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Determinantal Properties of Generalized Circulant Hadamard Matrices

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DETERMINANTAL PROPERTIES OF GENERALIZED CIRCULANT HADAMARD MATRICES*

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Abstract. The derivation of analytical formulas for the determinant and the minors of a given matrix is in general a difficult and challenging problem. The present work is focused on calculating minors of generalized circulant Hadamard matrices. The determinantal properties are studied explicitly, and generic theorems specifying the values of all the minors for this class of matrices are derived. An application of the derived formulae to an interesting problem of numerical analysis, the growth problem, is also presented.

Key words. Hadamard matrices, Gaussian elimination, Growth problem, Determinant calculus, Symbolic computations.

AMS subject classifications. 65F05, 15A15, 65F40, 65G50, 05B20.

1. Introduction and motivation of the problem. Determinants and principal minors arise in numerous fields due to their immediate connections to solving linear systems, matrix inversions, handling eigenvalue problems and so forth. Various applications of determinant theory include self-validating algorithms, the detection of P-matrices [6], the interval matrix analysis, the determinantal assignment problem [12] and the specification of pivot patterns of matrices [10].

The importance of determinants motivates their extensive research, which concerns in particular the challenging issue of their efficient evaluation. Direct methods based on various LU-factorizations of a given square matrix A of order n require a heavy computational cost of $\mathrm{O}(2^n \cdot n^3)$ [6]. Therefore, whenever possible, it is useful to develop analytical formulae for determinants and minors, taking advantage of the structural properties of matrices.

The analytical approach was successful in case of Hadamard matrices (Definition 1.1 below; see [8] for further details and applications), for which analytical expressions for their determinant and principal minors up to certain orders were found.

Definition 1.1. A *Hadamard matrix* H of order n has entries ± 1 and satisfies $HH^T = H^T H = nI_n$ for I_n denoting the identity matrix of order n.

Definition 1.1 immediately implies the important property that every two distinct rows or columns of a Hadamard matrix are mutually orthogonal. Using the orthogonality of any three rows, one can easily demonstrate that the order of a Hadamard matrix can only be $n = 1, 2$, or $n \equiv 0 \bmod 4$. As mentioned above, the special structure of Hadamard matrices allowed to characterize their minors of certain orders. Achieved results are summarized in the following theorem.

Theorem 1.2. [9, 13] *Let us consider a Hadamard matrix of order n. Its $n-j$ minors for $j = 1, 2, 3, 4$ are specified as follows: The $n-1$ minors are $\pm n^{\frac{n}{2}-1}$. The $n-2$ minors are 0 or $2n^{\frac{n}{2}-2}$. The $n-3$ minors are 0 or $4n^{\frac{n}{2}-3}$. The $n-4$ minors are 0, $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$.*

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The existence of analytical expressions for minors of Hadamard matrices, as shown in Theorem 1.2, encourages the study of variations of Hadamard matrices, as it may lead to analytical formulae for all or some of their minors as well. Besides, there is another important motivation, which stems from a theory of the so-called growth factor. The growth factor of a matrix A, denoted by $g(A)$, measures the stability of Gaussian elimination applied to the system $A \cdot x = b$. (We will provide exact definition of $g(A)$ and more details in Section 3.) The growth factor is usually substantially smaller than the order of the given matrix, making the Gaussian elimination stable. However, this empiric rule is broken by Hadamard matrices, which have, according to numerical results and Cryer's Conjecture [1], the growth factor equal to their order. Furthermore, Hadamard matrices are the only known examples of matrices with such property.

These surprising facts about the growth factor, together with the explicit results of Theorem 1.2, strongly motivate the study of determinantal properties of other variations of Hadamard matrices. The research may lead to analytical formulae for all or some of their minors, as well as contribute to proving Cryer's Conjecture, which has further consequences on the theory of numerical stability of solving linear systems.

In this paper, we will focus on a variation of Hadamard matrices that is introduced in Definition 1.3.

Definition 1.3. We call a circulant matrix C a *generalized circulant Hadamard matrix (GCH matrix) with diagonal* d if the off-diagonal entries of C take values from $\{-1, 1\}$, the diagonal entries are equal to $d \in \mathbb{R}$ and the rows of C are mutually orthogonal.

Definition 1.3 extends the notion of circulant Hadamard matrices, which correspond to $d \in \{1, -1\}$. The structure of GCH matrices was studied in [14]. It was proved that if $2d \in \mathbb{N}_0$, then there exists a GCH matrix with diagonal d of order $n = 2d + 2$, and its first row is equal to one of the following vectors:

$$(d, -1, -1, -1, \ldots, -1), \tag{1.1a}$$

$$(d, 1, -1, 1, -1, 1, \ldots, -1, 1), \tag{1.1b}$$

$$(d, -1, 1, 1, -1, -1, 1, 1, -1, \ldots, -1, -1, 1, 1), \tag{1.1c}$$

$$(d, 1, 1, -1, -1, 1, 1, -1, -1, \ldots, -1, 1, 1, -1). \tag{1.1d}$$

Furthermore, it was conjectured ibidem that vectors (1.1) cover all existing GCH matrices with nonnegative diagonal $d \geq 0$; i.e., the first row of any GCH matrix with $d \geq 0$ is equal to one of the vectors (1.1a)–(1.1d). This conjecture was proved except for d being odd integer and $C \neq C^T$.

We will calculate the minors of generalized circulant Hadamard matrices, providing analytical formulae and stating explicit theorems concerning the convergence of the quotients of subsequent minors. This result is immediately applied to the growth problem, which characterizes the stability of Gaussian elimination in numerical analysis.

