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CONE-CONSTRAINED RATIONAL EIGENVALUE PROBLEMS*

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Abstract. This work deals with the eigenvalue analysis of a rational matrix-valued function subject to complementarity constraints induced by a polyhedral cone K. The eigenvalue problem under consideration has the general structure

$$\left(\sum_{k=0}^{d} \lambda^k A_k + \sum_{k=1}^{m} \frac{p_k(\lambda)}{q_k(\lambda)} B_k\right) x = y, \quad K \ni x \perp y \in K^*,$$

where K^* denotes the dual cone of K. The unconstrained version of this problem has been discussed in [Y.F. Su and Z.J. Bai. Solving rational eigenvalue problems via linearization. *SIAM J. Matrix Anal. Appl.*, 32:201–216, 2011.] with special emphasis on the implementation of linearization-based methods. The cone-constrained case can be handled by combining Su and Bai's linearization approach and the so-called facial reduction technique. In essence, this technique consists in solving one unconstrained rational eigenvalue problem for each face of the polyhedral cone K.

Key words. Nonlinear eigenvalue problem, Rational matrix-valued function, Complementarity problem, Polyhedral cone, Linearization method, Facial reduction technique.

AMS subject classifications. 15A18, 15A22, 15A39, 65H17.

1. Introduction.

Let $\mathbb{M}_n$ be the linear space of real matrices of order n. The real version of the polynomial eigenvalue problem consists in finding the values of $\lambda \in \mathbb{R}$ for which the system $P(\lambda)x = 0$ admits a nonzero solution $x \in \mathbb{R}^n$. Here, $P : \mathbb{R} \to \mathbb{M}_n$ is a matrix-valued function of polynomial type

$$P(\lambda) := \sum_{k=0}^{d} \lambda^k A_k,$$

d is a positive integer and the A_k's are matrices of order n. The real spectrum of P is given by

$$\sigma(P) := \{\lambda \in \mathbb{R} : \det[P(\lambda)] = 0\}.$$

We leave complex eigenvalues out of the discussion. Instead of the unconstrained system $P(\lambda)x = 0$, in this work, we analyze a cone-constrained equilibrium model

$$K \ni x \perp \Phi(\lambda)x \in K^* \tag{1.1}$$

involving a possibly non-polynomial matrix-valued function $\Phi : \Lambda \to \mathbb{M}_n$ defined on a subset of the real line. We say that n and Λ are the order and domain of Φ, respectively. The symbol $\perp$ stands for orthogonality relative to the usual inner product of $\mathbb{R}^n$, K is a closed convex cone, and K^* is the dual cone of K. We write sometimes the model (1.1) in the primal-dual form

$$\Phi(\lambda)x = y, \quad K \ni x \perp y \in K^* \tag{1.2}$$

and view (x, y) as a couple of complementarity vectors. By an obvious reason, we refer to x as primal vector and y as dual vector.

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Definition 1. Let K be a closed convex cone in $\mathbb{R}^n$ and $\Phi : \Lambda \to \mathbb{M}_n$ with $\Lambda \subseteq \mathbb{R}$. We say that $\lambda \in \Lambda$ is a K-eigenvalue of Φ if (1.1) holds for some nonzero $x \in \mathbb{R}^n$. Such a vector x is called a K-eigenvector of Φ. The set of K-eigenvalues of Φ is denoted by $\sigma(\Phi, K)$ and it is called the K-spectrum of Φ.

Definition 1 is similar to Definition 2.9 in Seeger [17], but K-spectra of matrix-valued functions are not quite the same mathematical objects as K-spectra of matrices. The link between these objects is reflected by the transfer formula

$$\sigma(\Phi, K) = \{\lambda \in \Lambda : 0 \in S(\Phi(\lambda), K)\}, \tag{1.3}$$

where $S(A, K)$ denotes the K-spectrum of a matrix $A \in \mathbb{M}_n$. If the cone K is the whole space $\mathbb{R}^n$, then (1.1) reduces to $\Phi(\lambda)x = 0$ and

$$\sigma(\Phi, \mathbb{R}^n) = \sigma(\Phi) := \{\lambda \in \Lambda : \det[\Phi(\lambda)] = 0\}$$

is the real spectrum of Φ. The cone-constrained eigenvalue problem (1.1) has been studied in the literature for affine and quadratic matrix-valued functions:

$$\begin{aligned}\Phi_1(\lambda) &:= A_0 + \lambda A_1,\\ \Phi_2(\lambda) &:= A_0 + \lambda A_1 + \lambda^2 A_2.\end{aligned}$$

For getting acquainted with the theory of cone-constrained affine eigenvalue problems the reader may consult Seeger [17], Seeger and Torki [19], Pinto da Costa and Seeger [14, 15], and references therein. Cone-constrained quadratic eigenvalue problems are considered in Seeger [18], Brás et al. [4, 5], Fernandes et al. [7], Iusem et al. [9, 10], and Niu et al. [13].

Two concrete examples of eigenvalue problem of type (1.1) arising in mechanics are solved in Pinto da Costa el al. [16]. In both examples, the convex cone K is polyhedral and Φ is non-polynomial. We would like to underline that if Φ and K are absolutely general, then the set $\sigma(\Phi, K)$ may have a very complicated structure and its numerical computation could practically be impossible. In this work, we study the eigenvalue problem (1.1) under the following two hypotheses:

$$K \text{ is a polyhedral cone in } \mathbb{R}^n, \tag{1.4}$$

$$\Phi \text{ is a rational matrix-valued function of order } n. \tag{1.5}$$

Such particular framework is flexible enough to cover a great variety of cone-constrained eigenvalue problems arising in various fields of mathematics and engineering. To avoid trivialities it is implicitly understood in (1.4) that K spans a linear subspace of dimension at least two. Hypothesis (1.5) means that Φ has the form

$$\Phi(\lambda) = \begin{bmatrix} \frac{p_{1,1}(\lambda)}{q_{1,1}(\lambda)} & \cdots & \frac{p_{1,n}(\lambda)}{q_{1,n}(\lambda)} \\ \vdots & & \vdots \\ \frac{p_{n,1}(\lambda)}{q_{n,1}(\lambda)} & \cdots & \frac{p_{n,n}(\lambda)}{q_{n,n}(\lambda)} \end{bmatrix}, \tag{1.6}$$

where each entry of (1.6) is a quotient of two coprime scalar polynomials, the one in the denominator being not identically zero. Recall that two scalar polynomials are coprime if they do not have a common factor. After carrying out an Euclidean division in each entry of (1.6) and rearranging terms, we obtain the representation

$$\Phi(\lambda) = \underbrace{\sum_{k=0}^{d} \lambda^k A_k}_{\text{polynomial part}} + \underbrace{\sum_{k=1}^{m} \frac{p_k(\lambda)}{q_k(\lambda)} B_k}_{\text{purely rational part}}, \tag{1.7}$$

