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NON-SPARSE COMPANION MATRICES*

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Abstract. Given a polynomial $p(z)$, a companion matrix can be thought of as a simple template for placing the coefficients of $p(z)$ in a matrix such that the characteristic polynomial is $p(z)$. The Frobenius companion and the more recently-discovered Fiedler companion matrices are examples. Both the Frobenius and Fiedler companion matrices have the maximum possible number of zero entries, and in that sense are sparse. In this paper, companion matrices are explored that are not sparse. Some constructions of non-sparse companion matrices are provided, and properties that all companion matrices must exhibit are given. For example, it is shown that every companion matrix realization is non-derogatory. Bounds on the minimum number of zeros that must appear in a companion matrix, are also given.

Key words. Companion matrix, Fiedler companion matrix, Non-derogatory matrix, Characteristic polynomial.

AMS subject classifications. 15A18, 15B99, 11C20, 05C50.

1. Introduction.

The *Frobenius companion matrix* to the polynomial

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \tag{1.1}$$

is the matrix

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}. \tag{1.2}$$

In general, a *companion matrix* to $p(z)$ is an $n \times n$ matrix $C = C(p)$ over $\mathbb{R}[a_1, a_2, \ldots, a_n]$ with $n^2 - n$ entries in $\mathbb{R}$ and the remaining entries variables $-a_1, -a_2, \ldots, -a_n$ such that the characteristic polynomial of C is $p(z)$. We say that A is a *realization* of a companion matrix C if A is obtained from C by replacing each of the variable entries by real numbers.

Example 1.1. The matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & 1 & 0 \\ 0 & -a_2 & 0 & 1 \\ -a_4 & -a_3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_2 & -a_1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -a_4 & -a_3 & -4 & -2 \end{bmatrix}$$

are companion matrices to the characteristic polynomial $p(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$.

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In 2003, Fiedler [8] introduced a new way to construct companion matrices that has piqued interest in the general structure of companion matrices (see, for example, [4, 11, 5]), including extensions to matrix pencils (see, for example, [1, 3]). As noted in [4], a companion matrix must have at least $2n-1$ nonzero entries. A companion matrix is *sparse* if it has exactly $2n-1$ nonzero entries. The first matrix in Example 1.1 is sparse, the second is not sparse. The sparse companion matrices were characterized in [4] as described here in Theorem 1.2. (Note that two matrices A and B are *equivalent*, denoted $A \equiv B$, if B can be obtained from A by any combination of transposition, permutation similarity or diagonal similarity.)

Theorem 1.2 ([4]). *A matrix A is an $n \times n$ sparse companion matrix if and only if A is equivalent to a unit lower Hessenberg matrix C such that*

- $C_{j,j} = -a_1$ *for some* j, $1 \le j \le n$,
- $-a_{k+1}$ *is on the* k*th subdiagonal of* C, *for* $1 \le k \le n-1$, *and*
- $-a_{k+1}$ *is in the rectangular submatrix* R *whose upper right corner is* $C_{j,j} = -a_1$ *and whose lower left corner is* $C_{n,1} = -a_n$.

In this paper, we explore properties of companion matrices by focusing on the non-sparse companion matrices. A few non-sparse companion matrices were noted in [4] as examples. We determine some general properties of non-sparse companion matrices and describe some particular constructions. A companion matrix is a *superpattern* of a sparse companion matrix if it can be obtained from the sparse companion matrix by changing some of the zero entries to nonzero entries. We answer a question of [4] by describing some companion matrices that are not superpatterns of the sparse companion matrices (see Examples 2.2 and 6.7).

While Theorem 1.2 states that every sparse companion matrix is equivalent to some Hessenberg matrix, we will see that not every companion matrix has a Hessenberg form. For example, in Section 5, we demonstrate that there exist $n \times n$ companion matrices with fewer than $\frac{5}{2}n$ zero entries, however a Hessenberg matrix must have at least $\frac{n^2-3n+2}{2}$ zero entries. Also, as noted in [4], a key property of sparse companion matrices is that the digraph of a sparse companion must contain a Hamilton cycle. This property is not required in a non-sparse companion matrix, as can be observed in Example 2.2.

We start the next section by outlining some of the notation and required technical definitions, along with a known result connecting the coefficients of the characteristic polynomial of a matrix with the cycles in an associated digraph. In Section 3, we describe some properties of companion matrices. In particular, we observe that not only are Hessenberg companion matrix realizations non-derogatory, but all companion matrix realizations are non-derogatory. We conclude the section with a structural result for companion matrices. In Section 4, we list various examples of non-sparse companion matrices and describe the non-sparse $n \times n$ matrices for small n. In Section 5, we develop a result that restricts the structure of superpatterns of Fiedler companion matrices, concluding that there is no companion matrix that is a superpattern of a Frobenius companion matrix. In Section 6, we describe a way to construct a companion matrix by starting with an arbitrary non-derogatory nilpotent matrix. In Section 7, we explore how far a non-sparse companion matrix can be from being sparse. In particular, we provide some bounds on the minimum number of zero entries required in a companion matrix. We conclude in Section 8 with an observation about the structure of the companion matrices that we have explored, leaving it as an open question whether this structure is a necessary feature of every companion matrix.

While we explore constructions and properties of non-sparse companion matrices in this paper, in future it would be helpful to determine if these non-sparse companion can be useful in some of the many contexts

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in which the classical companion matrix has been used. For example, sparse companion matrices have been useful for providing bounds on the roots of a polynomial (see, e.g., [13] and [15]). It is natural to ask if there are applications in which the structure of the non-sparse companion matrices might be more desirable than the structure of the sparse companion matrices.

2. Technical definitions and notation. Throughout the paper, we let $\mathbf{e}_j$ denote the jth standard basis vector and let $J_r = \begin{bmatrix} \mathbf{0} & \mathbf{e}_1 & \cdots & \mathbf{e}_{r-1} \end{bmatrix}$ be an $r \times r$ Jordan block. We also let N_i denote the ith row of a matrix N.

The kth *subdiagonal* of an $n \times n$ matrix A consists of the entries $\{A_{k+1,1}, A_{k+2,2}, \ldots, A_{n,n-k}\}$, for $0 \leq k \leq n-1$. Note that the 0th subdiagonal is usually called the main diagonal of a matrix. For $1 \leq k \leq m$, the kth *diagonal* of an $n \times m$ matrix R consists of the entries $\{A_{1,m-k+1}, A_{2,m-k+2}, \ldots, A_{k,m}\}$ and for $m+1 \leq k \leq n+m-1$, the kth diagonal consists of the entries $\{A_{k-m+1,1}, A_{k-m+2,2}, \ldots, A_{n,n-(k-m)}\}$.

One tool that is often used to classify permutationally equivalent matrices is the digraph of the matrix. Given an $n \times n$ matrix A, the *digraph* $D(A)$ has vertex set $\{v_1, v_2, \ldots, v_n\}$ with arc set $\{(v_i, v_j) : A_{i,j} \neq 0\}$. Note that two matrices are permutationally equivalent if and only if their labeled digraphs are isomorphic. For $1 \leq k \leq n$, a (simple) *cycle of length* k (or k-cycle) in a digraph D is a sequence of k arcs $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})$, sometimes denoted as $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_k} \to v_{i_1}$, with k distinct vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. If $D(A)$ has cycle $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})$ and $A = [A_{ij}]$ then $(-1)^{k-1} A_{v_{i_1}, v_{i_2}} A_{v_{i_2}, v_{i_3}} \cdots A_{v_{i_k}, v_{i_1}}$ is the associated *cycle product*. A *composite* cycle of length k is a set of vertex disjoint cycles with lengths summing to k (with an associated cycle product being the product of the cycle products of the individual cycles). The digraph cycles of $D(A)$ can be used to describe the characteristic polynomial of A (see, for example, [2, 6]):

Lemma 2.1. *If $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$ is the characteristic polynomial of A, then the sum of all the composite cycle products of A of length k is $(-1)^k a_k$.*

An n-cycle in a digraph with n vertices is called a *Hamilton* cycle. As can be seen by Theorem 1.2, the digraph of every sparse $n \times n$ companion matrix must have a Hamilton cycle that includes a_n. However, this feature is not required in a non-sparse companion matrix:

Example 2.2. The digraph of the companion matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & -a_1 & 1 \\ -a_5 & 0 & -a_3 & -a_2 & 0 \end{bmatrix}$$

does not have a Hamilton cycle. That is, there is no simple 5-cycle in $D(A)$.

3. Properties of companion matrices. Recall that a real matrix is said to be *non-derogatory* if each eigenspace has dimension one, or, equivalently, the minimal polynomial is the same as the characteristic polynomial of the matrix. We will say that a matrix A over $\mathbb{R}[a_1, a_2, \ldots, a_n]$ is non-derogatory if every realization of A is non-derogatory. It was observed in [4] that each sparse companion matrix is equivalent to a Hessenberg matrix, and hence is non-derogatory. Below we observe that while some non-sparse companion matrices are not equivalent to a Hessenberg matrix, every companion matrix is still non-derogatory. By [12, Theorem 3.3.15], this implies that every companion matrix realization is similar to a Hessenberg matrix, in

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fact, similar to a Frobenius companion matrix realization. While we consider only real matrices here, this result is in fact independent of the field used.

THEOREM 3.1. *If C is a companion matrix, then C is non-derogatory.*

Proof. Suppose A is an $n \times n$ companion matrix realization having some eigenvalue λ with two linearly independent λ-eigenvectors. Then there is a λ-eigenvector $\mathbf{x}$ such that $\mathbf{x}_i = 0$ for any specified coordinate position i. Let i be the column of A that includes the variable a_n. Let $\mathbf{x}$ be a λ-eigenvector of A with $\mathbf{x}_i = 0$.