The paper is organized as follows. In Sections 2.1–2.4, we study explicitly the minors of generalized circulant Hadamard matrices, whose first rows are given by vectors (1.1). For each type of those matrices, we derive formulae for their principal minors. Section 3 is devoted to the growth problem. We describe the notion of growth factor and provide its analytic evaluation for the class of GCH matrices. In particular, although Hadamard matrices attain a large growth factor, which is conjectured to be equal to their dimension [1], we prove that the growth factor of generalized circulant Hadamard matrices can approach the smallest value that the growth factor can take, i.e., the value of 1. Numerical examples are reported in Section 4, and Section 5 contains concluding remarks.

Throughout the paper, the symbol I_k represents the identity matrix of order k, the superscript ^{T} denotes

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the transpose, the symbol $\simeq$ means "approximately equal to" and the symbol $\ll$ means "much smaller than". A principal minor of order j of a given matrix will be denoted as $A(j)$.

2. Minors of generalized circulant Hadamard matrices.

2.1. Case I: Matrices with first row $(d, -1, -1, -1, \dots, -1)$**.** Let C_n be a circulant $n \times n$ matrix having the first row in the form $(d, -1, -1, -1, \dots, -1)$ with $d = \frac{n}{2} - 1$. For every $j \le n$, denote as $C_n^{(j)}$ the upper left $j \times j$ submatrix of C_n. Possible $j \times j$ minors of C_n, up to permutations of rows and columns, are

$$A(j) := \det(C_n^{(j)}) \quad \text{(leading principal minor)}, \tag{2.2}$$

$$B(j) := \begin{vmatrix} & & & -1 \\ & C_n^{(j-1)} & & \vdots \\ & & & -1 \\ -1 & \cdots & -1 & -1 \end{vmatrix}, \tag{2.3}$$

and also

$$\begin{vmatrix} & & & -1 & -1 \\ & C_n^{(j-2)} & & \vdots & \vdots \\ & & & -1 & -1 \\ -1 & \cdots & -1 & -1 & -1 \\ -1 & \cdots & -1 & -1 & -1 \end{vmatrix}, \quad \dots, \quad \begin{vmatrix} C_n^{(1)} & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -1 \end{vmatrix},$$

which are equal to 0.

Now we derive analytical formulae for the minors $A(j)$ and $B(j)$ of a GCH matrix C_n.

PROPOSITION 2.1. *The leading principal minor of C_n of order j equals*

$$A(j) = (d+1)^{j-1}(d+1-j).$$

Proof. We have

$$C_n^{(j)} - (d+1)I_j = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -1 \end{pmatrix}.$$

Recall that $A(j) = \det(C_n^{(j)})$ (equation (2.2)). The matrix $C_n^{(j)} - (d+1)I_j$ has an eigenvalue 0 with multiplicity $j-1$ and an eigenvalue $-j$ with multiplicity 1. Therefore, $C_n^{(j)}$ has eigenvalues $d+1$ and $d+1-j$ with multiplicities $j-1$ and 1, respectively. Hence,

$$\det(C_n^{(j)}) = (d+1)^{j-1}(d+1-j). \qquad \square$$

PROPOSITION 2.2. *The minor $B(j)$, defined in equation (2.3), equals*

$$B(j) = -(d+1)^{j-1}.$$

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Proof. We notice that each minor $B(j)$ is equal to

$$\begin{vmatrix} d & -1 & \cdots & -1 & -1 \\ -1 & d & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & d & -1 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix} = \begin{vmatrix} d+1 & 0 & \cdots & 0 & 0 \\ 0 & d+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d+1 & 0 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix} = -(d+1)^{j-1}. \qquad \square$$

2.2. Case II: Matrices with first row $(d, 1, -1, 1, \dots, -1, 1)$. In this section, let C_n be a circulant $n \times n$ matrix having the first row in the form $(d, 1, -1, 1, \dots, 1)$ with $d = \frac{n}{2} - 1$. Consider a diagonal $n \times n$ matrix $D = \operatorname{diag}(1, -1, 1, -1, 1, -1, \dots, 1, -1)$. We notice that DC_nD is an $n \times n$ circulant matrix with the first row $(d, -1, -1, \dots, -1)$, which we have already examined in Section 2.1. Minors of C_n are thus either equal to the minors evaluated in Propositions 2.1 and 2.2, or equal to them up to the sign.

2.3. Case III: Matrices with first row $(d, -1, 1, 1, -1, -1, 1, 1, -1, \dots, -1, -1, 1, 1)$.

PROPOSITION 2.3. *Let C_n be a circulant matrix of order n with the first row given as*

$$(d, -1, 1, 1, -1, -1, 1, 1, -1, \dots, -1, -1, 1, 1).$$

The leading principal minor of C_n of order j satisfies

$$A(j) = \begin{cases} (d+1)^{j-1}\left(d+1-j+\frac{j^2}{2(d+1)}\right) & \text{if } j \text{ is even;} \\ (d+1)^{j-1}\left(d+1-j+\frac{j^2-1}{2(d+1)}\right) & \text{if } j \text{ is odd.} \end{cases} \tag{2.4}$$

