal{U}_s}2>0$. Thus, h_s depends on α for any orientation δ. ∎

EXAMPLE 2.5. Notice that over characteristic p, if there are exactly p cycles all of the same length then there may be an orientation δ such that the α-δ-characteristic polynomial does not depend on α. A concrete example of this is given in Figure 3.

Given a graph G and an orientation δ, we say a route $U\in\mathcal{U}_k$ is *contributing* if there exists a routing $\overrightarrow{U}\sim U$ such that $\delta(\overrightarrow{U})=k$.

LEMMA 2.6. *If $U\in\mathcal{U}_k$ is contributing, then U does not have a component consisting of a single edge.*

Proof. Suppose $U\in\mathcal{U}_k$ contains a component consisting of a single edge e and $\overrightarrow{U}\sim U$. Further let $\overrightarrow{U}=e\cup\overrightarrow{U}_0$. Then $\delta(\overrightarrow{U})=\delta(e)+\delta(\overrightarrow{U}_0)=\delta(\overrightarrow{U}_0)\leq k-2$. ∎

THEOREM 2.7. *Given a graph G, there exists an orientation δ such that $\alpha^k+\alpha^{-k}$ appears in the coefficient h_k in the α-characteristic polynomial if and only if there exists a route $U\in\mathcal{U}_k$ such that U does not contain a component consisting of a single edge.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

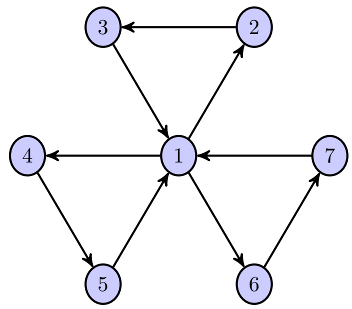


FIGURE 3. *The characteristic polynomial in $\mathbb{F}_3$ is $x^7 + 2x$.*

Proof. Suppose that $\alpha^k + \alpha^{-k}$ appears in the coefficient h_k in the α-characteristic polynomial. Then there must be a contributing route $U \in \mathcal{U}_k$, hence U must not contain a component consisting of a single edge by Lemma 2.6.

To show that if there exists a route $U \in \mathcal{U}_k$ such that U contains no edges then there exists an orientation δ such that $\alpha^k+\alpha^{-k}$ appears in h_k, we will construct an orientation δ such that there is only one contributing cycle in $\mathcal{U}_k$. Choose $U \in \mathcal{U}_k$ that contains no edges and label the cycles $C_1, C_2, \dots, C_p$. Let $\overrightarrow{U} = \overrightarrow{C_1} \cup \overrightarrow{C_2} \cup \cdots \cup \overrightarrow{C_p}$ be a routing of U. Then label the vertices in $\overrightarrow{C_1}$ as $1, 2, \dots, |C_1|$, the vertices in $\overrightarrow{C_2}$ as $|C_1|+1, |C_1|+2, \dots, |C_1|+|C_2|$, and so on until we label the vertices in $\overrightarrow{C_p}$ as $\sum_{i=1}^{p-1}|C_i|+1, \sum_{i=1}^{p-1}|C_i|+2, \dots, \sum_{i=1}^{p}|C_i|$. Let δ be defined by

$$\delta(ij) = \begin{cases} -1 & \text{if } i > j \text{ except if } j = \sum_{k=1}^{s}|C_k|+1 \text{ and } i = \sum_{k=1}^{s+1}|C_k| \text{ for some } s \in \{0,1,\dots,p-1\}, \\ -1 & \text{if } i = \sum_{k=1}^{s}|C_k|+1 \text{ and } j = \sum_{k=1}^{s+1}|C_k| \text{ for some } s \in \{0,1,\dots,p-1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\delta(\overrightarrow{U}) = k$ and $\delta(\overleftarrow{U}) = -k$ but for any other routings $\overrightarrow{U_1} \in \mathcal{U}_k$, $|\delta(\overrightarrow{U_1})| < k$, and thus, cancellation does not occur. ∎