Let A' be the matrix obtained from A by replacing entry a_n with $a_n + 1$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and $A'\mathbf{x} = \lambda\mathbf{x}$. That is, λ is an eigenvalue of A and A'. Thus, λ satisfies the characteristic polynomial of A, so $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$. And λ satisfies the characteristic polynomial of A', so $\lambda^n + a_1\lambda^{n-1} + \cdots + (a_n + 1) = 0$. But these two equations are contradictory. Therefore, A can not have two independent λ-eigenvectors. $\square$

In the next three lemmas, we note some properties of non-derogatory nilpotent matrices.

LEMMA 3.2. *If A is a non-derogatory nilpotent block triangular matrix of the form*

$$A = \left[\begin{array}{c|c} N & H \\ \hline O & W \end{array}\right],$$

then both diagonal blocks N and W are nilpotent and non-derogatory.

Proof. Suppose N is $n \times n$ and W is $w \times w$. Observe that

$$A^k = \left[\begin{array}{c|c} N^k & \sum_{a=0}^{k} N^{k-a} H W^{a-1} \\ \hline O & W^k \end{array}\right]$$

and, since $A^{n+w} = 0$, this implies that N and W are nilpotent. It follows that

$$A^{n+w-1} = \left[\begin{array}{c|c} O & \sum_{a=0}^{n+w-1} N^{n+w-1-a} H W^{a-1} \\ \hline O & O \end{array}\right].$$

Since A is non-derogatory, $\sum_{a=0}^{n+w-1} N^{n+w-1-a} H W^{a-1} \neq O$. But

$$\begin{aligned} \sum_{a=0}^{n+w-1} N^{n+w-1-a} H W^{a-1} &= \sum_{a>w} N^{n+w-1-a} H W^{a-1} + \sum_{a<w} N^{n+w-1-a} H W^{a-1} + N^{n-1} H W^{w-1} \\ &= N^{n-1} H W^{w-1}. \end{aligned}$$

Therefore, both N and W are non-derogatory. $\square$

LEMMA 3.3. *Suppose N is an $r \times r$ non-derogatory nilpotent matrix. Then there exists a nonsingular matrix S such that $S^{-1} N S = J_r$. Further, if*

$$S = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_r \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_r \end{bmatrix},$$

then $N^{r-k} = \sum_{j=1}^{k} \mathbf{s}_j \mathbf{t}_{r-k+j}$ *for* $1 \leq k \leq r-1$.

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Proof. Since N is non-derogatory and nilpotent, the Jordan decomposition for N is SJS^{-1} for some invertible matrix S. It follows that

$$\begin{aligned} N^{r-k} = SJ^{r-k}S^{-1} &= S\begin{bmatrix}\mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_1 & \cdots & \mathbf{e}_k\end{bmatrix}S^{-1} \\ &= \begin{bmatrix}\mathbf{0} & \cdots & \mathbf{0} & \mathbf{s}_1 & \cdots & \mathbf{s}_k\end{bmatrix}S^{-1} \\ &= \mathbf{s}_1\mathbf{t}_{r-k+1} + \mathbf{s}_2\mathbf{t}_{r-k+2} + \cdots + \mathbf{s}_k\mathbf{t}_r \\ &= \sum_{j=1}^{k}\mathbf{s}_j\mathbf{t}_{r-k+j}. \qquad \square \end{aligned}$$

.

Remark 3.4. Note that, for $k = 1$, Lemma 3.3 implies that $N^{r-1} = \mathbf{s}_1\mathbf{t}_r$.

Lemma 3.5. *Suppose N is an $r \times r$ non-derogatory nilpotent matrix. If $N^{r-1}\mathbf{x} \neq \mathbf{0}$ and*

$$\begin{bmatrix}(N^{r-k})_r \\ \vdots \\ (N^{r-1})_r\end{bmatrix}\mathbf{x} = \begin{bmatrix}0 \\ \vdots \\ 0\end{bmatrix}$$

for some k, $1 \leq k < r$, then $(N^{r-k})_r = (N^{r-k+1})_r = \cdots = (N^{r-1})_r = \mathbf{0}^T$.

Proof. By Lemma 3.3, we may let $S = [S_{i,j}]$ be a matrix such that $S^{-1}NS = J_r$. Since $N^{r-1}\mathbf{x} \neq \mathbf{0}$, by Remark 3.4, we have $\mathbf{t}_r\mathbf{x} \neq 0$. Thus, $(N^{r-1})_r\mathbf{x} = S_{r,1}\mathbf{t}_r\mathbf{x} = 0$ implies that $S_{r,1} = 0$; and hence, $(N^{r-1})_r = \mathbf{0}^T$. As an inductive hypothesis, suppose that $S_{r,1} = S_{r,2} = \cdots = S_{r,j} = 0$. Then, using Lemma 3.3,

$$(N^{r-(j+1)})_r\mathbf{x} = S_{r,1}\mathbf{t}_{r-j}\mathbf{x} + S_{r,2}\mathbf{t}_{r-j+1}\mathbf{x} + \cdots + S_{r,j}\mathbf{t}_{r-1}\mathbf{x} + S_{r,(j+1)}\mathbf{t}_r\mathbf{x} = S_{r,(j+1)}\mathbf{t}_r\mathbf{x} = 0.$$

Thus, $S_{r,(j+1)} = 0$; and hence, $(N^{r-(j+1)})_r = \mathbf{0}^T$. $\square$

We will use the following connection with the adjugate of a nilpotent matrix that could be derived from [14] but is explicitly demonstrated in [10].

Lemma 3.6 ([10, Lemma 3.3]). *If N is an $r \times r$ nilpotent matrix, then*

$$\operatorname{adj}(zI - N) = N^{r-1} + zN^{r-2} + \cdots + z^{r-1}I.$$

Remark 3.7. We will repeatedly use a succinct expression for the characteristic polynomial of a matrix using an adjugate (see, for example, [7]) that can be derived via cofactor expansion. In particular, if

$$A = \left[\begin{array}{c|c} N & \mathbf{x} \\ \hline \mathbf{y}^T & a \end{array}\right],$$

then

$$\det(zI - A) = (z - a)\det(zI - N) - \mathbf{y}^T\operatorname{adj}(zI - N)\mathbf{x}.$$

We next apply some of the above observations about non-derogatory nilpotent matrices to obtain properties of companion matrices.

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Lemma 3.8. *Let* $\mathrm{R} = \mathbb{R}[a_2, \ldots, a_n]$*. If* A *is an* $n \times n$ *companion matrix, then* A *is equivalent to*

$$\left[\begin{array}{c|c} N & \mathbf{x} \\ \hline \mathbf{y}^T & -a_1 \end{array}\right] \tag{3.3}$$

for some $(n-1) \times (n-1)$ *matrix* N *over* R*, and some* $\mathbf{x}, \mathbf{y} \in \mathrm{R}^{n-1}$*. Further,*

1. N *is nilpotent and non-derogatory, and*
2. *if* $\mathbf{y}_{n-1} = -a_2$*, then* $N_{n-1} = \mathbf{0}^T$ *and* $\mathbf{x}_{n-1} = 1$*.*

Proof. Since a_1 appears on the main diagonal of A, by equivalency we may assume that $A_{n,n} = -a_1$. Consider the characteristic polynomial of A:

$$\begin{aligned} \det(zI - A) &= (z + a_1)\det(zI - N) - \mathbf{y}^T \operatorname{adj}(zI - N)\mathbf{x} \\ &= z\det(zI - N) + a_1 \det(zI - N) - \mathbf{y}^T \operatorname{adj}(zI - N)\mathbf{x}. \end{aligned}$$

Note that the first and third summands contain no terms with a_1 as a factor. Because A is a companion matrix, $a_1 \det(zI - N) = a_1 z^{n-1}$. So, $\det(zI - N) = z^{n-1}$. Thus, N is nilpotent. Then, by Lemma 3.6,

$$\det(zI - A) = z^n + a_1 z^{n-1} - \mathbf{y}^T \left(N^{n-2} + zN^{n-3} + \cdots + z^{n-3}N + z^{n-2}I\right)\mathbf{x}.$$

Since A is a companion matrix, the constant term is $a_n = -\mathbf{y}^T N^{n-2}\mathbf{x}$. Thus, $N^{n-2}\mathbf{x} \neq \mathbf{0}$, and so in particular N^{n-2} is not the zero matrix, and hence, N is non-derogatory. Further, the coefficient of z^{n-2} is a_2 and we must have $-\mathbf{y}^T z^{n-2} I\mathbf{x} = a_2 z^{n-2}$. Thus, $-\mathbf{y}^T\mathbf{x} = a_2$.

If $\mathbf{y}_{n-1} = -a_2$, then $\mathbf{x}_{n-1} = 1$ and $a_2 \mathbf{e}_{n-1}^T \left(N^{n-2} + zN^{n-3} + \cdots + z^{n-3}N\right)\mathbf{x} = 0$. Thus,

$$\begin{bmatrix} N_{n-1} \\ (N^2)_{n-1} \\ \vdots \\ (N^{n-2})_{n-1} \end{bmatrix} \mathbf{x} = \mathbf{0}$$

and, by Lemma 3.5, $N_{n-1} = \mathbf{0}^T$. ∎

Because the elements on the main diagonal of A must add to $-a_1$ and each variable is allowed to appear only once in the matrix, no variable other than a_1 can occur on the main diagonal of A.

Lemma 3.9. *For every companion matrix* A*, the only variable on the main diagonal of* A *is* a_1*.*

By Lemma 3.9, a variable other than a_1 that occurs in a composite 2-cycle can do so only by occurring in a simple 2-cycle. Since that variable occurs just once in the matrix, it must in fact then occur in a unique simple 2-cycle. Hence, using Lemma 2.1, we have the following.