Proof. Let $C_n^{(j)}$ be the upper left $j \times j$ submatrix of C_n. Then,

$$C_n^{(j)} - (d+1)I_j = \begin{pmatrix} -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & \cdots \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & \cdots \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & \cdots \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Since $\operatorname{rank}(C_n^{(j)} - (d+1)I_j) = 2$, the matrix $C_n^{(j)} - (d+1)I_j$ has an eigenvalue 0 of multiplicity $j-2$. Furthermore, it is easy to check that the vectors

$$(1, \mathrm{i}, -1, -\mathrm{i}, 1, \mathrm{i}, -1, -\mathrm{i}, \dots)^T \quad \text{and} \quad (1, -\mathrm{i}, -1, \mathrm{i}, 1, -\mathrm{i}, -1, \mathrm{i}, \dots)^T$$

are eigenvectors of $C_n^{(j)} - (d+1)I_j$. They correspond to eigenvalues

- $-\frac{j}{2}(1+\mathrm{i})$ and $-\frac{j}{2}(1-\mathrm{i})$, respectively, for even j;
- $-\frac{j}{2} - \mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2 - \frac{1}{2}}$ and $-\frac{j}{2} + \mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2 - \frac{1}{2}}$, respectively, for odd j.

Therefore, $C_n^{(j)}$ has an eigenvalue $d+1$ with multiplicity $j-2$ and two simple eigenvalues, given as

- $d+1-\frac{j}{2}(1+\mathrm{i})$ and $d+1-\frac{j}{2}(1-\mathrm{i})$, respectively, for even j;
- $d+1-\frac{j}{2} - \mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2 - \frac{1}{2}}$ and $d+1-\frac{j}{2} + \mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2 - \frac{1}{2}}$, respectively, for odd j.

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Consequently, if j is even, the determinant of $C_n^{(j)}$ is

$$(d+1)^{j-2}\left(d+1-\frac{j}{2}(1+\mathrm{i})\right)\left(d+1-\frac{j}{2}(1-\mathrm{i})\right)=(d+1)^{j-1}\left(d+1-j+\frac{j^2}{2(d+1)}\right).$$

If j is odd, the determinant of $C_n^{(j)}$ equals

$$(d{+}1)^{j-2}\left(d+1-\frac{j}{2}-\mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2-\frac{1}{2}}\right)\left(d+1-\frac{j}{2}+\mathrm{i}\sqrt{\left(\frac{j}{2}\right)^2-\frac{1}{2}}\right)=(d{+}1)^{j-1}\left(d+1-j+\frac{j^2-1}{2(d+1)}\right).$$

Finally, $A(j)=\det(C_n^{(j)})$ gives formula (2.4). ∎

The next result presents a majorization inequality between the principal minors and the minors of the matrix C_n.

PROPOSITION 2.4. *Let C_n be a circulant matrix of order n with the first row*

$$(d,-1,1,1,-1,-1,1,1,-1,\ldots,-1,-1,1,1).$$

The absolute value of every minor of C_n of order j formed by the elements of C_n from rows $1,2,\ldots,j$ and columns $1,2,\ldots,j-1,\ell$ for a certain $\ell>j$ is less or equal to the absolute value of $A(j)$, given by formula (2.4). Thus, the principal minors majorize the minors of C_n.

Proof. Any $j\times j$ minor of C_n satisfying the assumptions takes the form

$$B(j)=\begin{vmatrix} d & -1 & 1 & 1 & -1 & -1 & \cdots & & \\ 1 & d & -1 & 1 & 1 & -1 & \cdots & & \\ 1 & 1 & d & -1 & 1 & 1 & \cdots & & \\ -1 & 1 & 1 & d & -1 & 1 & \cdots & & \\ -1 & -1 & 1 & 1 & d & -1 & \cdots & & \\ 1 & -1 & -1 & 1 & 1 & d & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & & & & d & \\ & & & & & & & & \pm 1 \end{vmatrix}$$

$$=\begin{vmatrix} d & -1 & 1 & 1 & -1 & -1 & \cdots & & \\ 1 & d & -1 & 1 & 1 & -1 & \cdots & & \\ 1+d & 0 & d+1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1+d & 0 & d+1 & 0 & 0 & \cdots & 0 & 0 \\ -(1+d) & 0 & 0 & 0 & d+1 & -1 & \cdots & 0 & 0 \\ 0 & -(1+d) & 0 & 0 & 0 & d+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & & & & d+1 & 0 \\ & & & & & & & 0 & 0 \end{vmatrix}.$$

The last row begins with one of the pairs $(1+d,0)$, $(0,1+d)$, $(-(1+d),0)$, $(0,-(1+d))$; the remaining terms in the row vanish. Taking advantage of the fact that the last row has only one nonzero term, we arrive at the result $|B(j)|=(d+1)^{j-1}$.

In the next step, we compare $|B(j)|$ with $|A_j(j)|$ given by equation (2.4). We distinguish 4 possible cases. Recall that $d=\frac{n}{2}-1$.