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where $A_0, \ldots, A_d, B_1, \ldots, B_m$ are matrices of order n and

$$\begin{cases} p_k \text{ and } q_k \text{ are coprime scalar polynomials,} \\ \text{the degree of } p_k \text{ is smaller than the degree of } q_k, \\ \text{the leading coefficient of } q_k \text{ is equal to } 1, \end{cases} \tag{1.8}$$

for all $k \in \mathbb{N}_m := \{1, \ldots, m\}$. The last requirement in (1.8) is a normalization condition that does not entail a loss of generality. The domain of (1.7) is given by

$$\Lambda := \{\lambda \in \mathbb{R} : q_1(\lambda) \neq 0, \ldots, q_m(\lambda) \neq 0\}.$$

Said in other words, the domain Λ leaves aside the poles of the purely rational part

$$S(\lambda) := \sum_{k=1}^{m} \frac{p_k(\lambda)}{q_k(\lambda)} B_k. \tag{1.9}$$

Note that (1.7) includes a polynomial matrix-valued function as particular case. Adding an expression like (1.9) to a polynomial matrix-valued function enlarges considerably the field of applications of the theory of K-spectra.

Example 1. An interesting example of rational matrix-valued function arising in mechanical engineering is

$$\Phi(\lambda) = K - \lambda M + \sum_{k=1}^{m} \frac{\lambda}{\lambda - \omega_k} Q_k, \tag{1.10}$$

where the ω_k's are positive parameters, K and M are symmetric matrices, and the Q_k's are symmetric matrices of low rank. Note that (1.10) admits the representation (1.7) with $d = 1$ and

$$A_0 := K + \sum_{k=1}^{m} Q_k, \quad A_1 := -M, \quad B_k := Q_k, \quad p_k(\lambda) := \omega_k, \quad \text{and} \quad q_k(\lambda) := \lambda - \omega_k.$$

See Voss [22, 23] for a physical interpretation of (1.10).

A battery of examples of rational eigenvalue problems arising in real life applications can be found in Mehrmann and Voss [12, Section 1] and in Betcke et al. [3]. The book of Kaczorek [11] is a rich source of information concerning the general theory of rational matrix-valued functions. Various numerical methods for solving unconstrained rational eigenvalue problems have been proposed in the literature. In Section 3, we explain how to adapt the linearization-based method of Su and Bai [21] to a cone-constrained setting.

2. Characterization of K-spectra in a polyhedral setting. The next theorem characterizes the K-spectrum of a possibly non-rational matrix-valued function. We represent the polyhedral cone K as intersection of finitely many half-spaces, say

$$K = \{x \in \mathbb{R}^n : W^\top x \succeq 0\}, \tag{2.11}$$

where $W = [w_1, \ldots, w_r]$ is a matrix whose columns are nonzero n-dimensional vectors and $u \succeq 0$ means that each component of u is nonnegative. Without loss of generality we assume that no column of W is a nonnegative linear combination of the remaining ones. Before stating Theorem 1 we need to introduce some

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notation. We write $u \succ 0$ to indicate that u is a vector whose components are all positive. For a subset J of $\mathbb{N}_r := \{1, \ldots, r\}$, the symbol $|J|$ stands for the cardinality of J and $\bar{J}$ refers to the set-complement of J with respect to $\mathbb{N}_r$. For a nonempty subset J of $\mathbb{N}_r$, let W_J be the matrix whose columns are $\{w_j : j \in J\}$. Finally, let $\mathbb{J}_W$ be defined by

$$J \in \mathbb{J}_W \quad \Leftrightarrow \quad J \subseteq \mathbb{N}_r,\ 1 \leq |J| \leq n-1, \text{ and } \operatorname{rank}(W_J) = |J|. \tag{2.12}$$

The full rank condition in (2.12) is a short way of saying that $\{w_j : j \in J\}$ are linearly independent vectors.

Theorem 1. *Let K be a polyhedral cone as in (2.11) and $\Phi : \Lambda \to \mathbb{M}_n$ with $\Lambda \subseteq \mathbb{R}$. Then $\lambda \in \Lambda$ is a K-eigenvalue of Φ if and only if either one of the following conditions is true:*

(a) There exists a nonzero vector $x \in \mathbb{R}^n$ such that

$$\Phi(\lambda)x = 0, \quad W^\top x \succeq 0. \tag{2.13}$$

(b) There exist $x \in \mathbb{R}^n$, $J \in \mathbb{J}_W$ and $\eta \in \mathbb{R}^{|J|}$ such that

$$\begin{bmatrix} \Phi(\lambda) & W_J \\ W_J^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ -\eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.14}$$

$$W_{\bar{J}}^\top x \succeq 0, \tag{2.15}$$

$$\eta \succ 0. \tag{2.16}$$

Proof. Let $\lambda \in \Lambda$ be a K-eigenvalue of Φ and x be an associated K-eigenvector. In particular, $W^\top x \succeq 0$ and $y := \Phi(\lambda)x$ belongs to the dual cone

$$K^* = \{Wz : z \succeq 0\}.$$

If $y = 0$, then (a) holds and we are done. Suppose that $y \neq 0$. In such a case, $y = W_J\,\eta$ for some nonempty index set $J \subseteq \mathbb{N}_r$ and some vector $\eta \in \mathbb{R}^{|J|}$ whose components are all positive. Thanks to the conic version of Caratheodory's theorem (cf. [24, Section 6.1]), we may suppose that the columns of W_J are linearly independent, in which case $|J| \leq n$. We know already that $W_J^\top x \succeq 0$ and $W_{\bar{J}}^\top x \succeq 0$. But

$$\langle x, y\rangle = 0 \ \Leftrightarrow\ \langle x, W_J \eta\rangle = 0 \ \Leftrightarrow\ \langle W_J^\top x, \eta\rangle = 0 \ \Leftrightarrow\ W_J^\top x = 0.$$

Hence, we can write

$$W_J^\top x = 0, \quad W_{\bar{J}}^\top x \succeq 0, \quad \Phi(\lambda)x - W_J\,\eta = 0, \quad \eta \succ 0, \tag{2.17}$$

which is precisely the system (2.14)–(2.16). Observe that $|J| \neq n$, because otherwise the first condition in (2.17) would imply that $x = 0$. In conclusion, $J \in \mathbb{J}_W$ and (b) holds. Conversely, let x be as in (a) or (b). In either case we see that x is a K-eigenvector of Φ and λ is an associated K-eigenvalue. ∎

The matrix-valued function Φ in Theorem 1 does not need to be of rational type, but the cone K must be polyhedral and represented as in (2.11). The next small dimensional example illustrates how Theorem 1 works in practice.