Lemma 3.10. *For every companion matrix* A*, the only variable that is on a simple* 2*-cycle of* $D(A)$ *is* a_2*, and* a_2 *appears on exactly one* 2*-cycle in* $D(A)$*.*

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Theorem 3.11. *Let* $\mathrm{R} = \mathbb{R}[a_3, \ldots, a_n]$ *with* $n \geq 3$. *Suppose* A *is an* $n \times n$ *companion matrix. Then* A *is equivalent to*

$$\left[\begin{array}{c|cc} N & \mathbf{x} & \mathbf{0} \\ \hline \mathbf{u}^T & -a_1 & 1 \\ \mathbf{v}^T & -a_2 & 0 \end{array}\right] \tag{3.4}$$

for some $(n-2) \times (n-2)$ *nilpotent non-derogatory matrix* N *over* R, *and some* $\mathbf{x}, \mathbf{u}, \mathbf{v} \in \mathrm{R}^{n-2}$ *such that* $\mathbf{x}$ *is orthogonal to* $\mathbf{u}$ *and* $N^{n-3}\mathbf{x} \neq \mathbf{0}$.

Proof. Let A be an $n \times n$ companion matrix. By Lemma 3.8, part 1, A is equivalent to a matrix

$$B = \left[\begin{array}{c|c} N' & * \\ \hline * & -a_1 \end{array}\right]$$

for some $(n-1) \times (n-1)$ nilpotent, non-derogatory matrix N'. By Lemma 3.10, $D(A)$ must have a simple 2-cycle of weight a_2 and there can be only one such cycle. Since N' is nilpotent, this 2-cycle cannot be contained in $D(N')$. Thus, we may assume $-a_2$ appears in row n of B and then, by permutation equivalence, that $B_{n,n-1} = -a_2$. Then, by Lemma 3.8, part 2, $B_{n-1} = \mathbf{e}_n^T$. Hence, by Lemma 3.2, the leading $(n-2)\times(n-2)$ principal submatrix of B is nilpotent and non-derogatory. Thus, by transposing B and permuting the last two rows/columns, we can observe that A is equivalent to a matrix having the form (3.4) for some $(n-2)\times(n-2)$ nilpotent non-derogatory matrix N over R, and some $\mathbf{x}, \mathbf{u}, \mathbf{v} \in \mathrm{R}^{n-2}$.

To observe the claimed restrictions on $\mathbf{x}$, consider the characteristic polynomial of A computed by cofactor expansion along the nth column of (3.4):

$$\begin{aligned}
\det(zI - A) &= (-1)(-1)^{2n-1} \det \left[\begin{array}{c|c} zI - N & -\mathbf{x} \\ \hline -\mathbf{v}^T & a_2 \end{array}\right] + z(-1)^{2n} \det \left[\begin{array}{c|c} zI - N & -\mathbf{x} \\ \hline -\mathbf{u}^T & z + a_1 \end{array}\right] \\
&= \left(a_2 z^{n-2} - \mathbf{v}^T \operatorname{adj}(zI - N)\mathbf{x}\right) + z\left((z + a_1)z^{n-2} - \mathbf{u}^T \operatorname{adj}(zI - N)\mathbf{x}\right) \\
&= z^n + a_1 z^{n-1} + a_2 z^{n-2} - z\mathbf{u}^T \operatorname{adj}(zI - N)\mathbf{x} - \mathbf{v}^T \operatorname{adj}(zI - N)\mathbf{x} \\
&= z^n + a_1 z^{n-1} + a_2 z^{n-2} - \left(z\mathbf{u}^T + \mathbf{v}^T\right)\left(N^{n-3} + zN^{n-4} + \cdots + z^{n-4}N + z^{n-3}I\right)\mathbf{x}, \qquad (3.5)
\end{aligned}$$

using Lemma 3.6. Since A is a companion matrix, we need $\mathbf{u}^T z^{n-2} I \mathbf{x} = 0$. Thus, $\mathbf{x}$ is orthogonal to $\mathbf{u}$. Further, since the constant term of the characteristic polynomial of A is a_n, it follows from (3.5) that $\mathbf{v}^T N^{n-3}\mathbf{x} \neq 0$, and in particular, $N^{n-3}\mathbf{x} \neq \mathbf{0}$. ∎

4. Examples of non-sparse companion matrices. In this section, we describe some examples of non-sparse companion matrices. Note that any 2×2 companion matrix is equivalent to a Frobenius companion matrix and hence is sparse. We will observe that there is also no 3×3 non-sparse companion matrix, and classify the 4×4 non-sparse companion matrices.

Theorem 4.1. *There are no non-sparse* 3×3 *companion matrices.*

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Proof. Suppose A is a 3×3 companion matrix. By Theorem 3.11, A is equivalent to

$$\begin{bmatrix} 0 & x & 0 \\ u & -a_1 & 1 \\ v & -a_2 & 0 \end{bmatrix}$$

for some x, u, v with $ux = 0$. Then $\det(zI - A) = (z + a_1)z^2 + a_2 z - vx$. Thus, $vx = -a_3$, and hence, $u = 0$. Therefore, A is a sparse companion matrix. ∎

We next show that, structurally, there is essentially only one 4×4 non-sparse companion matrix. It is a companion matrix that is a superpattern of a sparse companion matrix.

Theorem 4.2. *If A is a non-sparse 4×4 companion matrix, then A is equivalent to*

$$\begin{bmatrix} -b & 1 & 0 & 0 \\ -b^2 & b & 1 & 0 \\ -a_3 & 0 & -a_1 & 1 \\ -a_4 & 0 & -a_2 & 0 \end{bmatrix} \tag{4.6}$$

for some $b \in \mathbb{R}$ with $b \neq 0$.

Proof. Suppose A is a 4×4 companion matrix. By Theorem 3.11, we can assume that A has the structure given in (3.4) for some 2×2 nilpotent, non-derogatory matrix N and some vectors $\mathbf{u}, \mathbf{x}, \mathbf{v}$ with $\mathbf{u}^T \mathbf{x} = 0$ and $N\mathbf{x} \neq \mathbf{0}$.

We first show that a_3 does not appear in N. Otherwise, we have (since N is nilpotent) that $-a_3$ is the only nonzero entry of N, and so, without loss of generality,

$$zI - A = \begin{bmatrix} z & \boxed{a_3} & -x_1 & 0 \\ 0 & z & \boxed{-x_2} & 0 \\ -u_1 & -u_2 & z + a_1 & \boxed{-1} \\ \boxed{-v_1} & -v_2 & a_2 & z \end{bmatrix}.$$

Then the highlighted transversal is the only one that is nonzero and constant with respect to z. This contradicts the fact that A is a companion matrix. Hence, a_3 must not appear in N.

By (3.5), $\det(zI - A) = z^4 + a_1 z^3 + a_2 z^2 - (z\mathbf{u}^T + \mathbf{v}^T)(N + zI)\mathbf{x}$. Thus,

$$-a_3 = (\mathbf{u}^T N + \mathbf{v}^T)\mathbf{x} \quad \text{and} \tag{4.7}$$

$$-a_4 = \mathbf{v}^T N \mathbf{x}. \tag{4.8}$$

Up to equivalence, there are three cases for consideration:

1. Suppose $x_2 = -a_3$. By Lemma 3.10, $u_2 = 0$. Equation (4.8) implies that $(N^T)_2 \mathbf{v} = 0$ and $N^T \mathbf{v} \neq 0$. By Lemma 3.5, $(N^T)_2 = \mathbf{0}$. Then (4.7) implies $v_2 = 1$. Also, since the trace of N is zero, $N_{11} = 0$. Meanwhile, $\mathbf{u}^T \mathbf{x} = 0$ implies $u_1 x_1 = 0$, while (4.7) gives $v_1 x_1 = 0$. However, $N\mathbf{x} \neq \mathbf{0}$ implies $x_1 \neq 0$. Therefore, $u_1 = v_1 = 0$ and A is a sparse companion matrix.
2. Suppose $v_2 = -a_3$. Then (4.8) implies $N_2 \mathbf{x} = 0$ and, by Lemma 3.5, $N_2 = \mathbf{0}$. We again have $N_{11} = 0$ since the trace of N is zero. Moreover, (4.7) requires $x_2 = 1$, and so, (4.8) implies $v_1 N_{12} = -a_4$. Thus, $N_{12} \neq 0$ and $v_1 \neq 0$. Then (4.7) implies $x_1 = u_1 = 0$. It follows that $u_2 = 0$ since $\mathbf{u}^T \mathbf{x} = 0$. Thus, A is a sparse companion matrix.

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3. Suppose $u_2 = -a_3$. By Lemma 3.10, $x_2 = 0$. Equation (4.7) implies $N_{21}x_1 = 1$, and hence, $x_1 \neq 0$. But then $u_1 = 0$ since $\mathbf{u}^T\mathbf{x} = 0$. Further, $v_1 = 0$ by (4.7), and (4.8) implies that $v_2 = -a_4$. Letting $b = N_{11}$ and $c = N_{21}$, we have, since N is nilpotent, that $c \neq 0$, and hence, A is equivalent to

$$\left[\begin{array}{cc|cc} b & -b^2/c & 1/c & 0 \\ c & -b & 0 & 0 \\ \hline 0 & -a_3 & -a_1 & 1 \\ 0 & -a_4 & -a_2 & 0 \end{array}\right] \equiv \left[\begin{array}{cc|cc} b & -b^2 & 1 & 0 \\ 1 & -b & 0 & 0 \\ \hline 0 & -a_3 & -a_1 & 1 \\ 0 & -a_4 & -a_2 & 0 \end{array}\right] \equiv \begin{bmatrix} -b & 1 & 0 & 0 \\ -b^2 & b & 1 & 0 \\ -a_3 & 0 & -a_1 & 1 \\ -a_4 & 0 & -a_2 & 0 \end{bmatrix}. \qquad \square$$

While there exists only one non-sparse 4×4 companion matrix, there exist many 5×5 examples. We describe a couple of cases in Example 4.3. Note that Examples 2.2 and 5.1 are also 5×5 non-sparse companion matrices.