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- If $j \leq \frac{n}{2} - 1$, equation (2.4) gives

$$|A(j)| \geq (d+1)^{j-1}(d+1-j) \geq (d+1)^{j-1}\left(\frac{n}{2} - (\frac{n}{2} - 1)\right) = (d+1)^{j-1} = |B(j)|.$$

- If $j = \frac{n}{2}$, j is even (because n is a multiple of 4). Then

$$|A(j)| = (d+1)^{j-1}\left(d+1-j+\frac{j^2}{2(d+1)}\right) = (d+1)^{j-1}\left(d+1-\frac{n}{2}+\frac{n}{4}\right) = (d+1)^{j-1}\cdot\frac{n}{4}.$$

  Therefore, for all $n \geq 4$, we have $|A(j)| \geq (d+1)^{j-1} = |B(j)|$.
- If $j = \frac{n}{2} + 1$, j is odd (because n is a multiple of 4). Then

$$|A(j)| = (d+1)^{j-1}\left(d+1-j+\frac{j^2-1}{2(d+1)}\right)) = (d+1)^{j-1}\left(\frac{n}{2}-\frac{n}{2}-1+\frac{n}{4}+1\right) = (d+1)^{j-1}\cdot\frac{n}{4}.$$

  Hence, $|A(j)| \geq (d+1)^{j-1} = |B(j)|$ for every $n \geq 4$.
- If $j \geq \frac{n}{2} + 1$, we have

$$\begin{aligned}|A(j)| &\geq (d+1)^{j-1}\left(d+1-j+\frac{j^2-1}{2(d+1)}\right) = (d+1)^{j-1}\left(\frac{n}{2}-j+\frac{j^2-1}{n}\right)\\ &= (d+1)^{j-1}\left(\frac{n}{4}-\frac{1}{n}+\frac{1}{n}\left(j-\frac{n}{2}\right)^2\right) \geq (d+1)^{j-1}\left(\frac{n}{4}-\frac{1}{n}+\frac{1}{n}\right) = (d+1)^{j-1}\cdot\frac{n}{4},\end{aligned}$$

  i.e., $|A(j)| \geq (d+1)^{j-1} = |B(j)|$ for all $n \geq 4$. □

2.4. Case IV: Matrices with first row $(d, 1, 1, -1, -1, 1, 1, -1, -1, \ldots, -1, 1, 1, -1)$**.** In this section, let C_n be a circulant $n \times n$ matrix having the first row in the form $(d, 1, 1, -1, -1, \ldots, 1, -1)$ with $d = \frac{n}{2} - 1$. Consider a diagonal $n \times n$ matrix $D = \text{diag}(1, -1, 1, -1, 1, -1, \ldots, 1, -1)$. It is easy to see that DC_nD is an $n \times n$ circulant matrix with the first row $(d, -1, 1, 1, -1, \ldots, -1, 1, 1)$, which we have already examined in Section 2.3. Minors of C_n are thus equal to the minors evaluated in Propositions 2.3 and 2.4, possibly taken with the opposite sign.

3. Application to the growth problem.

3.1. Description of the problem. Consider a linear system of the form $A \cdot x = b$, where $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is nonsingular. Gaussian elimination (GE) [2, 7] is the simplest way to solve such a system, and thus, it is used as a standard method for solving it on computers. However, since GE becomes less precise if any pivot is close to zero, a search for the element with maximum absolute value is performed. If the search is done in the respective column, then the procedure is called *GE with partial pivoting*; if the search is performed in the whole respective lower right submatrix, we call the procedure *GE with complete pivoting*. Let $A^{[k]} = (a_{ij}^{[k]})$ denote the matrix obtained after the first k pivoting operations, so $A^{[n-1]}$ is the final upper triangular matrix. Note that the diagonal of $A^{[n-1]}$ is formed from the pivots chosen during the calculation.

Traditionally, backward error analysis for GE is expressed in terms of the *growth factor*

$$g(A) = \frac{\max_{i,j,k} |a_{ij}^{[k]}|}{\max_{i,j} |a_{ij}|},$$

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which involves all the elements $a_{ij}^{[k]}$, $k = 1, 2, \dots, n$ that occur during the elimination. The growth factor $g(A)$ actually measures how large the entries become during the process of elimination, and so it governs the stability of GE. That is why the study of the growth factor is important.

Matrices with the property that no exchanges are actually needed during GE with complete pivoting are called *completely pivoted (CP)* or *feasible*. Equivalently, a matrix is CP if the rows and columns are ordered so that GE without pivoting satisfies the requirements for complete pivoting; in other words, if the element with maximal absolute value in each elimination step appears on the diagonal position. For a CP matrix A we have

$$g(A) = \frac{\max\{|p_1|, |p_2|, \dots, |p_n|\}}{|a_{11}|}, \tag{3.5}$$

where $p_1, p_2, \dots, p_n$ are the pivots of A. The study of the values attained by $g(A)$ and the specification of pivot patterns are referred to as *the growth problem.*

Lemma 3.1. [1, 3, 4] *Let A be a CP matrix.*

(i) *The magnitude of the pivots appearing after application of GE operations on A is given by*

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \dots, n, \quad A(0) = 1, \tag{3.6}$$

where $A(j)$ denotes the principal minor of A of order j.

(ii) *The maximum $j \times j$ leading principal minor of A, when the first $j-1$ rows and columns are fixed, is $A(j)$.*

We see from Lemma 3.1 that a calculation of minors is useful for the study of pivot structures, and thus to tackle the growth problem for CP matrices.

Numerical experiments show that the growth factor that appears in practice is typically substantially smaller than the order of the given matrix. In [15, p. 213] it is stated that no real matrix A had been discovered for which $g(A) > n$. This conjecture became one of the most famous problems in numerical analysis [7] and has been investigated by many mathematicians. The conjecture was finally resolved by Gould [5] who found an example of a 13×13 matrix for which the growth was 13.0205. Furthermore, Cryer [1] conjectured that for real $n \times n$ matrices whose entries are in $[-1, 1]$ it holds that $g(A) \leq n$, with equality if and only if A is a Hadamard matrix. The part of Cryer's conjecture regarding Hadamard matrices remains open until nowadays. This, in particular, gives rise to an interesting problem of evaluation of the growth factor of Hadamard matrices [7, page 181].