LEMMA 2.8. *Let $\mathbb{F}$ be a field and α be an indeterminate. If the α-spectrum of a graph G is independent of α for all orientations δ, then G is acyclic.*

Proof. Let s be the length of the shortest cycle. Then U_s contains a cycle, and hence, by Theorem 2.7, there exists an orientation such that $\alpha^s + \alpha^{-s}$ appears in h_s. ∎

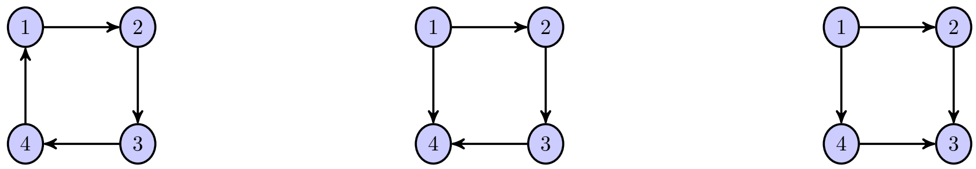
THEOREM 2.9. *Let $\mathbb{F}$ be a field and α be an indeterminate. Given a graph G, the following statements are equivalent:*

(a) G is acyclic;
(b) G has only one α-spectrum, which is independent of both α and the orientation δ.

Proof. If G contains no cycles, then each connected component is a tree, and thus, by Lemma 2.3, the α-spectrum of G is independent of α. The converse is Lemma 2.8. ∎

EXAMPLE 2.10. If we require α to be a non-zero element in $\mathbb{F}$ then Theorem 2.9 no longer holds. For example, consider the finite field $\mathbb{Z}_3$ and let G be a 4 cycle. In general, for a 4 cycle, there are 3 orientations that give rise to different α-δ-characteristic polynomials. However, if we require α to be a non-zero element in $\mathbb{Z}_3$, then α must be 1 or 2. In either of these cases, $\alpha^2 = 1$, and hence, $\alpha^4 = 1$. Thus, all three characteristic polynomials reduce to $P(x) = x^4 - 4x^2$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela



$P_1(x) = x^4 - 4x^2 + 2 - \alpha^4 - \alpha^{-4}$ $\quad P_2(x) = x^4 - 4x^2 + 2 - \alpha^2 - \alpha^{-2}$ $\quad P_3(x) = x^4 - 4x^2$

FIGURE 4. *Characteristic polynomials for a 4 cycle.*

PROPOSITION 2.11. *Given a graph G, there is an orientation δ such that the α-δ-spectrum is independent of α if and only if G does not contain any odd cycles.*

Proof. If G contains an odd cycle, then by Lemma 2.4, the α-spectrum of G depends on α for all orientations δ.

If G does not contain any odd cycles, then G is bipartite. Let U_1, U_2 be the parts of G. Let $\delta(ij) = 1$ if $i \in U_1$, $j \in U_2$. Note for any dicycle $\overrightarrow{C}$ in $\overrightarrow{G}$, there must be the same number of edges from U_1 to U_2 as from U_2 to U_1, and hence, $\delta(C) = 0$. Thus, by equation (2.8), the coefficients in the characteristic polynomial are independent of α, and hence, the α-spectrum is independent of α. ∎

REMARK 2.12. There are α-cospectral graphs. In fact, A. Schwenk shows that almost all trees have a cospectral mate which is also a tree (under the standard adjacency matrix) [4]. Further, by Theorem 2.9, trees that are cospectral under the standard adjacency matrix are also α-cospectral.