Example 4.3. Suppose $c \neq 0$ and at least one of b or d is nonzero. Then the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -a_3 & -a_2 & -a_1 & 1 & 0 \\ 0 & 0 & 0 & c & 1 \\ -a_5 & -a_4 & 0 & -c^2 & -c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -a_2 & -a_1 & 1 & 0 & 0 \\ 0 & 0 & b & 1 & 0 \\ -a_4 & -a_3 & -b^2-d & -b & 1 \\ -a_5 & 0 & bd & d & 0 \end{bmatrix}$$

are non-sparse companion matrices.

Example 4.4. Suppose a, b, c, d and e are nonzero. Then

$$\left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline -a_2 & -a_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & b & 1 & 0 & 0 \\ -a_4 & -a_3 & c & -b & 1 & 0 \\ 0 & 0 & -b^3+b^2d-bc+cd-e & -b^2-c & d & 1 \\ -a_6 & -a_5 & -b^2d^2-cd^2+be+de & e & -d^2 & -d \end{array}\right], \quad \left[\begin{array}{ccc|cccc} -d & 1 & 0 & 0 & 0 & 0 & 0 \\ -d^2 & d & 1 & 0 & 0 & 0 & 0 \\ \hline -a_3 & 0 & -a_1 & 1 & 0 & 0 & 0 \\ -a_4 & 0 & -a_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 1 & 0 \\ -a_6 & 0 & 0 & 0 & -b^2-c & -b & 1 \\ -a_7 & 0 & -a_5 & 0 & bc & c & 0 \end{array}\right]$$

are non-sparse 6×6 and 7×7 companion matrices which are superpatterns of sparse companion matrices.

In the next example, the companion matrix has the property that it contains additional nonzero entries (relative to a sparse companion matrix) inside as well as outside the lower left submatrix R described in Theorem 1.2, unlike the previous examples.

Example 4.5. Suppose b, f and g are nonzero. Then

$$A = \left[\begin{array}{ccc|cccc} f & 1 & 0 & 0 & 0 & 0 & 0 \\ -f^2 & -f & 1 & 0 & 0 & 0 & 0 \\ \hline -a_3 & 0 & -a_1 & 1 & 0 & 0 & 0 \\ -a_4 & b & -a_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -b & 0 & g & 1 & 0 \\ -a_6 & 0 & bf+bg & 0 & -g^2 & -g & 1 \\ -a_7 & 0 & -a_5 & 0 & 0 & 0 & 0 \end{array}\right]$$

is a 7×7 non-sparse companion matrix.

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In many examples, if A is a companion matrix partitioned as

$$A = \left[\begin{array}{c|c|c} \quad N \quad & * & * \\ \hline \multicolumn{2}{c|}{R} & * \end{array}\right],$$

with R containing all of the variable entries, including $-a_1$ in its top right corner, then N is nilpotent. The next example shows, however, that N does not have to be nilpotent in this situation.

Example 4.6. The 9×9 non-sparse companion matrix

$$\left[\begin{array}{rrrrr|rrrr} 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & -2 & 1 & 0 & 5 & -1 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & -3 & 0 & 0 \\ 1 & -12 & -6 & 7 & -3 & 0 & -20 & 4 & 0 \\ \hline -a_5 & 0 & -a_3 & 0 & -a_1 & 1 & 0 & 0 & 0 \\ -a_6 & 1 & -a_4 & 1 & -a_2 & 0 & -1 & 0 & 0 \\ 1 & -6 & -3 & 4 & -2 & 0 & -11 & 2 & 0 \\ -a_8 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ -a_9 & 0 & -a_7 & 0 & 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

has the property that the leading 4×4 principal submatrix N is *not* a nilpotent matrix.

5. Superpatterns of Fiedler companion matrices. A *Fiedler* companion matrix A is a sparse companion matrix which is equivalent to a unit lower Hessenberg for which the variable entries form a lattice path starting in the lower left corner and ending on the main diagonal. In particular, the *lattice path* property means that if $-a_{k-1}$ is in position (i, j) of A then $-a_k$ is in position $(i + 1, j)$ or $(i, j - 1)$ for $2 \le k \le n$. The first matrix in Example 1.1 is a Fiedler companion matrix (see also Example 5.1 with $b = 0$). The second matrix in Example 1.1 is a superpattern of a sparse companion matrix, but is not a superpattern of a Fiedler companion matrix; the variable entries in the second matrix do not form a lattice path. As described in [4], the Fiedler companion matrices defined here are equivalent to the matrices developed by Fiedler in [8]. In this section, we explore companion matrices that are superpatterns of a Fiedler companion matrix.

Example 5.1. For any choice of $b \in \mathbb{R}$, $b \neq 0$, the companion matrix

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline b & -a_2 & -a_1 & 1 & 0 \\ 0 & -a_3 & 0 & 0 & 1 \\ -a_5 & -a_4 & -b & 0 & 0 \end{array}\right]$$

is a proper superpattern of a Fiedler companion matrix.

We begin with some helpful preliminary lemmas.

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LEMMA 5.2. *Suppose* $\mathbf{x} \in \mathbb{R}^r$, $\mathbf{y} \in \mathbb{R}^w$, *with* $w = n - r - 1$, *and*

$$A = \left[\begin{array}{c|c|c} N & \mathbf{x} & * \\ \hline \multicolumn{2}{c|}{} & \mathbf{y}^T \\ \cline{3-3} \multicolumn{2}{c|}{O - a_n\mathbf{e}_{w+1}\mathbf{e}_1^T} & M^T \end{array}\right]$$

for some $r \times r$ *matrix* N *and* $w \times w$ *matrix* M. *If* $\det(zI - A) = z^n + a_n$, *then* N *and* M *are nilpotent and non-derogatory with* $N^{r-1}\mathbf{x} \neq \mathbf{0}$ *and* $M^{w-1}\mathbf{y} \neq \mathbf{0}$.

Proof. Since $\det(zI - A) = z^n$ when $a_n = 0$, N and M must be nilpotent. Note that every transversal of $zI - A$ includes at least one term from the bottom left block. Since there are only two nonzero entries in that block, namely a_n and z, we can write

$$\det(zI - A) = (z)z^{n-1} - a_n\left(\mathbf{e}_1^T\left(\operatorname{adj}(zI - N)\mathbf{x}\right)\right)\left(\mathbf{e}_w^T\left(\operatorname{adj}(zI - M)\mathbf{y}\right)\right) + a_n z f(z)$$

for some function $f(z)$. Since N and M are nilpotent, Lemma 3.6 gives

$$\left[\mathbf{e}_1^T\left(\operatorname{adj}(zI - N)\mathbf{x}\right)\right]\left[\mathbf{e}_w^T\left(\operatorname{adj}(zI - M)\mathbf{y}\right)\right] =$$

$$\left(\mathbf{e}_1^T(N^{r-1} + zN^{r-2} + \cdots + z^{r-2}N + z^{r-1}I)\mathbf{x}\right)\left(\mathbf{e}_w^T(M^{w-1} + zM^{w-2} + \cdots + z^{w-1}I)\mathbf{y}\right).$$

Focusing on the constant term in $\det(zI - A)$, we have $\left(\mathbf{e}_1^T(N^{r-1})\mathbf{x}\right)\left(\mathbf{e}_w^T(M^{w-1})\mathbf{y}\right) = -1$. In particular, $N^{r-1}\mathbf{x} \neq \mathbf{0}$ and $M^{w-1}\mathbf{y} \neq \mathbf{0}$. Thus, N and M are non-derogatory. ◻

LEMMA 5.3. *Suppose* $1 < k \leq n$, $\mathbf{x} \in \mathbb{R}^r$, $\mathbf{y} \in \mathbb{R}^w$ *and*

$$A = \left[\begin{array}{c|c|c|c} N & \mathbf{x} & Q & \\ \cline{1-3} \multicolumn{1}{c|}{O - a_k\mathbf{e}_{k-1}\mathbf{e}_r^T} & \multicolumn{2}{c|}{J_{k-1}} & * \\ \cline{4-4} & \multicolumn{2}{c|}{} & \mathbf{y}^T \\ \hline \multicolumn{3}{c|}{O - a_n\mathbf{e}_w\mathbf{e}_1^T} & M^T \end{array}\right]$$

for some $r \times r$ *matrix* N *and* $w \times w$ *matrix* M *with* $1 \leq r \leq n - k + 1$, *and* $w = n - r - k + 1$. *If* $k \neq n$ *and* $\det(zI - A) = z^n + a_k z^{n-k} + a_n$, *or* $k = n$ *and* $\det(zI - A) = z^n + a_n$, *then* $N_r = \mathbf{0}^T$, $\mathbf{x}_r = 1$, *and* $Q_r = \mathbf{0}^T$.