Currently Hadamard matrices are the only known matrices that experimentally attain growth factor equal to their order, but even the class of Hadamard matrices has not been fully examined yet from the point of view of the growth problem. A certain progress was achieved in [9], where analytical formulae for some pivot values of CP Hadamard matrices were given. More specifically, the first 5 and the last 4 pivots for any Hadamard matrix were determined. This effort allowed to characterize the growth of Hadamard matrices up to order 16 [11], but still left the Cryer's Conjecture unsolved. With regard to these facts, it is natural to examine the growth factor on matrix classes that are related to Hadamard matrices.

In the present section, we will consider the class of generalized circulant Hadamard matrices and calculate the corresponding growth factor, employing the results of Section 2.

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First of all, note that in this matrix class, the $(1,1)$ entry is dominant (having the greatest absolute value) in all cases except for $n=2, d=0$ and $n=3, d=\frac{1}{2}$. These two cases are readily solved by performing GE directly, which leads to the following result:

- If $n=2$, then $p_1=p_2=1$; thus, $g(C)=1$.
- If $n=3$, then $p_1=1, p_2=1.5, p_3=2.25$; thus, $g(C)=2.25$.

From now on let $n\geq 4$. Since the $(1,1)$ entry of C is dominant, we have $p_1=d$. The calculation of the growth factor of C is then performed as follows.

1. Set $S_1=\{1\}$.
2. For all $j=2,\ldots,n$, find $i\in\{1,\ldots,n\}\backslash S_{j-1}$ such that the minor $\det(C_{\{1,\ldots,j\},S_j})$, corresponding to rows $1,\ldots,j$ and columns given by the set $S_j=S_{j-1}\cup\{i\}$, has maximal possible absolute value.
3. The pivots resulting from GE with complete pivoting are given by the formula
$$p_j=\frac{\det(C_{\{1,\ldots,j\},S_j})}{\det(C_{\{1,\ldots,j-1\},S_{j-1}})}.$$
4. The sought growth factor of C is $g(C)=\frac{\max\{|p_1|,\ldots,|p_n|\}}{d}$.

3.2. Specification of pivots for Case I. The structure of generalized circulant Hadamard matrices allowed us to derive analytical formulae for all their minors in Section 2. Using those results, we can find the quotients of subsequent minors, and therefore proceed to expressing the pivot values in Gaussian elimination.

Proposition 3.2. *The GE with complete pivoting applied on C_n having the first row $(d,-1,-1,\ldots,-1)$ leads to pivots p_j, $j=1,\ldots,n$, the absolute values of which are given as follows.*

- *For all $j\leq\frac{n}{2}-1$ and $j\geq\frac{n}{2}+2$, we have*
$$|p_j|=\frac{n}{2}\cdot\frac{n-2j}{n-2j+2}\,. \tag{3.7}$$
- *If n is even, then*
$$|p_{\frac{n}{2}}|=|p_{\frac{n}{2}+1}|=\frac{n}{2}\,.$$
- *If n is odd, then*
$$|p_{\frac{n-1}{2}}|=\frac{n}{3},\quad |p_{\frac{n+1}{2}}|=\frac{n}{2},\quad |p_{\frac{n+3}{2}}|=\frac{3n}{4}\,.$$

Proof. For $j=1$, we have trivially $p_1=d=\frac{n}{2}-1$, which is consistent with equation (3.7). If $1<j\leq\frac{n}{2}-1$, Propositions 2.1 and 2.2 together with $n=2d+2$ imply the inequality $|A(j)|\geq|B(j)|$; hence,

$$p_j=\frac{A(j)}{A(j-1)}=(d+1)\cdot\frac{d+1-j}{d+1-(j-1)}=\frac{n}{2}\cdot\frac{\frac{n}{2}-j}{\frac{n}{2}-j+1}\quad\text{for all } j\leq\tfrac{n}{2}-1.$$

Let us proceed to $\frac{n}{2}-1<j<\frac{n}{2}+2$. We need to distinguish even and odd n.

- Let n be even. Then $A(\frac{n}{2})=0$ and $B(\frac{n}{2})=-(d+1)^{\frac{n}{2}-1}$, i.e., $|B(\frac{n}{2})|>|A(\frac{n}{2})|$. Hence, we get $|p_{\frac{n}{2}}|$:

$$|p_{\frac{n}{2}}|=\left|\frac{B(\frac{n}{2})}{A(\frac{n}{2}-1)}\right|=\left|-(d+1)\cdot\frac{1}{d+1-(\frac{n}{2}-1)}\right|=\left|-\frac{n}{2}\cdot\frac{1}{\frac{n}{2}-\frac{n}{2}+1}\right|=\frac{n}{2}\,.$$

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In order to find $|p_{\frac{n}{2}+1}|$, we express $A(\frac{n}{2}+1) = (d+1)^{\frac{n}{2}}(d+1-\frac{n}{2}-1) = -(d+1)^{\frac{n}{2}}$ and $B(\frac{n}{2}+1) = -(d+1)^{\frac{n}{2}}$. Since $|A(\frac{n}{2}+1)| = |B(\frac{n}{2}+1)|$, we conclude that

$$|p_{\frac{n}{2}+1}| = \left|\frac{A(\frac{n}{2}+1)}{B(\frac{n}{2})}\right| = d+1 = \frac{n}{2}\,.$$