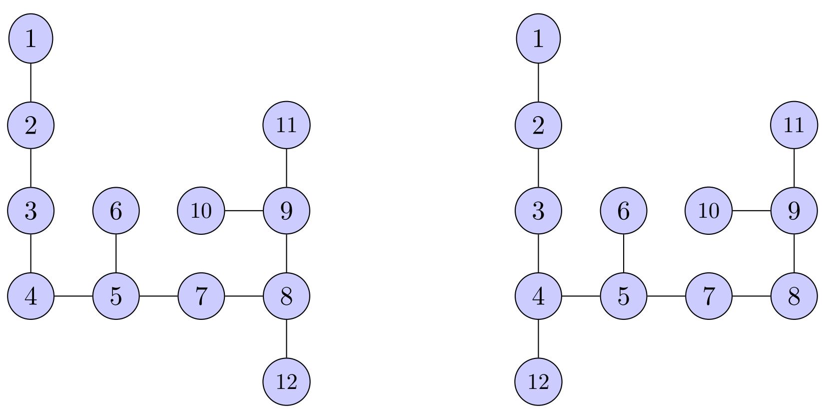


FIGURE 5. *An example of cospectral trees.*

Similar to [4], we can extend the cospectral tree in Figure 5 by attaching any graph G to vertex 12 to produce a pair of α-cospectral graphs.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

3. The average α-characteristic polynomial. In this section, we will consider averaging the α-δ-characteristic polynomials over all the possible orientations of a graph. It is often convenient to use $\hat{\alpha} = \frac{1}{2}(\alpha + \alpha^{-1})$. Again if $\mathbb{F} = \mathbb{C}$ and $|\alpha| = 1$, then $\hat{\alpha} = \mathrm{Re}(\alpha)$. We will begin by looking at the average α-characteristic for a cycle.

THEOREM 3.1. *The average α-characteristic polynomial for a cycle of length n is given by*

$$C_n^\alpha(x) = C_n(x) + 2 - 2\hat{\alpha}^n,$$

where $C_n(x)$ is the standard characteristic polynomial for a cycle of length n and $\hat{\alpha} = \frac{1}{2}(\alpha + \alpha^{-1})$.

Proof. Using Theorem 2.1, the only term that does not come from a collection of disjoint edges is the contribution from the entire cycle. The contribution in the α-characteristic polynomial is the same as in the standard characteristic polynomial for edges. Thus, the only difference between the average α-characteristic polynomial and the standard characteristic polynomial comes from the contribution of the entire cycle. That is

$$C_n^\alpha(x) = C_n(x) + d,$$

where d is the difference between the contribution of the entire cycle in the standard characteristic polynomial and in the α-characteristic polynomial.

In the standard characteristic polynomial, the entire cycle contributes a -2 to the characteristic polynomial.

For the contribution in the α-characteristic polynomial, let us fix a forward direction (e.g. clockwise). Notice that for an orientation, it only matters the number of edges oriented in the forward direction. If the cycle has r edges oriented in the forward direction, then the contribution for that orientation will be $-\alpha^{2r-n} - \alpha^{-(2r-n)}$. For a given r there are $\binom{n}{r}$ orientations with r edges oriented in the forward direction. Thus, the average contribution from the entire cycle will be

$$\begin{aligned}
-\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(\alpha^{2r-n} + \alpha^{-(2r-n)}) &= -\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\alpha^{2r-n} - \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\alpha^{-(2r-n)} \\
&= -\frac{2}{2^n}\sum_{r=0}^{n}\binom{n}{r}\alpha^{2r-n} \\
&= -\frac{2\alpha^{-n}}{2^n}\sum_{r=0}^{n}\binom{n}{r}(\alpha^2)^r \\
&= -\frac{2\alpha^{-n}}{2^n}(1+\alpha^2)^n \\
&= -2\left(\frac{\alpha^{-1}+\alpha}{2}\right)^n \\
&= -2\hat{\alpha}^n.
\end{aligned}$$

The second line follows from changing the dummy variable r to $n - r$ in the second sum and using the fact that $\binom{n}{n-r} = \binom{n}{r}$. The fourth line follows from the binomial formula. All other lines are basic algebraic manipulation. ∎

Using this we can now get a Harary-Sachs result for the average α-characteristic polynomial.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