Proof. Applying Lemma 5.2 with $a_k = 0$, we see that N and M are nilpotent and non-derogatory, and $N^{r-1}\mathbf{x} \neq \mathbf{0}$. Suppose $Q = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_{k-2}\end{bmatrix}$. Suppose $a_k \neq 0$, and if $n \neq k$ suppose also that $a_n = 0$. By cofactor expansion,

$$\det(zI - A) = z^w\left(z^{r+k-1} + a_k\mathbf{e}_r^T\operatorname{adj}(zI - N)\mathbf{x} + a_k\mathbf{e}_r^T\operatorname{adj}(zI - N)\sum_{i=1}^{k-2}\mathbf{q}_i z^i\right).$$

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Given $\det(zI-A) = z^n + a_k z^{n-k} + a_n$, it follows that

$$\mathbf{e}_r^T \operatorname{adj}(zI-N)\mathbf{x} + \mathbf{e}_r^T \operatorname{adj}(zI-N)\sum_{i=1}^{k-2} \mathbf{q}_i z^i = z^{r-1}. \tag{5.9}$$

Expanding (5.9) with Lemma 3.6 yields

$$\left[\sum_{i=1}^{r} (N^{r-i})_r z^{i-1}\right]\left[\mathbf{x} + \sum_{i=1}^{k-2} \mathbf{q}_i z^i\right] = z^{r-1}. \tag{5.10}$$

Considering the constant term of (5.10), we have $(N^{r-1})_r \mathbf{x} = 0$. Thus, by Lemma 3.5, $(N^{r-1})_r = \mathbf{0}^T$. Therefore, (5.10) simplifies to

$$\left[\sum_{i=2}^{r} (N^{r-i})_r z^{i-1}\right]\left[\mathbf{x} + \sum_{i=1}^{k-2} \mathbf{q}_i z^i\right] = z^{r-1}. \tag{5.11}$$

Considering the coefficient of z in (5.11), we have $(N^{r-2})_r \mathbf{x} = 0$. Again, by Lemma 3.5, $(N^{r-2})_r = \mathbf{0}^T$. Continuing this process inductively, we observe that $(N^{r-j})_r = \mathbf{0}^T$ for $j = 1, \ldots, r-1$. And, in particular, $N_r = \mathbf{0}^T$. Now (5.11) simplifies to

$$\mathbf{e}_r^T z^{r-1}\left[\mathbf{x} + \sum_{i=1}^{k-2} \mathbf{q}_i z^i\right] = z^{r-1}.$$

It follows that

$$\mathbf{e}_r^T \begin{bmatrix} \mathbf{x} & \mathbf{q}_1 & \cdots & \mathbf{q}_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Therefore, $\mathbf{x}_r = 1$ and $Q_r = \mathbf{0}^T$. ◻

In the next proof, we use the fact that the matrix obtained from A by reflecting its entries across the anti-diagonal is given by PA^TP, where P is the reverse permutation matrix, i.e., $P = \left[\begin{smallmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{smallmatrix}\right]$.

THEOREM 5.4. *Suppose $1 \le m < n$. Suppose R is an $(n-m)\times(m+1)$ matrix with exactly n nonzero entries, namely $a_1, a_2, \ldots, a_n$, such that $R_{n-m,1} = -a_n$ and $R_{1,m+1} = -a_1$ and the variable entries form a lattice path. If*

$$A = \left[\begin{array}{c|c} * & * \\ \hline R & * \end{array}\right]$$

is an $n \times n$ companion matrix then A is a sparse companion matrix.

Proof. We aim to show that replacing the variables a_i with zero for $1 \le i \le n$ in A gives the matrix J_n. Let A_k denote the matrix A with $a_i = 0$ for all $i \ne k$. For some k with $2 \le k \le n$, suppose A_k has the

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structure

$$A_k = \left[\begin{array}{cc|c} N & * & \multirow{2}{*}{$*$} \\ \hline O - a_k\mathbf{e}_{k-1}\mathbf{e}_r^T & J_{k-1} & \\ \hline \multicolumn{2}{c|}{O} & * \end{array}\right] \tag{5.12}$$

for some $r \times r$ matrix N with $1 \le r < n$. In this case, we can observe from Lemma 5.3 that the first $r+k-1$ entries of the rth row of A_k consist of r zeros followed by a one and $k-2$ zeros, and hence,

$$A_k = \left[\begin{array}{cc|c} N' & * & \multirow{2}{*}{$*$} \\ \hline O & J_k - a_k\mathbf{e}_k\mathbf{e}_1^T & \\ \hline \multicolumn{2}{c|}{O} & * \end{array}\right] \tag{5.13}$$

for some $(r-1) \times (r-1)$ matrix N'. By the assumption that the entries a_1 through a_n form a lattice path, a_2 is either to the left of or below a_1 in A, and so either A_2 or PA_2^TP has the structure (5.12), where P is the reverse permutation matrix. Then by the above observation, either A_2 or PA_2^TP has structure (5.13) and so, again by the lattice path property of R, either A_3 or PA_3^TP will have structure (5.12). In fact, it follows inductively that either A_k or PA_k^TP has structure (5.12) for $2 \le k \le n$. In particular, A_n has structure (5.13). Thus, it follows that A is sparse. ◻

Corollary 5.5. *If A is a companion matrix which is a superpattern of a Fiedler companion matrix, then A has a nonzero, non-variable entry in the rectangle R.*

Example 5.1 provides a superpattern of a Fiedler companion matrix with the additional nonzero entries (relative to the Fiedler matrix) in the rectangle R. We do not have any examples of a companion matrix which is a superpattern of a Fiedler companion matrix with any additional nonzero entries appearing outside the rectangle R. In fact, we wonder if such an example is possible.

Since the Frobenius companion matrix is a Fiedler companion matrix, Corollary 5.5 implies that there is no companion matrix that is a superpattern of a Frobenius companion matrix.

Corollary 5.6. *There is no companion matrix that is a superpattern of the Frobenius companion matrix.*

6. A class of non-sparse companion matrices. In this section, we describe a way to construct a $2r \times 2r$ non-sparse companion matrix by starting with any $r \times r$ non-derogatory nilpotent matrix as an initial building block. We first present two helpful lemmas.

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Lemma 6.1. *Suppose N is an $r \times r$ non-derogatory nilpotent matrix such that, in particular, $(N^{r-1})_j \neq \mathbf{0}^T$ for some j with $1 \leq j \leq r$. If $S = \{\mathbf{e}_j^T, N_j, (N^2)_j, \cdots, (N^{r-1})_j\}$, then S is a linearly independent set.*

Proof. Note that $(N^{r-1})_j$ is a left eigenvector of N. Further,

$$S = \{\mathbf{e}_j^T, N_j, (N^2)_j, \cdots, (N^{r-1})_j\} = \{\mathbf{e}_j^T, \mathbf{e}_j^T N, \mathbf{e}_j^T N^2, \cdots, \mathbf{e}_j^T N^{r-1}\}.$$

Hence, S is a cycle of generalized eigenvectors of N of length r and so is linearly independent (see, e.g., [9, p. 488]). We include an argument of this claim for completeness. Suppose $c_1\mathbf{e}_j^T + c_2\mathbf{e}_j^T N + \cdots + c_r\mathbf{e}_j^T N^{r-1} = \mathbf{0}^T$. Then

$$\mathbf{e}_j^T(c_1 I + c_2 N + \cdots + c_r N^{r-1}) = \mathbf{0}^T. \tag{6.14}$$

Multiplying equation (6.14) by N^{r-1} on the right yields $c_1(N^{r-1})_j = \mathbf{0}^T$, and hence, $c_1 = 0$. We can repeat this process by multiplying (6.14) on the right by N^{r-2}, N^{r-3}, and so on, to get $c_2 = c_3 = \cdots = c_r = 0$. This implies that S is linearly independent. ∎

Lemma 6.2. *Let N be an $r \times r$ non-derogatory nilpotent matrix and suppose $1 \leq j \leq r$. The matrix*

$$B = \left[\begin{array}{c|c} N & \mathbf{x} \\ \hline a\mathbf{e}_j^T & b \end{array}\right]$$

has characteristic polynomial $x^{r+1} - bx^r - a$ if and only if $(N^{r-1})_j \neq \mathbf{0}^T$ and $\mathbf{x}$ is the unique solution to

$$\begin{bmatrix} \mathbf{e}_j^T \\ N_j \\ (N^2)_j \\ \vdots \\ (N^{r-2})_j \\ (N^{r-1})_j \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{6.15}$$

Proof. By Lemma 3.6,

$$\mathbf{e}_j^T[\mathrm{adj}(zI - N)] = (N^{r-1})_j + z(N^{r-2})_j + z^2(N^{r-3})_j + \cdots + z^{r-2}N_j + z^{r-1}\mathbf{e}_j^T. \tag{6.16}$$

Thus, cofactor expansion along row $r+1$ gives

$$\begin{aligned} \det(zI - B) &= (z-b)\det(zI - N) - a\mathbf{e}_j^T \,\mathrm{adj}(zI - N)\mathbf{x} \\ &= (z-b)z^r - a\left((N^{r-1})_j + z(N^{r-2})_j + z^2(N^{r-3})_j + \cdots + z^{r-2}N_j + z^{r-1}\mathbf{e}_j^T\right)\mathbf{x} \\ &= z^{r+1} - bz^r - a(N^{r-1})_j\mathbf{x} + az(N^{r-2})_j\mathbf{x} + az^2(N^{r-3})_j\mathbf{x} + \cdots + az^{r-2}N_j\mathbf{x} + az^{r-1}\mathbf{e}_j^T\mathbf{x}. \end{aligned}$$

From this we see that $\det(zI - B) = z^{r+1} - bz^r - a$ if and only if $\mathbf{x}$ is a solution to (6.15) (and hence, $(N^{r-1})_j \neq \mathbf{0}^T$). Moreover, when this is the case, Lemma 6.1 shows that $\mathbf{x}$ is the unique such solution. ∎

If N is an $r \times r$ non-derogatory nilpotent matrix, then $(N^{r-1})_j \neq \mathbf{0}^T$ for some j with $1 \leq j \leq r$. Hence, the following result allows one to construct a companion matrix by starting with *any* non-derogatory nilpotent matrix.