Example 2. Let $K = \{x \in \mathbb{R}^2 : x_1 \geq x_2 \geq 0\}$. This polyhedral cone is expressible as in (2.11) with

$$W = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

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Let Φ be the rational matrix-valued function given by

$$\Phi(\lambda) = \begin{bmatrix} -\frac{\lambda(\lambda+2)}{12} & -\frac{\lambda}{\lambda-3} \\ \\ \frac{\lambda+2}{\lambda^2+1} & \lambda^2-4 \end{bmatrix}. \tag{2.18}$$

By computing the roots of

$$\det[\Phi(\lambda)] = -\frac{\lambda^2(\lambda+2)(\lambda-1)(\lambda^3-2\lambda^2-5\lambda+4)}{12(\lambda-3)(\lambda^2+1)},$$

we get the real spectrum

$$\sigma(\Phi) = \{-2.00000,\ -1.85577,\ 0.00000,\ 0.67836,\ 1.00000,\ 3.17741\}.$$

Table 1 displays seven triplets (λ, x, y) solving (1.2). By working out the case (a) of Theorem 1, we get the triplets numbered $1, 2, 3, 5$, and 6. We now work out the case (b) of Theorem 1: the index set $J = \{1\}$ yields the triplet numbered 7, whereas $J = \{2\}$ yields the triplet numbered 4. The value $\lambda = 0$ appears twice in Table 1, but repetitions are not counted in $\sigma(\Phi, K)$. Summarizing,

$$\sigma(\Phi, K) = \{-2.00000,\ -1.85577,\ 0.00000,\ 0.67836,\ 1.00000,\ 1.53349\}. \tag{2.19}$$

This example shows that the K-spectrum and the real spectrum of Φ are not comparable in general.

Table 1
Spectral analysis of Example 2.

Nr.	λ	primal vector		dual vector		facial dimension	
		x_1	x_2	y_1	y_2	primal	dual
1	-2	1	0	0	0	1	0
2	-1.85577	17.1344	1	0	0	2	0
3	0	2	1	0	0	2	0
4	0	1	0	0	2	1	1
5	0.67836	1.9298	1	0	0	2	0
6	1	2	1	0	0	2	0
7	1.53349	1	1	0.5941	-0.5941	1	1

A comment on the last two columns of Table 1 is in order. Suppose that (λ, x, y) solves (1.2). The primal vector x is nonzero and belongs to a face of K. Let $F_x(K)$ be the unique face of K that contains x in its relative interior. The facial dimension of the primal vector x is defined as the dimension of the linear space spanned by $F_x(K)$. Analogously, the facial dimension of the dual vector y is the dimension of the linear space spanned by $F_y(K^*)$.

2.1. Cardinality issues. The next corollary is a direct consequence of Theorem 1. For notational convenience we write

$$\Gamma_J(\lambda) := \det \begin{bmatrix} \Phi(\lambda) & W_J \\ W_J^\top & 0 \end{bmatrix}.$$

Note that Γ_J has the same domain as Φ. Furthermore, if Φ is rational, then Γ_J is rational.

Corollary 1. *Let K be a polyhedral cone as in (2.11) and $\Phi : \Lambda \to \mathbb{M}_n$ with $\Lambda \subseteq \mathbb{R}$. Then a necessary condition for $\lambda \in \Lambda$ to be a K-eigenvalue of Φ is that*

$$\det[\Phi(\lambda)] = 0 \quad \text{or} \quad \Gamma_J(\lambda) = 0 \ \text{ for some } J \in \mathbb{J}_W.$$

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Said in other words,

$$\sigma(\Phi, K) \;\subseteq\; \sigma(\Phi) \bigcup \left(\bigcup_{J\in \mathbb{J}_W} \{t \in \Lambda : \Gamma_J(t) = 0\} \right). \tag{2.20}$$

The set on the right-hand side of (2.20) is a union of finitely many real spectra. Such upper bound is coarse in general, because the inequality constraints in (2.13) and in (2.15)–(2.16) are being neglected. Anyway, the set on the right-hand side of (2.20) can be used to identify potential candidates for membership in $\sigma(\Phi, K)$. The next proposition concerns the finiteness of the K-spectrum of a rational matrix-valued function.

Proposition 1. *Let K be a polyhedral cone as in (2.11) and $\Phi : \Lambda \to \mathbb{M}_n$ with $\Lambda \subseteq \mathbb{R}$. Assume that*

$$\det \circ \Phi \;\; \textit{is not identically zero on } \Lambda, \tag{2.21}$$

$$\forall J \in \mathbb{J}_W,\, \Gamma_J \;\textit{is not identically zero on } \Lambda. \tag{2.22}$$

Then $\sigma(\Phi, K)$ has finite cardinality.

Proof. The composition $\det \circ \Phi$ and the Γ_J's are rational functions on Λ. Assumption (2.21) implies that $\sigma(\Phi)$ is finite and assumption (2.22) implies that each zero-set

$$\Gamma_J^{-1}(\{0\}) := \{t \in \Lambda : \Gamma_J(t) = 0\}$$

is finite. The upper bound (2.20) completes the proof of the proposition. ∎

We state below a variant of Proposition 1 involving the facial structure of K. Let $\mathcal{F}(K)$ be the set of nonzero faces of K. Each face F of a polyhedral cone K is yet another polyhedral cone. The dimension of the face F is understood as the dimension of span$F$, the linear subspace spanned by F. If H is a linear subspace of $\mathbb{R}^n$, then $\mathbb{M}(H)$ denotes the set of matrices of size $n \times \dim H$ whose columns form a basis of H.

Proposition 2. *Let K be a polyhedral cone in $\mathbb{R}^n$ and $\Phi : \Lambda \to \mathbb{M}_n$ be a rational matrix-valued function. Then*

$$\sigma(\Phi, K) \;\subseteq\; \bigcup_{F\in\mathcal{F}(K)} \sigma(\Phi, \mathrm{span}F). \tag{2.23}$$

In particular, under the assumption

$$\begin{cases} \textit{for all } F \in \mathcal{F}(K), \;\textit{there exists } V \in \mathbb{M}(\mathrm{span}F) \;\textit{s.t.} \\ \det[V^\top \Phi(\,\cdot\,)V] \;\textit{is not identically zero on } \Lambda, \end{cases} \tag{2.24}$$

the set $\sigma(\Phi, K)$ has finite cardinality.