THEOREM 3.2. *Let $\hat{\alpha} = \frac{1}{2}(\alpha + \alpha^{-1})$ and G be a graph. The average α-characteristic polynomial of G is given by*

$$P(x) = x^n + h_1^{avg} x^{n-1} + \cdots + h_{n-1}^{avg} x + h_n^{avg} \tag{3.11}$$

in which the coefficients are given by

$$h_k^{avg} = \sum_{U \in \mathcal{U}_k} (-1)^{\#(U)} \prod_{\substack{C_i \in U \\ |C_i| \geq 3}} 2\hat{\alpha}^{|C_i|}, \tag{3.12}$$

where $\#(U)$ is the number of components in U.

Proof. Averaging (2.8) over all orientations, we have that

$$h_k^{\text{avg}} = \frac{1}{2^{|E|}} \sum_{\delta} \sum_{U \in \mathcal{U}_k} (-1)^{\#U} \prod_{\substack{C \in U \\ |C| \geq 3}} (\alpha^{\delta(C)} + \alpha^{-\delta(C)}),$$

where $|E|$ is the number of edges in G. As these sums are finite we can switch the order, thus

$$h_k^{\text{avg}} = \frac{1}{2^{|E|}} \sum_{U \in \mathcal{U}_k} (-1)^{\#U} \sum_{\delta} \prod_{\substack{C \in U \\ |C| \geq 3}} (\alpha^{\delta(C)} + \alpha^{-\delta(C)}). \tag{3.13}$$

For a fixed $U \in \mathcal{U}_k$, the orientation of edges not in U does not affect the inner sum. To simplify further, we fix a $U = \{C_1, C_2, \ldots, C_p, e_1, e_2, \ldots, e_s\}$, where each C_i is a cycle (of length at least 3) and each e_i is a pairing of two vertices. Such a U contains $k - s$ edges, thus there are $|E| - k + s$ edges which do not contribute. So for such a U,

$$\sum_{\delta} \prod_{\substack{C \in U \\ |C| \geq 3}} (\alpha^{\delta(C)} + \alpha^{-\delta(C)}) = 2^{|E|-k+s} \sum_{\delta|_U} \prod_{i=1}^{p} (\alpha^{\delta|_U(C_i)} + \alpha^{-\delta|_U(C_i)}),$$

where $\delta|_U$ represents δ restricted to U. Then we can break up the sum over all orientations of U into sums over its components. Thus,

$$\begin{aligned}
\sum_{\delta|_U} \prod_{i=1}^{p} \left(\alpha^{\delta|_U(C_i)} + \alpha^{-\delta|_U(C_i)}\right) &= \sum_{\delta|_{e_1}} \sum_{\delta|_{e_2}} \cdots \sum_{\delta|_{e_s}} \sum_{\delta|_{C_1}} \sum_{\delta|_{C_2}} \cdots \sum_{\delta|_{C_p}} \prod_{C_i \in U} (\alpha^{\delta|_U(C_i)} + \alpha^{-\delta|_U(C_i)}) \\
&= 2^s \prod_{C_i \in U} \sum_{\delta|_{C_i}} (\alpha^{\delta|_{C_i}(C_i)} + \alpha^{-\delta|_{C_i}(C_i)}) \\
&= 2^{k-s} \prod_{C_i \in U} \sum_{\delta|_{C_i}} \frac{1}{2^{|C_i|}} (\alpha^{\delta|_{C_i}(C_i)} + \alpha^{-\delta|_{C_i}(C_i)}).
\end{aligned}$$

The innermost sum is the exact contribution of the cycle c_i which is $2\hat{\alpha}^{|C_i|}$. Thus,

$$\sum_{\delta} \prod_{\substack{C \in U \\ |C| \geq 3}} (\alpha^{\delta(C)} + \alpha^{-\delta(C)}) = 2^{|E|} \prod_{C_i \in U} 2\hat{\alpha}^{|C_i|}.$$

Substituting into equation (3.13), we have

$$h_k^{\text{avg}} = \sum_{U \in \mathcal{U}_k} (-1)^{\#U} \prod_{\substack{C \in U \\ |C| \geq 3}} 2\hat{\alpha}^{|C|}.$$

□

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

Theorem 3.2 can be used to establish the following corollaries.