- Let n be odd. It is easy to check that $|A(\frac{n-1}{2})| < |B(\frac{n-1}{2})|$; hence, we get the pivot

$$|p_{\frac{n-1}{2}}| = \left|\frac{B(\frac{n-1}{2})}{A(\frac{n-3}{2})}\right| = \left|-(d+1)\cdot\frac{1}{d+1-\frac{n-3}{2}}\right| = \left|-\frac{n}{2}\cdot\frac{1}{\frac{n}{2}-\frac{n-3}{2}}\right| = \frac{n}{3}.$$

Similarly, one can easily verify the relation $|A(\frac{n+1}{2})| < |B(\frac{n+1}{2})|$, which implies

$$|p_{\frac{n}{2}+1}| = \left|\frac{B(\frac{n+1}{2})}{B(\frac{n-1}{2})}\right| = d+1 = \frac{n}{2}\,.$$

Finally, we have $|A(\frac{n+3}{2})| > |B(\frac{n+3}{2})|$, and hence,

$$|p_{\frac{n+3}{2}}| = \left|\frac{A(\frac{n+3}{2})}{B(\frac{n+1}{2})}\right| = \left|(d+1)\cdot\frac{d+1-\frac{n+3}{2}}{-1}\right| = \left|\frac{n}{2}\cdot\frac{\frac{n}{2}-\frac{n+3}{2}}{-1}\right| = \frac{3n}{4}.$$

It remains to examine the case $j \geq \frac{n}{2}+2$. One can again easily check the validity of the relation $|A(j)| > |B(j)|$ for all $j \geq \frac{n}{2}+2$. Consequently,

$$|p_j| = \left|\frac{A(j)}{A(j-1)}\right| = \left|(d+1)\cdot\frac{d+1-j}{d+1-(j-1)}\right| = \frac{n}{2}\cdot\frac{\frac{n}{2}-j}{\frac{n}{2}+1-j} \quad \text{for all } j \geq \frac{n}{2}+2.$$

Theorem 3.3. *The growth factor of the circulant $n \times n$ matrix C_n with the first row equal to*

$$\left(\frac{n}{2}-1, -1, -1, -1, \ldots, -1\right)$$

is

$$g(C_n) = \begin{cases} \frac{2n}{n-2} & \text{if } n \text{ is even;} \\ \frac{5n}{3(n-2)} & \text{if } n \text{ is odd.} \end{cases} \tag{3.8}$$

Thus, $g(C_n) \simeq 2$ for large even n and $g(C_n) \simeq 5/3$ for large odd n.

Proof. Let us use Proposition 3.2. If n is even, we have $|p_j| \leq |p_{\frac{n}{2}+2}| = n$ for all $j = 1, \ldots, n$. If n is odd, we have $|p_j| \leq |p_{\frac{n+5}{2}}| = \frac{5n}{6}$ for all $j = 1, \ldots, n$. Applying formula (3.5), we obtain $g(C_n) = \frac{n}{d} = \frac{2n}{n-2}$ and $g(C_n) = \frac{5n}{6d} = \frac{5n}{3(n-2)}$, respectively. ∎

3.3. Specification of pivots for Case II.

Proposition 3.4. *Let C_n be a circulant $n \times n$ matrix with the first row $(\frac{n}{2}-1, 1, -1, 1, -1, 1, \ldots, -1, 1)$. The growth factor of C_n is*

$$g(C_n) = \begin{cases} \frac{2n}{n-2} & \text{if } n \text{ is even;} \\ \frac{5n}{3(n-2)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider a diagonal $n \times n$ matrix $D = \mathrm{diag}(1, -1, 1, -1, 1, -1, \ldots, 1, -1)$. It is easy to see that DC_nD is an $n \times n$ circulant matrix with the first row $(d, -1, -1, \ldots, -1)$, which we have already examined in Section 2.1. Absolute values of minors of C_n are thus equal to the absolute values of minors evaluated in Proposition 2.1 and 2.2. Consequently, GE with complete pivoting applied on C_n leads to the pivots whose absolute values coincide with the pivots obtained in Proposition 3.2. In particular, the growth factor of C_n obeys formula (3.8) from Theorem 3.3. ∎

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3.4. Specification of pivots for Case III.

Proposition 3.5. *GE with complete pivoting applied on C_n with the first row*

$$\left(\frac{n}{2}-1,-1,1,1,-1,-1,1,1,-1,\dots,-1,-1,1,1\right)$$

leads to pivots

$$p_j = \begin{cases} \frac{n}{2}\cdot\frac{\frac{n}{2}-j+\frac{j^2}{n}}{\frac{n}{2}+1-j+\frac{j(j-2)}{n}} & \text{if } j \text{ is even;} \\ \frac{n}{2}\cdot\frac{\frac{n}{2}-j+\frac{j^2-1}{n}}{\frac{n}{2}+1-j+\frac{(j-1)^2}{n}} & \text{if } j \text{ is odd.} \end{cases} \tag{3.9}$$

Proof. Proposition 2.4 implies that the GE with complete pivoting coincides with GE with partial pivoting. The partial pivoting leads to $p_j = A(j)/A(j-1)$. Then Proposition 2.3 gives formula (3.9). ◻

Theorem 3.6. *The growth of an $n\times n$ circulant matrix C_n with the first row equal to*

$$\left(\frac{n}{2}-1,-1,1,1,-1,-1,1,1,-1,\dots,-1,-1,1,1\right)$$

is

$$g(C_n)=\frac{n^2}{(n-2)^2}\,.$$

In particular, $\lim_{n\to\infty} g(C_n)=1$.