Proof. Theorem 3.4 in Seeger and Torki [19] asserts that

$$S(A, K) \;\subseteq\; \bigcup_{F\in\mathcal{F}(K)} S(A, \mathrm{span}F)$$

for all $A \in \mathbb{M}_n$. This general inclusion for K-spectra of matrices and the transfer formula (1.3) yield (2.23). Now, suppose that assumption (2.24) is in force. Pick any $F \in \mathcal{F}(K)$ and select a matrix V as in (2.24).

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Let ℓ be the dimension of F. By using the change of variables $x = Vu$, we see that the subspace-constrained equilibrium model

$$x \in \mathrm{span}F, \quad \Phi(\lambda)x \in [\mathrm{span}F]^{\perp}$$

admits a nonzero solution $x \in \mathbb{R}^n$ if and only if the unconstrained system $V^{\top}\Phi(\lambda)Vu = 0$ admits a nonzero solution $u \in \mathbb{R}^{\ell}$. In other words,

$$\sigma(\Phi, \mathrm{span}F) = \{t \in \Lambda : \det[V^{\top}\Phi(t)V] = 0\}. \tag{2.25}$$

The zero-set (2.25) is finite because the rational function $\det[V^{\top}\Phi(\,\cdot\,)V]$ is not identically zero on Λ. This completes the proof of the proposition. □

Example 3. Consider again Φ and K as in Example 2. The polyhedral cone under consideration has three nonzero faces:

$$F_1 = K, \quad F_2 = \{x \in K : x_1 = x_2\}, \quad F_3 = \{x \in K : x_2 = 0\}.$$

As bases for the linear subspaces spanned by these faces, we use the columns of the matrices

$$V_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_2 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V_3 := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

respectively. We get

$$\sigma(\Phi, \mathrm{span}F_1) = \sigma(\Phi) = \{-2.00000,\ -1.85577,\ 0.00000,\ 0.67836,\ 1.00000,\ 3.17741\}. \tag{2.26}$$

On the other hand, the sets

$$\sigma(\Phi, \mathrm{span}F_2) = \{-2.11009,\ 1.53349,\ 3.49665\}, \tag{2.27}$$

$$\sigma(\Phi, \mathrm{span}F_3) = \{-2.00000,\ 0.00000\} \tag{2.28}$$

are formed with the roots of

$$\det[V_2^{\top}\Phi(\lambda)V_2] = \frac{\lambda+2}{\lambda^2+1} + \lambda^2 - 4 - \frac{\lambda}{\lambda-3} - \frac{\lambda(\lambda+2)}{12},$$

$$\det[V_3^{\top}\Phi(\lambda)V_3] = -\frac{\lambda(\lambda+2)}{12},$$

respectively. Hence, the K-eigenvalues of Φ are to be sought among the elements of (2.26), (2.27), and (2.28). This observation is consistent with what we obtained in (2.19). In this example, the inclusion (2.23) is strict because -2.11009, 3.17741, and 3.49665, are on the right-hand side of (2.23) but not on the left-hand side.

3. Pareto spectra of rational matrix-valued functions. In this section, we assume that K is the n-dimensional Pareto cone, i.e., $K = \mathbb{R}^n_+$. The equilibrium model (1.2) takes the more familiar form

$$\Phi(\lambda)x = y, \quad 0 \preceq x \perp y \succeq 0, \tag{3.29}$$

where nonnegativity of a vector is understood in the componentwise sense. The Pareto cone is the most popular polyhedral cone used to formulate complementarity constraints. The set

$$\Pi(\Phi) := \sigma(\Phi, \mathbb{R}^n_+)$$

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is called Pareto spectrum of Φ and the elements of $\Pi(\Phi)$ are called Pareto eigenvalues of Φ. Since the n-dimensional Pareto cone has 2^n-1 nonzero faces, an exhaustive computation of the Pareto spectrum of a rational matrix-valued function of order n requires to solve 2^n-1 unconstrained rational eigenvalue problems. The next theorem explains the details, but we need first to introduce some notation. The symbol $\mathcal{J}_n$ stands for the set of nonempty subsets of $\mathbb{N}_n := \{1,\ldots,n\}$. Given a matrix $A\in\mathbb{M}_n$, we write A^{J_1,J_2} to indicate the submatrix of A obtained by keeping only the rows indexed by $J_1\in\mathcal{J}_n$ and the columns indexed by $J_2\in\mathcal{J}_n$. In particular, $A^J := A^{J,J}$ is a principal submatrix of A. For alleviating notation, we also write

$$\Phi^{J_1,J_2}(\lambda) := [\Phi(\lambda)]^{J_1,J_2} \quad \text{and} \quad \Phi^{J}(\lambda) := [\Phi(\lambda)]^{J,J}.$$

Theorem 1. *Let $\Phi:\Lambda\to\mathbb{M}_n$ with $\Lambda\subseteq\mathbb{R}$. Then $\lambda\in\Lambda$ is a Pareto eigenvalue of Φ if and only if there exist an index set $J\in\mathcal{J}_n$ and a vector $u\in\mathbb{R}^{|J|}$ such that*

$$\Phi^J(\lambda)u = 0, \tag{3.30}$$

$$u \succ 0, \tag{3.31}$$

$$\Phi^{\bar{J},J}(\lambda)u \succeq 0. \tag{3.32}$$

Proof. It suffices to combine [17, Theorem 4.1] and relation (1.3). □

If J is the whole set $\mathbb{N}_n$, then $\bar{J}$ is empty and the slackness condition (3.32) must be dropped of course. In general, if λ is a Pareto eigenvalue of Φ produced by an index set J, then an associated Pareto eigenvector $x\in\mathbb{R}^n$ is obtained by setting

$$x_j = \begin{cases} u_j & \text{if } j\in J, \\ 0 & \text{if } j\in \bar{J}. \end{cases}$$

Theorem 1 has a number of easy consequences. The next corollary is just one of them. Note that if Φ is a rational matrix-valued function of order n, then Φ^J is a rational matrix-valued function of order $|J|$.

Corollary 2. *Let $\Phi:\Lambda\to\mathbb{M}_n$ with $\Lambda\subseteq\mathbb{R}$. Then*

$$\Pi(\Phi) \subseteq \bigcup_{J\in\mathcal{J}_n} \sigma\left(\Phi^J\right). \tag{3.33}$$

In particular, the Pareto spectrum of Φ has finite cardinality if we assume that, for each $J\in\mathcal{J}_n$, the composite function $f^J := \det\circ\Phi^J$ is not identically zero on Λ.

We call f^J the characteristic function of Φ^J. Strictly speaking, Φ^J and f^J are defined on a domain Λ^J possibly bigger than Λ, but for notational simplicity we consider only their restrictions to Λ. If the characteristic function of at least one Φ^J is identically zero, then the Pareto spectrum of Φ may be uncountable.