COROLLARY 3.3. *The average α-characteristic polynomial uniquely determines the number of cycles of length* $3, 4$ *and* 5.

Proof. By Theorem 3.2, the coefficients of the average α-characteristic polynomials are polynomials in $\hat{\alpha}$. Let

$$h_k^{\text{avg}} = \sum_{i=0}^{k} h_i^{(k)} \hat{\alpha}^i. \tag{3.14}$$

The only cycles that contribute to $h_3^{(3)}$ are cycles of length 3, each which contributes a factor of -2, thus the number of cycles of length 3 in G is given by $-\frac{1}{2}h_3^{(3)}$. Similarly, the number of cycles of length 4 is $-\frac{1}{2}h_4^{(4)}$ and the number of cycles of length 5 is $-\frac{1}{2}h_5^{(5)}$. ∎

EXAMPLE 3.4. Notice that the coefficient $h_k^{(k)}$ in equation (3.14) does not determine the number of cycles of length $k \geq 6$. For example, let $G = C_3 \cup C_3 \cup C_6 \cup C_6$ be the collection of two disjoint 3 cycles with two disjoint 6 cycles. Notice that each of the the 6 cycles contribute a -2 to $h_6^{(6)}$ and the collection of the two 3 cycles contribute a $+4$. Thus, $h_6^{(6)} = 0$ even though G contains two 6 cycles.

While there is no easy way to determine the number of cycles of length greater than 5, it is easy to determine the parity of that number.

COROLLARY 3.5. *Let c_k be the number of cycles of length $k \geq 3$ in a graph G and the coefficients in the average α-characteristic polynomial be given by $h_k^{avg} = \sum_{i=0}^{k} h_i^{(k)} \hat{\alpha}^i$. Then*

$$c_k \equiv \frac{h_k^{(k)}}{2} \mod 2.$$

Proof. Notice that for $U \in \mathcal{U}_k$ to contribute to $h_k^{(k)}$, there are not any pairings of vertices in U. Further if U contains s cycles, the contribution of U to $h_k^{(k)}$ is $(-2)^s$. Hence, $\frac{h_k^{(k)}}{2} \mod 2$ will only have contributions from cycles of length exactly k. ∎

COROLLARY 3.6. *The average α-characteristic polynomial uniquely determines the number of matchings on k vertices.*

Proof. Again, let

$$h_k^{\text{avg}} = \sum_{i=0}^{k} h_i^{(k)} \hat{\alpha}^i.$$

The matching polynomial

$$M_G(x) = \sum_{k \geq 0} (-1)^k m_k x^{n-2k}$$

is given by

$$M_G(x) = \sum_{k \geq 0} h_0^{(k)} x^{n-k},$$

where m_k is the number of matchings on k vertices. This can be obtained by setting $\hat{\alpha} = 0$ in the average α characteristic polynomial. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 365-375, August 2019.
http://repository.uwyo.edu/ela

REFERENCES

[1] M. Cavers, S.M. Cioaba, S. Fallat, D.A. Gregory, W.H. Haemers, S.J. Kirkland, J.J. McDonald, and M. Tsatsomeros. Skew-adjacency matrices of graphs. *Linear Algebra Appl.*, 436:4512–4529, 2012.

[2] D. Cvetkovic, M. Doob, and H. Sachs. *Spectra of Graphs.* Academic Press, New York, 1980.

[3] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, Cambridge, 1985.

[4] A.J. Schwenk. Almost all tree are cospectral. *New Directions in Graph Theory*, Academic Press, New York, 275–307, 1973.

[5] B. Shader and W. So. Skew spectra of oriented graphs. *Electron. J. Combin.*, 16(1):Article Number N32, 2009.