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THEOREM 6.3. *Suppose N is an $r\times r$ non-derogatory nilpotent matrix with $(N^{r-1})_j \neq \mathbf{0}^T$ for some j, $1\le j\le r$. Let*

$$A = \left[\begin{array}{ccc|c|ccc} & N & & \mathbf{x} & & O & \\ \hline & -a_{r+1} & & -a_1 & & & \\ O & -a_{r+2} & O & \vdots & & I & \\ & \vdots & & -a_{r-1} & & & \\ & -a_{2r} & & -a_r & 0 & \cdots & 0 \end{array}\right], \tag{6.17}$$

with a_{2r} in the jth column of A. Then there exists a vector $\mathbf{x}$, with $\mathbf{x}_j = 0$, such that A is a companion matrix. In particular, this $\mathbf{x}$ is the unique solution to equation (6.15).

Proof. We employ the digraph interpretation of the characteristic polynomial described in Lemma 2.1. Based on the principal submatrix

$$\begin{bmatrix} -a_1 & 1 & & & \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ -a_r & & & & \end{bmatrix},$$

each a_i for $1\le i\le r$ appears on a unique simple cycle, the weight of which is a_i. Moreover, every simple cycle disjoint from that one includes only arcs within N. It follows, since N is nilpotent, that for $\ell > i$, the sum of the composite ℓ-cycles that include a_i is zero. Meanwhile, since A has a zero block in the top right corner, any nonzero composite ℓ-cycle including a_{r+m} for some $m\ge 1$, must involve vertex v_{r+1} and must have $\ell \ge m+1$ (the smallest potential cycle being $v_{r+1}\to v_{r+2}\to\cdots\to v_{r+m}\to v_j\to v_{r+1}$). Define

$$B = \left[\begin{array}{c|c} N & \mathbf{x} \\ \hline -a_{r+1}\mathbf{e}_j^T & -a_1 \end{array}\right].$$

If we let C_k be the sum of the composite k-cycles in B involving a_{r+1} when $a_{r+1}\neq 0$, then the sum of the composite ℓ-cycles in A involving a_{r+m} is $\frac{a_{r+m}}{a_{r+1}}C_{\ell-m+1}$. Now, following Lemma 6.2, let $\mathbf{x}$ be the unique vector such that $C_k = 0$ for $2\le k\le r$ and $C_{r+1} = a_{r+1}$. (Note that the first equation in the system of equations corresponding to (6.15) implies $\mathbf{x}_j = 0$.) Then the sum of the composite ℓ-cycles in A involving a_{r+m} with $m+1\le \ell\le r+m-1$ is zero, and the sum is a_{r+m} for $\ell = r+m$. Finally, A is nilpotent if $a_1 = a_2 = \cdots = a_{2r} = 0$, since

$$\left[\begin{array}{c|c|c} N & \mathbf{x} & O \\ \hline O & \multicolumn{2}{c}{J_r} \end{array}\right]$$

is block triangular with nilpotent blocks. So the sum of the composite ℓ-cycles having constant weight (i.e.,

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not involving a_i for any i) is zero. Therefore, the characteristic polynomial of A is $z^{2r}+a_1z^{2r-1}+a_2z^{2r-2}+\cdots+a_{2r-1}z+a_{2r}$ and so A is a companion matrix. ∎

Example 6.4. Taking $N=\begin{bmatrix} 1 & 1 & 1\\ -1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$ gives $(N^2)_1 \neq \mathbf{0}^T$. Hence, taking $j=1$ in Theorem 6.3, and solving (6.15) for $\mathbf{x}$, we obtain the companion matrix

$$A=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0\\ -1 & -1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ -a_4 & 0 & 0 & -a_1 & 1 & 0\\ -a_5 & 0 & 0 & -a_2 & 0 & 1\\ -a_6 & 0 & 0 & -a_3 & 0 & 0\end{bmatrix}.$$

We note that A is not a superpattern of a sparse companion matrix.

Example 6.5. Taking $N=J_r$ with a_{2r} in the first column of A in (6.17) will force $\mathbf{x}$ to be $\mathbf{e}_r$ when solving (6.15). Then, with $n=2r$, Theorem 6.3 gives A equivalent to the sparse companion matrix $C_n\left(\frac{n}{2},\frac{n}{2}\right)$ defined in [4, p. 269].

Example 6.6. Two examples with N full:

$$A=\begin{bmatrix} -3 & 2 & -4 & 0 & 0 & 0\\ \frac12 & -1 & 1 & -1 & 0 & 0\\ 3 & -2 & 4 & -\frac12 & 0 & 0\\ -a_4 & 0 & 0 & -a_1 & 1 & 0\\ -a_5 & 0 & 0 & -a_2 & 0 & 1\\ -a_6 & 0 & 0 & -a_3 & 0 & 0\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix} -3 & 2 & -4 & 2 & 0 & 0\\ \frac12 & -1 & 1 & 0 & 0 & 0\\ 3 & -2 & 4 & -1 & 0 & 0\\ 0 & -a_4 & 0 & -a_1 & 1 & 0\\ 0 & -a_5 & 0 & -a_2 & 0 & 1\\ 0 & -a_6 & 0 & -a_3 & 0 & 0\end{bmatrix}.$$

Observe that if $(N^{r-1})_j \neq \mathbf{0}^T$ for some j, then there is a permutation matrix P such that $((P^TNP)^{r-1})_1 \neq \mathbf{0}^T$. Thus, each matrix A given by Theorem 6.3 is permutationally equivalent to a matrix in which the column of variables including a_{2r} is the first column. In particular, B is equivalent to

$$\begin{bmatrix} -1 & \frac12 & 1 & 0 & 0 & 0\\ 2 & -3 & -4 & 2 & 0 & 0\\ -2 & 3 & 4 & -1 & 0 & 0\\ -a_4 & 0 & 0 & -a_1 & 1 & 0\\ -a_5 & 0 & 0 & -a_2 & 0 & 1\\ -a_6 & 0 & 0 & -a_3 & 0 & 0\end{bmatrix}.$$

Example 6.7. We can generalize Example 6.4 to obtain a class of companion matrices that are not superpatterns of sparse companion matrices. Consider the $r\times r$ matrix

$$N=\left[\begin{array}{cc|c} 1 & 1 & \mathbf{e}_1^T\\ -1 & -1 & \mathbf{0}^T\\ \hline & & \\ O & & J_{r-2}\\ & & \end{array}\right].$$

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We can observe that $(N^k)_1 = \mathbf{e}_{k+1}^T + \mathbf{e}_{k+2}^T$ for $2 \le k < r-1$ and $(N^{r-1})_1 = \mathbf{e}_r^T$. It follows that the unique solution to (6.15) is given by $\mathbf{x}^T = \begin{bmatrix} 0 & (-1)^{r-2} & (-1)^{r-3} & \cdots & -1 & 1 \end{bmatrix}$. The resulting $2r \times 2r$ companion matrix given by Theorem 6.3 is

$$A = \left[\begin{array}{c|c|c|c} \multicolumn{2}{c|}{N} & \mathbf{x} & O \\ \hline -a_{r+1} & & -a_1 & \\ -a_{r+2} & O & \vdots & I_{r-1} \\ \vdots & & -a_{r-1} & \\ -a_{2r} & & -a_r & \mathbf{0}^T \end{array}\right],$$

and A is not a superpattern of a sparse companion matrix. In fact, A does not have a Hamilton cycle.

We saw in Corollary 5.6 that there exists no companion matrix that is a proper superpattern of the Frobenius companion matrix. We make use of this corollary to provide a converse to the companion matrix construction of Theorem 6.3:

Theorem 6.8. *Suppose N is an $r \times r$ matrix and*

$$A = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{N} & \multicolumn{2}{c}{H} \\ \hline -a_{r+1} & & -a_1 & \\ \vdots & O & \vdots & W \\ -a_{2r} & & -a_r & \end{array}\right]$$

is a $2r \times 2r$ companion matrix. Then A has the form described in Theorem 6.3.

Proof. Note that if $a_0 = a_1 = \cdots = a_{2r} = 0$, then by Theorem 3.1 the matrix A is nilpotent and non-derogatory. Thus, by Lemma 3.2, both N and $\left[\begin{array}{c|c} \mathbf{0} & W \end{array}\right]$ are nilpotent, non-derogatory matrices. Further, note that because N is nilpotent (and the only nonzero entries in the bottom left block are variables), the bottom right block must be an $r \times r$ companion matrix. Thus, by Corollary 5.6, we have $W = \left[\begin{array}{c} I \\ \hline \mathbf{0}^T \end{array}\right]$. Finally, we claim $H = \left[\begin{array}{c|c} \mathbf{x} & O \end{array}\right]$. Suppose $H = \begin{bmatrix} \mathbf{x} & \mathbf{h}_2 & \cdots & \mathbf{h}_r \end{bmatrix}$. Applying Lemma 6.2 with B given by the leading principal $(r+1) \times (r+1)$ submatrix of A, we can deduce that $(N^{r-1})_1 \ne \mathbf{0}^T$ and that $\mathbf{e}_1^T \operatorname{adj}(zI - N)\mathbf{x} = 1$ for some unique $\mathbf{x}$. Consider the submatrix

$$Y = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{N} & \mathbf{x} & \mathbf{h}_2 \\ \hline -a_{r+1} & \mathbf{0}^T & -a_1 & 1 \\ -a_{r+2} & \mathbf{0}^T & -a_2 & 0 \end{array}\right].$$

By cofactor expansion, we obtain

$$\det(zI - Y) = z\det(zI - B) + a_2\det(zI - N) + a_{r+2} + (a_{r+2}(z + a_1) - a_2 a_{r+1})\mathbf{e}_1^T \operatorname{adj}(zI - N)\mathbf{h}_2,$$