Example 2. Consider the rational matrix-valued function of order 3 given by

$$\Phi(\lambda) = \begin{bmatrix} \lambda & -1 & -1 \\ \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \lambda \\ -1 & 1 & 2 \end{bmatrix}.$$

If we write (3.30)–(3.32) for $J=\{1,2\}$, then we get the system

$$\begin{bmatrix} \lambda & -1 \\ \frac{1}{\lambda} & -\frac{1}{\lambda^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_1>0, \quad u_2>0, \quad -u_1+u_2\geq 0. \tag{3.34}$$

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Clearly, f^J is identically zero and (3.34) has a solution $u \in \mathbb{R}^2$ if and only if $\lambda \geq 1$. The Pareto spectrum of Φ contains then the interval $[1, \infty[$. By working out the case $J = \{1, 3\}$, we see that Φ admits $\lambda = 1/2$ as Pareto eigenvalue. Each remaining index set J leads to a system (3.30)–(3.32) that is unsolvable. Hence, $\Pi(\Phi) = \{1/2\} \cup [1, \infty[$. This situation is rather abnormal, because most rational matrix-valued funcions arising in practice have finitely many Pareto eigenvalues.

Theorem 1 suggests to identify the Pareto eigenvalues of a rational $\Phi : \Lambda \to \mathbb{M}_n$ by applying the so-called Facial Reduction Technique:

$$\text{(FRT)} \quad \begin{cases} \text{for each } J \in \mathcal{J}_n \text{ compute the real spectrum of } \Phi^J \text{ and,} \\ \text{for each } \lambda \in \sigma\left(\Phi^J\right), \text{ check whether the kernel of the matrix } \Phi^J(\lambda) \\ \text{contains a } u \succ 0 \text{ satisfying the slackness condition } \Phi^{\bar{J},J}(\lambda)u \succeq 0. \end{cases}$$

The FRT consists then in solving a collection of $2^n - 1$ unconstrained rational eigenvalue problems and checking, for each one of these problems, whether there exists a positive eigenvector satisfying a certain system of inequalities.

3.1. Linearization of rational matrix-valued functions. We now explain how the FRT works in practice when $\Phi : \Lambda \to \mathbb{M}_n$ is rational. We start by considering the index set $J = \mathbb{N}_n$. In this case, $\Phi^J = \Phi$ and the system (3.30)–(3.32) reduces to

$$\Phi(\lambda)x = 0, \quad x \succ 0, \tag{3.35}$$

which is a rational eigenvalue problem with eigenvalues constrained to the real line, except for the fact that the eigenvector $x \in \mathbb{R}^n$ must be positive. Often times Φ is given in the entrywise format (1.6) but, as mentioned before, an Euclidean polynomial division in each entry of Φ allows to separate the polynomial part and the purely rational part of Φ. So, we may assume that Φ is given in the form (1.7) with

$$\underbrace{d, (A_0, \ldots, A_d)}_{\text{data for } P}, \quad \underbrace{m, (p_1, q_1, B_1), \ldots, (p_m, q_m, B_m)}_{\text{data for } S}$$

readily available. Let d_k be the degree of q_k and r_k be the rank of B_k. We suppose that each B_k is given in a rank-revealing factorization

$$B_k = L_k R_k^\top, \tag{3.36}$$

where L_k, R_k are full rank matrices of size $n \times r_k$. In applications, the r_k's are usually much smaller than n. Since the degree of p_k is smaller than the degree of q_k, the rational function p_k/q_k can be represented as

$$\frac{p_k(\lambda)}{q_k(\lambda)} = a_k^\top (C_k + \lambda D_k)^{-1} b_k, \tag{3.37}$$

where a_k, b_k are column vectors in $\mathbb{R}^{d_k}$, C_k is a matrix of order d_k, and D_k is a nonsingular matrix of order d_k. In the parlance of control theory, the quadruple (a_k, b_k, C_k, D_k) is called a realization of the rational function p_k/q_k. Algorithms for constructing realizations of rational functions can be found in the specialized literature, cf. [20]. By substituting (3.36) and (3.37) into (1.9), we get

$$S(\lambda) = \sum_{k=1}^{m} a_k^\top (C_k + \lambda D_k)^{-1} b_k \, L_k R_k^\top .$$

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As mentioned in Su and Bai [21, Section 3], this can be rewritten in the more compact form

$$S(\lambda) = L(C + \lambda D)^{-1} R^\top,$$

where C, D are matrices of order

$$\kappa = \sum_{k=1}^{m} r_k d_k$$

and L, R are rectangular matrices of size $n \times \kappa$. Furthermore, D is nonsingular because each D_k is nonsingular. Summarizing, we may assume that $\Phi : \Lambda \to \mathbb{M}_n$ is given from the very beginning in the realization format

$$\Phi(\lambda) = \sum_{k=0}^{d} \lambda^k A_k \;+\; L(C + \lambda D)^{-1} R^\top, \tag{3.38}$$

$$\Lambda = \{\lambda \in \mathbb{R} : \det(C + \lambda D) \neq 0\}. \tag{3.39}$$

Note that the set-complement of (3.39) is finite because D is nonsingular. A small dimensional example is helpful to fix the ideas.

Example 3. Consider again the rational matrix-valued function Φ given by (2.18). We have

$$\Phi(\lambda) \;=\; \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & -4 \end{bmatrix}}_{A_0} + \lambda \underbrace{\begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix}}_{A_1} + \lambda^2 \underbrace{\begin{bmatrix} -\frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} \;+\; S(\lambda),$$

where the purely rational part written in rank-revealing format is

$$S(\lambda) \;=\; \frac{\lambda+2}{\lambda^2+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top + \frac{1}{\lambda-3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix}^\top. \tag{3.40}$$

For the first rational function in (3.40), we may consider for instance the realization

$$(a_1, b_1, C_1, D_1) = \left(\begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

For the second rational function in (3.40) we can take $(a_2, b_2, C_2, D_2) = (1, 1, -3, 1)$. Hence, Φ is representable as in (3.38) with purely rational part expressed in terms of the matrices

$$L = \begin{bmatrix} 0 & 0 & 1 \\ -1/2 & 3/2 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices C and D are not unique because a rational function has several realizations.