Proof. It is easy to check that the pivots p_j, given by formula (3.9), satisfy the inequalities

$$\begin{array}{ll} p_{j+2} > p_j & \text{for all } j=1,\dots,n-2; \\ p_{j+1} < p_j & \text{for all even } j=2,\dots,n-2. \end{array}$$

Hence, $p_j \le p_n$ for all $j=1,\dots,n$, where $p_n=\frac{n^2}{2(n-2)}$. Then $g(C_n)=\frac{p_n}{d}=\frac{n^2}{2(n-2)(\frac{n}{2}-1)}=\frac{n^2}{(n-2)^2}$. ◻

3.5. Specification of pivots for Case IV.

Proposition 3.7. *Let C_n be a circulant $n\times n$ matrix with the first row equal to*

$$\left(\frac{n}{2}-1,1,1,-1,-1,1,1,-1,-1,\dots,-1,1,1,-1\right).$$

The growth factor of C_n is

$$g(C_n)=\frac{n^2}{(n-2)^2}\,. \tag{3.10}$$

In particular, $\lim_{n\to\infty} g(C_n)=1$.

Proof. Consider a diagonal matrix $D=\operatorname{diag}(1,-1,1,-1,\dots,1,-1)$. Then DC_nD is the circulant matrix dicussed in Theorem 3.6. Using the same reasoning as in the proof of Proposition 3.4, we conclude that the growth factor of C_n coincides with the growth factor found in Theorem 3.6. ◻

4. Numerical results. In this section, we briefly present some numerical results validating the theoretical results of the previous sections. We performed the Gaussian elimination with complete pivoting of matrices having the first row listed in (1.1) on a computer using MATLAB application, and compared the pivots with the values p_j following from our formulae.

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4.1. Case I. Since the formulae obtained in Proposition 3.2 depend on the parity of n, we will consider both cases. At first let us take an even n, e.g., $n = 8$. In this case we have $d = 3$ and the first row of C_8 takes the form $(3, -1, -1, -1, -1, -1, -1, -1)$. A computer-aided calculation of the Gaussian elimination with complete pivoting gives

$$\mathrm{diag}(U) = (3.0000, 2.6667, 2.0000, -4.0000, -4.0000, 8.0000, 6.0000, 5.3333), \tag{4.11}$$

while Proposition 3.2 provides the absolute values

$$|p_1| = 3, \ \ |p_2| = \frac{8}{3}, \ \ |p_3| = 2, \ \ |p_4| = 4, \ \ |p_5| = 4, \ \ |p_6| = 8, \ \ |p_7| = 6, \ \ |p_8| = \frac{16}{3}\,.$$

We observe that the absolute values of the numerically computed diagonal terms of U are consistent with their prediction by Proposition 3.2. The growth factor equals $g(C_n) = 8/3 \simeq 2.6667$.

Let us also consider the case of large n. If $n = 100$, we obtain values $|p_j|$ that obey the formulae from Proposition 3.2, the largest absolute value being equal to 100. Consequently, the growth is $g(C_{100}) = \frac{100}{\frac{n}{2}-1} = \frac{100}{49} \simeq 2.0408$, as predicted by Theorem 3.3.

Now we proceed to an odd n, taking the example of $n = 9$. In this case we have $d = \frac{7}{2}$, and the matrix C_9 has the first row $(\frac{7}{2}, -1, -1, -1, -1, -1, -1, -1, -1)$. The Gaussian elimination gives

$$\mathrm{diag}(U) = (3.5000, 3.2143, 2.7000, -3.0000, -4.5000, -6.7500, 7.5000, 6.3000, 5.7857),$$

while Proposition 3.2 provides

$$|p_1| = \frac{7}{2}, \ \ |p_2| = \frac{45}{14}, \ \ |p_3| = \frac{27}{10}, \ \ |p_4| = 3, \ \ |p_5| = \frac{9}{2}, \ \ |p_6| = \frac{27}{4}, \ \ |p_7| = \frac{15}{2}, \ \ |p_8| = \frac{63}{10}, \ \ |p_9| = \frac{81}{14}\,.$$

We see again that the the absolute values of the numerically computed diagonal terms of U coincide with the result of Proposition 3.2. The growth factor takes the value $15/7 \simeq 2.1429$.

Let us also consider a large odd n, e.g., $n = 101$. Numerical computation gives values $|p_j|$ being in accord with Proposition 3.2. The largest one is approximately 84.167, and thus, the growth of C_{101} is approximately equal to $\frac{84.167}{\frac{n}{2}-1} \simeq 1,7003$, which is consistent with the value $\frac{5n}{3(n-2)} = \frac{505}{297} \simeq 1.7003$ from Theorem 3.3.

4.2. Case II. A matrix C_n having the first row $(d, 1, -1, 1, -1, 1, \ldots, -1, 1)$ obviously exists only for even orders n. Taking $n = 8$ as above and performing the Gaussian elimination of C_8 on a computer leads to a matrix U with the diagonal terms

$$\mathrm{diag}(U) = (3.0000, 2.6667, 2.0000, -4.0000, -4.0000, 8.0000, 6.0000, 5.3333).$$

They coincide with the pivots we found for matrix with the first row $(3, -1, -1, -1, -1, -1, -1, -1)$, cf. (4.11), and the growth factor is $8/3$. This result illustrates our observation made in the proof of Proposition 3.4.