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with B as in Lemma 6.2. Thus, since A is a companion matrix, it follows that $\mathbf{e}_1^T\,\mathrm{adj}(zI-N)\mathbf{h}_2 = 0$, and hence, by (6.16) and Lemma 6.1, it follows that $\mathbf{h}_2 = \mathbf{0}$. Inductively, we can now show that $\mathbf{h}_k = \mathbf{0}$ for each k with $3 \le k \le r$. In particular, assuming $\mathbf{h}_2 = \cdots = \mathbf{h}_{k-1} = \mathbf{0}$, any nonzero transversals containing a_{r+k+1} and a_k in the $(r+k+1)\times(r+k+1)$ leading principal submatrix of $zI - A$ must include the unit entries in rows $r+1$ to $r+k-1$. It follows that the sum of these transversals is $-a_k a_{r+k+1}\mathbf{e}_1^T\,\mathrm{adj}(zI-N)\mathbf{h}_k$, and hence, $\mathbf{h}_k = \mathbf{0}$. Therefore, $H = \left[\begin{array}{c|c} \mathbf{x} & O \end{array}\right]$. ∎

7. Minimum number of zero entries in a companion matrix. For a given n, an $n\times n$ sparse companion matrix has the most zero entries of all $n \times n$ companion matrices. Since we are exploring non-sparse companion matrices in this paper, it is natural to ask about the least number of zero entries possible in an $n\times n$ companion matrix. As such, let $\sigma(n)$ be the minimum number of zero entries possible in an $n\times n$ companion matrix. By Theorem 3.11, we know that $\sigma(n) \ge n-1$. In this section, we demonstrate that $\frac{3}{2}n - 2 \le \sigma(n) \le \frac{5}{2}n - 4$ when n is even. The construction that provides these bounds depends on n being even. Without further exploration, it is not clear if the bounds differ significantly for odd n.

Theorem 7.1. *If n is even, then $\sigma(n) \ge \frac{3}{2}n - 2$.*

Proof. Let $\mathrm{R} = \mathbb{R}[a_3,\dots,a_n]$ and let A be an $n\times n$ companion matrix with n even. As noted in Section 4, the only companion matrix up to equivalence is the Frobenius companion matrix if $n = 2$, and hence, $\sigma(2) = 1$. Likewise, Theorem 4.2 shows that $\sigma(4) = 6$. Suppose $n \ge 6$. By Theorem 3.11, we may assume

$$A = \left[\begin{array}{c|cc} N & \mathbf{x} & \mathbf{0} \\ \hline \mathbf{u}^T & -a_1 & 1 \\ \mathbf{v}^T & -a_2 & 0 \end{array}\right] \tag{7.18}$$

for some non-derogatory nilpotent matrix N over R, and some $\mathbf{x},\mathbf{u},\mathbf{v} \in \mathrm{R}^{n-2}$ with $N^{n-3}\mathbf{x} \ne \mathbf{0}$.

First suppose that $\mathbf{v}_j = a_3$ for some j. For each $1 \le k \le n-3$, we have from (3.5) that $\mathbf{v}^T N^k \mathbf{x} + \mathbf{u}^T N^{k+1}\mathbf{x} = -a_{k+3}$, so that, since a_3 appears in exactly one entry of A, we must have $(N^k)_j\mathbf{x} = 0$. Hence, by Lemma 3.5, $N_j = \mathbf{0}$. Thus, A has at least $2n-3$ zero entries.

Now suppose a_3 does not appear in $\mathbf{v}$. Then, by Lemma 3.10, the entry containing a_3 corresponds to some zero entry of A beyond those shown in (7.18). We now show that for k with $4 \le k \le n$, this property holds for a_k as well. If a_k does not appear in $\mathbf{v}$, then this again follows from Lemma 3.10. On the other hand, if $\mathbf{v}_j = a_k$ for some j, then, since we have from (3.5) that $\mathbf{v}^T I\mathbf{x} + \mathbf{u}^T N\mathbf{x} = -a_3$, we can deduce that $\mathbf{x}_j = 0$.

We now have that, for each i with $3 \le i \le n$, the entry containing a_i is accompanied by some corresponding zero entry of A in addition to those shown in (7.18). Note also that the only case in which a_{i_1} and a_{i_2} correspond to the same additional zero is that in which $\mathbf{u}_j = a_{i_1}$ and $\mathbf{v}_j = a_{i_2}$ for some j, since both imply $\mathbf{x}_j = 0$. In particular, at most two variables may correspond to the same additional zero, giving at least $\frac{1}{2}(n-2)$ additional zero entries. Thus, $\sigma(n) \ge (n-1) + \frac{1}{2}(n-2) = \frac{3}{2}n - 2$. ∎

We obtain an upper bound on $\sigma(n)$ for even n by constructing a particular $n\times n$ companion matrix, illustrated with Examples 7.6 and 7.7. Let r be even with $r \ge 4$. We start with a matrix $S = \left[\begin{array}{cccc} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_r \end{array}\right]$

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defined by

$$
\begin{aligned}
\mathbf{s}_1 &= -\mathbf{e}_1 - \mathbf{e}_r,\\
\mathbf{s}_{2k} &= \mathbf{e}_{2k} + 2\mathbf{e}_r \qquad \text{for } 1 \le k \le \frac{r-4}{2},\\
\mathbf{s}_{2k+1} &= -\sum_{j=1}^{2k+1} \mathbf{e}_j \qquad \text{for } 1 \le k \le \frac{r-4}{2},\\
\mathbf{s}_{r-2} &= \mathbf{e}_{r-2} - 2\mathbf{e}_r,\\
\mathbf{s}_{r-1} &= -\sum_{j=1}^{r} \mathbf{e}_j,\\
\mathbf{s}_r &= \mathbf{e}_1.
\end{aligned} \tag{7.19}
$$

Example 7.2. When $r = 8$,

$$
S = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & 0 & -1 & 1\\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0\\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ -1 & 2 & 0 & 2 & 0 & -2 & -1 & 0 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} 0 & 2 & -2 & 2 & -2 & -2 & 3 & -1\\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 1 & 2 & -3 & 2 & -2 & -2 & 3 & -1 \end{bmatrix}.
$$

In general, the matrix S is invertible since it has exactly one nonzero transversal. Let $S^{-1} = \begin{bmatrix} \mathbf{t}_1\\ \mathbf{t}_2\\ \vdots\\ \mathbf{t}_r \end{bmatrix}$. Then one can verify that

$$
\begin{aligned}
\mathbf{t}_1^T &= -2\mathbf{e}_{r-2} + 3\mathbf{e}_{r-1} - \mathbf{e}_r + 2\sum_{j=1}^{\frac{r-4}{2}} (\mathbf{e}_{2j} - \mathbf{e}_{2j+1}),\\
\mathbf{t}_{2k}^T &= \mathbf{e}_{2k} - \mathbf{e}_{2k+1} \qquad \text{for } 1 \le k \le \frac{r-2}{2},\\
\mathbf{t}_{2k+1}^T &= \mathbf{e}_{2k+3} - \mathbf{e}_{2k+1} \qquad \text{for } 1 \le k \le \frac{r-4}{2},\\
\mathbf{t}_{r-1}^T &= -\mathbf{e}_{r-1},\\
\mathbf{t}_r^T &= \mathbf{t}_1^T + \mathbf{e}_1 - \mathbf{e}_3.
\end{aligned}
$$

With $J = \begin{bmatrix} \mathbf{0} & \mathbf{e}_1 & \cdots & \mathbf{e}_{r-1} \end{bmatrix}$ and S as in (7.19), define N by

$$
N = SJS^{-1}. \tag{7.20}
$$

Note that N is nilpotent and non-derogatory. Now let

$$
\mathbf{x}^T = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & -1 \end{bmatrix}.
$$

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Then, for every $k \geq 0$,

$$N^k \mathbf{x} = \left(SJ^k\right)\left(S^{-1}\mathbf{x}\right),$$

and one can verify that

$$S^{-1}\mathbf{x} = \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 & 0 & -1 \end{bmatrix}^T.$$

Since $J^k = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_1 & \cdots & \mathbf{e}_{r-k} \end{bmatrix}$, we know that $SJ^k = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{s}_1 & \cdots & \mathbf{s}_{r-k} \end{bmatrix}$. Hence,

$$\begin{aligned} N^k\mathbf{x} &= \mathbf{s}_{r-k-2} - \mathbf{s}_{r-k} \qquad \text{for } 1 \leq k \leq r-3, \\ N^{r-2}\mathbf{x} &= -\mathbf{s}_2, \\ N^{r-1}\mathbf{x} &= -\mathbf{s}_1. \end{aligned} \tag{7.21}$$

THEOREM 7.3. *Let $r \geq 4$ be even. If N is the matrix given by (7.20) and $\mathbf{x}^T = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & -1 \end{bmatrix}$, then*

$$A = \left[\begin{array}{ccccc|cc} & & N & & & \mathbf{x} & \mathbf{0} \\ \hline -a_{r+1} & 0 & -a_{r-1} & 0 & \cdots & -a_1 & 1 \\ -a_{r+2} & 0 & -a_r & 0 & \cdots & -a_2 & 0 \end{array}\right]$$

is a companion matrix.