As shown in the next proposition, computing the real spectrum of the rational matrix-valued function (3.38) is equivalent to solving an affine eigenvalue problem

$$(\mathbb{A} + \lambda \mathbb{B}) z = 0 \tag{3.41}$$

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involving a pair of block structured matrices:

$$\mathbb{A} := \begin{bmatrix} A_{d-1} & A_{d-2} & \cdots & A_0 & L \\ -I_n & 0 & \cdots & 0 & \\ & \ddots & \ddots & \vdots & \\ & & -I_n & 0 & \\ & & & R^\top & -C \end{bmatrix}, \quad \mathbb{B} := \begin{bmatrix} A_d & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ & & & & -D \end{bmatrix}.$$

Written in full extent the affine eigenvalue problem (3.41) reads:

$$\begin{aligned} A_{d-1}z_1 + A_{d-2}z_2 + \ldots + A_0 z_d + L z_0 + \lambda A_d z_1 &= 0 \\ -z_1 + \lambda z_2 &= 0 \\ &\vdots \\ -z_{d-1} + \lambda z_d &= 0 \\ R^\top z_d - (C + \lambda D) z_0 &= 0. \end{aligned}$$

The matrices $\mathbb{A}$ and $\mathbb{B}$ are of order $dn + \kappa$. Note that $\mathbb{B}$ is nonsingular if and only if the leading matrix A_d is nonsingular.

Proposition 3 (Su and Bai, 2011). *Let $\Phi : \Lambda \to \mathbb{R}$ be given by (3.38). Let $\lambda \in \Lambda$.*

(a) If $z := (z_1, z_2, \ldots, z_d, z_0)^\top \in \mathbb{R}^{dn+\kappa}$ is a nonzero vector in the kernel of $\mathbb{A} + \lambda\mathbb{B}$, then $x := z_d$ is a nonzero vector in the kernel of $\Phi(\lambda)$.

(b) Conversely, if x is a nonzero vector in the kernel of $\Phi(\lambda)$, then

$$z := \left(\lambda^{d-1}x, \lambda^{d-2}x, \ldots, x, (C + \lambda D)^{-1} R^T x\right)^\top$$

is a nonzero vector in the kernel of $\mathbb{A} + \lambda\mathbb{B}$.

In conclusion, to check whether (3.35) holds for some pair $(\lambda, x) \in \Lambda \times \mathbb{R}^n$ amounts to check whether (3.41) has a solution $(\lambda, z) \in \Lambda \times \mathbb{R}^{dn+\kappa}$ with $z_d \succ 0$. The affine eigenvalue problem (3.41) can be handled with any eigensolver available in the literature. The case of a general index set J can be treated along the same lines. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and E_J be the matrix whose columns are the vectors $\{e_j : j \in J\}$. Since

$$\Phi^J(\lambda) = E_J^\top \Phi(\lambda) E_J,$$

a representation of Φ^J in realization format is

$$\Phi^J(\lambda) \;=\; \sum_{k=0}^{d} \lambda^k A_k^J \;+\; L_J (C + \lambda D)^{-1} R_J^\top,$$

where $A_k^J = E_J^\top A_k E_J$ is the principal submatrix of A_k induced by the index set J and

$$L_J := E_J^\top L, \quad R_J := E_J^\top R.$$

Hence, the rational eigenvalue problem (3.30) can be converted into an affine eigenvalue problem involving

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a pair of block structured matrices:

$$\mathbb{A}_J := \begin{bmatrix} A^J_{d-1} & A^J_{d-2} & \cdots & A^J_0 & L_J \\ -I_p & 0 & \cdots & 0 & \\ & \ddots & \ddots & \vdots & \\ & & -I_p & 0 & \\ & & & R_J^\top & -C \end{bmatrix}, \quad \mathbb{B}_J := \begin{bmatrix} A^J_d & & & & \\ & I_p & & & \\ & & \ddots & & \\ & & & I_p & \\ & & & & -D \end{bmatrix},$$

where I_p is the identity matrix of order $p := |J|$. The situation is essentially the same as with the case $J = \mathbb{N}_n$, except that now $\mathbb{A}_J$ and $\mathbb{B}_J$ are matrices of order $d\,|J| + \kappa$. In particular, the component z_d of the eigenvector z belongs to $\mathbb{R}^{|J|}$. Below we reformulate Theorem 1 for the particular case of a rational matrix-valued function given in realization format. In order to express the slackness condition (3.32) in terms of z we introduce the block structured matrices

$$\mathbb{A}_{\bar{J},J} := \begin{bmatrix} A^{\bar{J},J}_{d-1} & A^{\bar{J},J}_{d-2} & \cdots & A^{\bar{J},J}_0 & L_{\bar{J}} \end{bmatrix},$$
$$\mathbb{B}_{\bar{J},J} := \begin{bmatrix} A^{\bar{J},J}_{d} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $L_{\bar{J}} = E_{\bar{J}}^\top L$ and $A^{\bar{J},J} = E_{\bar{J}}^\top A E_J$ for all $A \in \mathbb{M}_n$.

THEOREM 4. *Let $\Phi : \Lambda \to \mathbb{R}$ be given by (3.38). Then $\lambda \in \Lambda$ is a Pareto eigenvalue of Φ if and only if there exist an index set $J \in \mathcal{J}_n$ and a vector $z \in \mathbb{R}^{d|J|+\kappa}$ such that*

$$(\mathbb{A}_J + \lambda \mathbb{B}_J) z = 0, \tag{3.42}$$
$$z_d \succ 0, \tag{3.43}$$
$$(\mathbb{A}_{\bar{J},J} + \lambda \mathbb{B}_{\bar{J},J}) z \succeq 0. \tag{3.44}$$

Proof. Condition (3.44) must be dropped of course when $J = \mathbb{N}_n$. This case is taken care by Proposition 3 and the fact that the positivity condition (3.31) is equivalent to the positivity condition (3.43). Suppose that J is strictly contained in $\mathbb{N}_n$. Let $z \in \mathbb{R}^{d|J|+\kappa}$ be as in (3.42)–(3.44) and define $u := z_d$. By using (3.42)–(3.43), we get

$$z_k = \lambda^{d-k} u \ \text{ for } \ k \in \mathbb{N}_d \tag{3.45}$$
$$z_0 = (C + \lambda D)^{-1} R^T u \tag{3.46}$$

and deduce that u is a positive vector in the kernel of $\Phi^J(\lambda)$. It remains to check that u satisfies the slackness condition (3.32). Condition (3.44) amounts to saying that

$$A^{\bar{J},J}_{d-1} z_1 + A^{\bar{J},J}_{d-2} z_2 + \cdots + A^{\bar{J},J}_0 z_d + L_{\bar{J}} z_0 + \lambda A^{\bar{J},J}_d z_1 \ \succeq\ 0. \tag{3.47}$$