4.3. Case III. A matrix C_n having the first row $(d, -1, 1, 1, -1, -1, 1, 1, -1, \ldots, -1, -1, 1, 1)$ exists only for the order n being a multiple of 4. Let us consider again $n = 8$, which corresponds to the circulant matrix having the first row $(3, -1, 1, 1, -1, -1, 1, 1)$. A computer-aided Gaussian elimination with complete pivoting gives a matrix U with the diagonal terms

$$\mathrm{diag}(U) = (3.0000, 3.3333, 3.2000, 4.0000, 4.0000, 5.0000, 4.8000, 5.3333),$$

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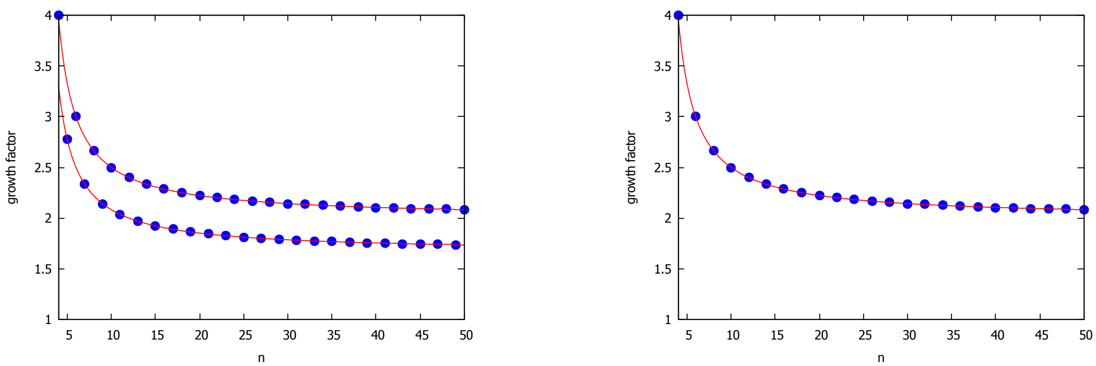

FIGURE 1. *The growth factor of circulant matrices C_n with the first row $(\frac{n}{2}-1,-1,-1,\ldots,-1)$ (left) and $(\frac{n}{2}-1,1,-1,1,-1,1\ldots,-1,1)$ (right) for n in the interval $[4,50]$. Notice that the latter type of C_n exists only for even n. As $n\to\infty$, the value $g(C_n)$ approaches 2 for even n and $5/3$ for odd n.*

while Proposition 3.5 provides

$$p_1=3,\ \ p_2=\frac{10}{3},\ \ p_3=\frac{16}{5},\ \ p_4=4,\ \ p_5=4,\ \ p_6=5,\ \ p_7=\frac{24}{5},\ \ p_8=\frac{16}{3}\,.$$

The results are apparently consistent. The growth factor equals $16/9\simeq 1.7778$.

4.4. Case IV. A matrix C_n having the first row $(d,1,1,-1,-1,1,1,-1,-1,\ldots,-1,1,1,-1)$ exists only for n being a multiple of 4. Taking $n=8$, i.e., a circulant matrix with the first row $(3,1,1,-1,-1,1,1,-1)$, one obtains the same numerical results as for the matrix with the first row $(3,-1,1,1,-1,-1,1,1)$, namely,

$$\operatorname{diag}(U)=(3.0000,3.3333,3.2000,4.0000,4.0000,5.0000,4.8000,5.3333).$$

This reflects the fact that the pivots of those matrices always coincide; see also the proof of Proposition 3.7. The growth factor is again $16/9\simeq 1.7778$.

Figures 1 and 2 illustrate the behaviour of the growth factor for each of the four types of matrices C_n whose first row is listed in (1.1).

5. Concluding remarks. In the present paper, we developed formulae for the minors of generalized circulant Hadamard matrices which were generated from four types of vectors. Some majorization results concerning the principal minors of these matrices were stated as well. Furthermore, the quotients of subsequent minors were expressed analytically. Based on the derived results, it was possible to specify the growth factor for all the matrices of this class. The results imply that these matrices have a stable behaviour when Gaussian elimination with complete pivoting is applied to them. Presented numerical examples support the produced theoretical pivot values and also validate the effectiveness of the obtained formulae.

Our results contribute to understanding the relation between the structural properties of Hadamard matrices and their large growth factor. Examining generalized circulant Hadamard matrices, we arrived at an explicit evidence that the unitarity and equal absolute values of the off-diagonal elements are by far not sufficient to achieve large $g(A)$. This finding forms a step on the path to localize the key property of Hadamard matrices, or a combination of properties, which is responsible for their large growth factor. The question is interesting and important to solve, because it would allow to describe other classes of matrices with unstable Gaussian elimination, which would have obvious practical consequences to numerical analysis.

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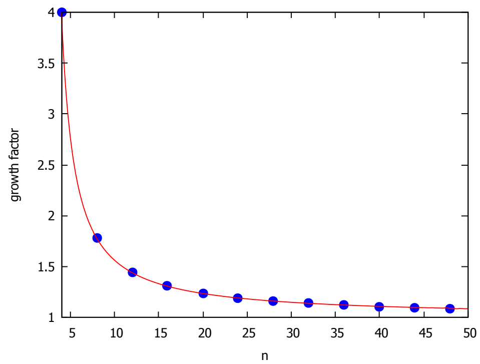

FIGURE 2. *The growth factor of circulant matrices with the first row* $(d,-1,1,1,-1,-1,1,1,-1,\dots,-1,-1,1,1)$ *or* $(d,1,1,-1,-1,1,1,-1,-1,\dots,-1,1,1,-1)$*. Notice that these matrices exist only for* n *being a multiple of* 4*. As* $n \to \infty$*, the value* $g(C_n)$ *approaches* 1*.*

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