By substituting (3.45)–(3.46) into (3.47) and rearranging terms, we get

$$\left(\sum_{k=0}^{d} \lambda^k A^{\bar{J},J}_k \ +\ L_{\bar{J}} (C + \lambda D)^{-1} R_J^\top \right) u \ \succeq\ 0.$$

On the other hand, a left multiplication in (3.38) by $E_{\bar{J}}^\top$ followed by right multiplication by E_J yield

$$\Phi^{\bar{J},J}(\lambda) \ =\ \sum_{k=0}^{d} \lambda^k A^{\bar{J},J}_k \ +\ L_{\bar{J}} (C + \lambda D)^{-1} R_J^\top.$$

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This shows that u satisfies (3.32). Conversely, suppose that $u \in \mathbb{R}^{|J|}$ satisfies the system (3.30)–(3.32). In such a case, a direct computation shows that

$$z := \left(\lambda^{d-1}u, \lambda^{d-2}u, \ldots, u, (C+\lambda D)^{-1}R^T u\right)^\top$$

satisfies (3.42)–(3.44). ∎

3.2. Analysis of a particular case. By way of application, we analyze a cone-constrained eigenvalue problem involving the rational matrix-valued function of Example 1. To be more precise, we solve the cone-constrained eigenvalue problem (3.29) for

$$\Phi(\lambda) := K - \lambda M + \sum_{k=1}^{m} \frac{\lambda}{\lambda - \omega_k}\, h_k h_k^\top, \tag{3.48}$$

where $\omega_1, \ldots, \omega_m$ are positive parameters, $h_1, \ldots, h_m$ are vectors of $\mathbb{R}^n$ and K, M are symmetric matrices of order n. We assume that M is positive definite. We explain in detail how to compute all the Pareto eigenvalues of such Φ. For a pedagogical reason we distinguish three phases:

Phase I. We separate the polynomial part and the purely rational part of Φ. After carrying out an Euclidean polynomial division we see that

$$P(\lambda) = K + \sum_{k=1}^{m} h_k h_k^\top - \lambda M, \tag{3.49}$$

$$S(\lambda) = \sum_{k=1}^{m} \frac{\omega_k}{\lambda - \omega_k}\, h_k h_k^\top. \tag{3.50}$$

Note that (3.49) is affine with $A_0 := K + \sum_{k=1}^{m} h_k h_k^\top$ as constant part and $A_1 := -M$ as linear part. Since M is positive definite, so is every principal submatrix of M. In particular, the characteristic function of each Φ^J is not identically zero. Corollary 2 implies that Φ has finitely many Pareto eigenvalues.

Phase II. The second phase consists in writing Φ in realization format. Clearly, the purely rational part (3.50) is expressible as $L(C+\lambda D)^{-1}R^\top$ with $C = -I_m$, $D := \mathrm{Diag}(1/w_1, \ldots, 1/w_m)$, and $R = L = H := [h_1, \ldots, h_m]$. Thus,

$$\Phi(\lambda) = A_0 + \lambda A_1 + H(-I_m + \lambda D)^{-1} H^\top.$$

Phase III. The third and final phase consists in solving the system (3.42)–(3.44) for each index set J taken from $\mathcal{J}_n$. There are $2^n - 1$ index sets in all. In this example, we have $d = 1$. Hence,

$$\mathbb{A}_J = \begin{bmatrix} A_0^J & H_J \\ H_J^\top & I_m \end{bmatrix}, \quad \mathbb{B}_J = \begin{bmatrix} A_1^J & 0 \\ 0 & -D \end{bmatrix}$$

$$\mathbb{A}_{\bar{J},J} = \begin{bmatrix} A_0^{\bar{J},J} & H_{\bar{J}} \end{bmatrix}, \quad \mathbb{B}_{\bar{J},J} = \begin{bmatrix} A_1^{\bar{J},J} & 0 \end{bmatrix},$$

and the system (3.42)–(3.44) becomes

$$\left(\begin{bmatrix} A_0^J & H_J \\ H_J^\top & I_m \end{bmatrix} + \lambda \begin{bmatrix} A_1^J & 0 \\ 0 & -D \end{bmatrix} \right) \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.51}$$

$$\left(\begin{bmatrix} A_0^{\bar{J},J} & H_{\bar{J}} \end{bmatrix} + \lambda \begin{bmatrix} A_1^{\bar{J},J} & 0 \end{bmatrix} \right) \begin{bmatrix} u \\ w \end{bmatrix} \succeq 0, \tag{3.52}$$

$$u \succ 0. \tag{3.53}$$

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Note that $\mathbb{A}_J$ and $\mathbb{B}_J$ are symmetric matrices of order $|J| + m$ and that $-\mathbb{B}_J$ is positive definite. Symmetry is not here a crucial property. We could have perfectly well started with asymmetric matrices K and M.

Of three phases mentioned above, the most expensive in computational cost is Phase III. In general, the number of systems of the type (3.42)–(3.44) that must we worked out is $2^n - 1$. This number increases exponentially with n and this fact is a recurrent nightmare of the theory of Pareto eigenvalues. A bothersome aspect of the Pareto cone $\mathbb{R}^n_+$ is that of having as much as $2^n - 1$ nonzero faces. If n is beyond a few dozens, then computing all the Pareto eigenvalues of a rational matrix-valued function is prohibitively expensive. Table 2 reports a numerical experiment with the following data:

$$K = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \omega_1 = 1, \quad \omega_2 = 4. \tag{3.54}$$

The real spectrum of the matrix-valued function Φ given by (3.48) has 5 elements, namely,

$$\sigma(\Phi) = \{0.17219,\ 0.57792,\ 0.87265,\ 2.37506,\ 4.50218\}.$$

This set is not to be confused with the Pareto spectrum of Φ, which has 11 elements in all. In this example, the sets $\sigma(\Phi)$ and $\Pi(\Phi)$ are not comparable and they have only two elements in common. Note that if we write (3.51)–(3.52) for $J = \{2\}$, then we get a system that is unsolvable. This explain why the index set $J = \{2\}$ produce no Pareto eigenvalue, cf. Table 2.

Table 2
Pareto eigenvalues of (3.48) with data given by (3.54).

J	λ	primal vector			dual vector		
		x_1	x_2	x_3	y_1	y_2	y_3
$\{1\}$	0.66667	1	0	0	0	0.33333	1
	2	1	0	0	0	3	1
$\{2\}$	none						
$\{3\}$	0.69722	0	0	1	1	1.21110	0
$\{1,2\}$	0.31015	1	6.90008	0	0	0	2.08405
	0.67100	6.51259	1	0	0	0	7.71415
	2.25262	3.65307	1	0	0	0	5.94221
$\{1,3\}$	0.79645	1	0	2.30215	0	3.46739	0
	2.04069	6.20466	0	1	0	21.16046	0
$\{2,3\}$	0.81215	0	1	2.49263	3.11693	0	0
$\{1,2,3\}$	0.87265	1	1.73270	4.17893	0	0	0
	2.37506	4.49424	1.40866	1	0	0	0

4. By way of conclusion. We convey the reader to Su and Bai [21, Section 5] for information on the cost of solving a classical rational eigenvalue problem. A word of caution is here appropriate: computing the Pareto spectrum of rational matrix-valued function Φ could be unaffordable if the order n is not of moderate size. This is because a complete enumeration of the Pareto eigenvalues of Φ requires to solve $2^n - 1$ classical rational eigenvalue problems. Furthermore, after solving an eigenvalue problem of the type (3.42), we should not forget to check the positivity condition (3.43) and the slackness condition (3.44). In some applications, it may happen that only some particular Pareto eigenvalues are of interest, for instance, those admitting an associated eigenvector in the relative interior of a facet of the Pareto cone. In such a case, we have to solve just n unconstrained rational eigenvalue problems because $\mathbb{R}^n_+$ has only n facets.

We end this work with some bibliographical comments and suggestions for further research. The linearization-based method of Su and Bai [21] is a trimmed linearization. The technique appeared in the

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context of polynomial eigenvalue problems with singular coefficient matrices in Byers et al. [6] and for rational eigenvalue problems in Su and Bai [21], see also the work of Alam and Behera [2] published a few years latter. A natural question to ask now is the following one: how to handle a cone-constrained eigenvalue problem like (1.1) when the matrix-valued function $\Phi : \Lambda \to \mathbb{M}_n$ is not rational? Such sort of situation arises in real life applications after all. For instance, Pinto da Costa et al. [16] were led to find all the values of λ in the open interval $\Lambda :=]0, \pi[$ for which the complementarity system

$$\begin{bmatrix} 24\phi_1(\lambda) & 12\phi_1(\lambda) & 0 & 6\phi_2(\lambda) \\ 12\phi_1(\lambda) & 24\phi_1(\lambda) & 6\phi_2(\lambda) & 0 \\ 0 & 6\phi_2(\lambda) & 8\phi_3(\lambda) & 2\phi_4(\lambda) \\ 6\phi_2(\lambda) & 0 & 2\phi_4(\lambda) & 8\phi_3(\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{bmatrix}, \tag{4.55}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \perp \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.56}$$

admits a nonzero solution $(x_1, x_2, x_3, x_4, y_1, y_2)$. The equilibrium model (4.55)–(4.56) arises in the buckling analysis of columns. The convex cone K behind (4.56) is of the product type $\mathbb{R}^2_+ \times \mathbb{R}^2$ and the entries of the matrix $\Phi(\lambda)$ in (4.55) depend on the so-called stability functions that, for a canonical finite element of length L, flexural stiffness EI and submitted to a compressive force P, are defined by

$$\begin{aligned} \phi_1(\lambda) &:= \frac{\lambda \cos\lambda}{\sin\lambda}\, \phi_2(\lambda), \\ \phi_2(\lambda) &:= \frac{\lambda^2}{3} \left[1 - \frac{\lambda\cos\lambda}{\sin\lambda}\right]^{-1}, \\ \phi_3(\lambda) &:= \frac{3}{4}\phi_2(\lambda) + \frac{1}{4}\frac{\lambda\cos\lambda}{\sin\lambda}, \\ \phi_4(\lambda) &:= \frac{3}{2}\phi_2(\lambda) - \frac{1}{2}\frac{\lambda\cos\lambda}{\sin\lambda}, \end{aligned}$$

with $\lambda := (L/2)\sqrt{P/EI}$ playing the role of a non-dimensional load parameter. Note that the ϕ_k's are non-rational functions. Pinto da Costa et al. [16] do not rely on linearization techniques for finding the eigenvalues of each Φ^J. These authors simply apply a standard bisection algorithm on Λ to find the roots of the characteristic function $f^J := \det \circ \Phi^J$. Their strategy is computationally viable because the domain Λ is a bounded interval and the order $n = 4$ is small. It is worthwhile mentioning that the matrix-valued function Φ in (4.55) can be written in the rational-trigonometric format

$$\Phi(\lambda) = \sum_{k=0}^{m} \frac{p_k(\lambda, \cos\lambda, \sin\lambda)}{q_k(\lambda, \cos\lambda, \sin\lambda)}\, A_k,$$

where $p_k : \mathbb{R}^3 \to \mathbb{R}$ and $q_k : \mathbb{R}^3 \to \mathbb{R}$ are multivariate polynomials. We are not aware of any special method in the literature for finding the eigenvalues of a rational-trigonometric matrix-valued function. This issue was raised in [16] and remains open for future research.

How to handle the case of a cone-constrained eigenvalue problem involving a smooth matrix-valued function $\Phi : \mathbb{R} \to \mathbb{M}_n$ on the whole real line and with an order n which is not necessarily small? If the cone K under consideration is the Pareto cone as in Section 3, then we see at least two possibilities:

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- Newton's method is a natural approach to compute eigenvalues or eigenpairs of nonlinear eigenvalue problems efficiently and accurately provided that good initial guesses are available. For a discussion on this technique, see Güttel and Tisseur [8] and references therein. If we consider this option, then what we can do is to apply Newton's method to each Φ^J separately. After finding a pair (λ, u) satisfying $\Phi^J(\lambda)u = 0$ and a suitable normalization condition for u, we must check the positivity condition (3.43) and the slackness condition (3.44). If either (3.43) or (3.44) is violated, then our candidate (λ, u) must be thrown out and we need to continue our search.
- Another simple idea is to adapt the technique of Adly and Seeger [1] from a linear pencil to a smooth Φ. This means applying the semismooth Newton method to the system

$$\begin{aligned}\Phi(\lambda)x - y &= 0,\\ \psi(x,y) &= 0,\\ \langle \mathbf{1}_n, x\rangle - 1 &= 0,\end{aligned}$$

  of $2n+1$ equations in the same number of unknown variables. Here, $\mathbf{1}_n$ is the n-dimensional vector of ones and $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the Fischer-Burmeister complementarity function or any other locally Lipschitz semismooth function satisfying

$$\psi(x,y) = 0 \quad \Leftrightarrow \quad 0 \preceq x \perp y \succeq 0.$$

Both options mentioned above work perfectly well if we are searching for one or a bunch of solutions λ. In a context as general as the present one, finding all the solutions λ is not realistic of course. The theory of cone-constrained nonlinear eigenvalue problems is still in the making and there is yet a long way to go before being able to handle efficiently the case of general matrix-valued functions of large order.

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