Proof. We first note that by (7.21), for odd ℓ with $1 \leq \ell < r$,

$$\begin{bmatrix} \mathbf{e}_\ell^T \\ N_\ell \\ (N^2)_\ell \\ \vdots \\ (N^{r-3})_\ell \\ (N^{r-2})_\ell \\ (N^{r-1})_\ell \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ S_{\ell,r-3} - S_{\ell,r-1} \\ S_{\ell,r-4} - S_{\ell,r-2} \\ \vdots \\ S_{\ell,1} - S_{\ell,3} \\ -S_{\ell,2} \\ -S_{\ell,1} \end{bmatrix} = \mathbf{e}_{r+1-\ell}.$$

Thus, for odd ℓ with $1 \leq \ell < r$,

$$N_\ell^k \mathbf{x} \neq 0 \quad \text{if and only if} \quad k = r - \ell, \tag{7.22}$$

and $N_\ell^{r-\ell}\mathbf{x} = 1$.

Note that each nonzero transversal of $zI - A$ contains at most one variable. Suppose k is odd and $3 \leq k \leq r+1$. The term of $\det(zI - A)$ containing a_k will be $za_k\mathbf{e}_{r+2-k}^T \operatorname{adj}(zI - N)\mathbf{x}$. However, by Lemma 3.6,

$$za_k\mathbf{e}_{r+2-k}^T \operatorname{adj}(zI-N)\mathbf{x} = za_k\mathbf{e}_{r+2-k}^T \left[(N^{r-1})\mathbf{x} + z(N^{r-2})\mathbf{x} + \cdots + z^{r-2}N\mathbf{x} + z^{r-1}\mathbf{x}\right].$$

Using (7.22), we see that

$$\begin{aligned} za_k\mathbf{e}_{r+2-k}^T \operatorname{adj}(zI-N)\mathbf{x} &= za_k[z^{r+1-k}(N^{k-2})_{r+2-k}\mathbf{x}] \\ &= a_k z^{r+2-k}. \end{aligned}$$

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Likewise, the term containing a_{k+1} in $\det(zI-A)$ will be $a_{k+1}\mathbf{e}_{r+2-k}^T \operatorname{adj}(zI-N)\mathbf{x}$ and so will be $a_{k+1}z^{r+1-k}$. The remaining nonzero transversals of $zI-A$ are

$$(a_2 + z(z+a_1))\det(zI-N) = z^{r+2} + a_1 z^{r+1} + a_2 z^r$$

since N is nilpotent. Therefore,

$$\begin{aligned}\det(zI-A) &= z^{r+2} + a_1 z^{r+1} + a_2 z^r + \sum_{\substack{k \text{ odd} \\ k=3}}^{r+1} (a_k z^{r+2-k} + a_{k+1} z^{r+1-k}) \\ &= z^{r+2} + a_1 z^{r+1} + a_2 z^r + a_3 z^{r-1} + a_4 z^{r-2} + \cdots + a_{r+1} z + a_{r+2}\end{aligned}$$

and so A is a companion matrix. $\square$

The next lemma shows that the matrix N defined in line (7.20) is a *full matrix*, that is, it has no zero entries. Moreover, the proof of this lemma gives the entries of N explicitly (see Examples 7.6 and 7.7).

Lemma 7.4. *For $r \geq 4$, the matrix N defined in (7.20) is a full matrix.*

Proof. Let

$$SJ = \begin{bmatrix} \mathbf{0} & \mathbf{s}_1 & \cdots & \mathbf{s}_{r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

Then

$$N = SJS^{-1} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix},$$

with

$$\mathbf{u}_1 = \begin{bmatrix} 0 & -1 & 0 & -1 & \cdots & 0 & -1 \end{bmatrix} = -\sum_{j=1}^{\frac{r}{2}} \mathbf{e}_{2j}^T,$$

$$\mathbf{u}_{2k} = [\,\underbrace{0 \ \cdots \ 0}_{2k} \ 1 \ -1 \ 0 \ -1 \ \cdots \ 0 \ -1\,] = \mathbf{e}_{2k+1}^T - \sum_{j=1}^{\frac{r-2k}{2}} \mathbf{e}_{2k+2j}^T, \quad 1 \leq k \leq \frac{r-2}{2},$$

$$\mathbf{u}_{2k+1} = [\,\underbrace{0 \ \cdots \ 0}_{2k} \ 0 \ -1 \ 0 \ -1 \ \cdots \ 0 \ -1\,] = -\sum_{j=1}^{\frac{r-2k}{2}} \mathbf{e}_{2k+2j}^T, \quad 1 \leq k \leq \frac{r-2}{2},$$

$$\mathbf{u}_r = \begin{bmatrix} 0 & -1 & 2 & 0 & 2 & 0 & \cdots & 2 & 0 & -2 & -1 \end{bmatrix} = -\mathbf{e}_2^T - 2\mathbf{e}_{r-1}^T - \mathbf{e}_r^T + 2\sum_{j=1}^{\frac{r-4}{2}} \mathbf{e}_{2j+1}^T,$$

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and

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{e}_r,\\
\mathbf{v}_3 &= -2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - 3\mathbf{e}_r \quad (\text{if } r > 4),\\
\mathbf{v}_{2k} &= 2\mathbf{e}_1 + \mathbf{e}_{2k} + 2\mathbf{e}_r, \quad 1 \le k \le \frac{r-4}{2},\\
\mathbf{v}_{2k+1} &= -2\mathbf{e}_1 + \mathbf{e}_{2k-1} - \mathbf{e}_{2k} - \mathbf{e}_{2k+1} - 2\mathbf{e}_r, \quad 2 \le k \le \frac{r-4}{2},\\
\mathbf{v}_{r-2} &= -2\mathbf{e}_1 + \mathbf{e}_{r-2} - 2\mathbf{e}_r,\\
\mathbf{v}_{r-1} &= 3\mathbf{e}_1 + \mathbf{e}_{r-3} - \mathbf{e}_{r-2} - \mathbf{e}_{r-1} + 3\mathbf{e}_r,\\
\mathbf{v}_r &= -\mathbf{e}_1 - \mathbf{e}_r.
\end{aligned}$$

We can now calculate the entries of N. Throughout the rest, let $1 \le i \le \frac{r-2}{2}$.

$$\mathbf{u}_1\mathbf{v}_1 = -1 \qquad \mathbf{u}_{2i}\mathbf{v}_1 = -1 \qquad \mathbf{u}_{2i+1}\mathbf{v}_1 = -1 \qquad \mathbf{u}_r\mathbf{v}_1 = -1$$

$$\mathbf{u}_1\mathbf{v}_3 = 4 \qquad \mathbf{u}_{2i}\mathbf{v}_3 = \begin{cases} 2 & \text{if } i = 1\\ 3 & \text{if } i > 1 \end{cases} \qquad \mathbf{u}_{2i+1}\mathbf{v}_3 = 3 \qquad \mathbf{u}_r\mathbf{v}_3 = 2.$$

For $1 \le k \le \frac{r-4}{2}$, $\mathbf{u}_1\mathbf{v}_{2k} = -3$,

$$\mathbf{u}_{2i}\mathbf{v}_{2k} = \mathbf{u}_{2i+1}\mathbf{v}_{2k} = \begin{cases} -3 & \text{if } i \le k-1\\ -2 & \text{if } i > k-1 \end{cases} \qquad \mathbf{u}_r\mathbf{v}_{2k} = \begin{cases} -3 & \text{if } k = 1\\ -2 & \text{if } k > 1. \end{cases}$$

For $2 \le k \le \frac{r-4}{2}$, $\mathbf{u}_1\mathbf{v}_{2k+1} = 3$,

$$\mathbf{u}_{2i}\mathbf{v}_{2k+1} = \begin{cases} 3 & \text{if } i < k-1\\ 4 & \text{if } i = k-1\\ 1 & \text{if } i = k\\ 2 & \text{if } i > k \end{cases} \qquad \mathbf{u}_{2i+1}\mathbf{v}_{2k+1} = \begin{cases} 3 & \text{if } i \le k-1\\ 2 & \text{if } i > k-1 \end{cases} \qquad \mathbf{u}_r\mathbf{v}_{2k+1} = 2.$$

And finally,

$$\begin{aligned}
\mathbf{u}_1\mathbf{v}_{r-2} &= 1 & \mathbf{u}_r\mathbf{v}_{r-2} &= 2\\
\mathbf{u}_1\mathbf{v}_{r-1} &= -2 & \mathbf{u}_r\mathbf{v}_{r-1} &= 1\\
\mathbf{u}_1\mathbf{v}_r &= 1 & \mathbf{u}_r\mathbf{v}_r &= 1
\end{aligned}$$

$$\mathbf{u}_{2i}\mathbf{v}_{r-2} = \begin{cases} 1 & \text{if } 2i < r-2\\ 2 & \text{if } 2i = r-2 \end{cases} \qquad \mathbf{u}_{2i+1}\mathbf{v}_{r-2} = \begin{cases} 1 & \text{if } 2i+1 \le r-3\\ 2 & \text{if } 2i+1 = r-1 \end{cases}$$

$$\mathbf{u}_{2i}\mathbf{v}_{r-1} = \begin{cases} -2 & \text{if } 2i < r-4\\ -1 & \text{if } 2i = r-4\\ -4 & \text{if } 2i = r-2 \end{cases} \qquad \mathbf{u}_{2i+1}\mathbf{v}_{r-1} = \begin{cases} -2 & \text{if } 2i+1 \le r-3\\ -3 & \text{if } 2i+1 = r-1 \end{cases}$$

$$\mathbf{u}_{2i}\mathbf{v}_r = 1 \qquad \mathbf{u}_{2i+1}\mathbf{v}_r = 1.$$

Thus, N is a full matrix with no zero entries. ∎

By Lemma 7.4, the matrix A of Theorem 7.3 has exactly $3n-7$ zero entries, giving an upper bound on $\sigma(n)$ when n is even:

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THEOREM 7.5. *If $n \geq 6$ is even, then $\sigma(n) \leq 3n - 7$.*

Next we provide two examples illustrating the companion matrices given by Theorem 7.3 for $n = 8$ and $n = 10$. In particular, these matrices demonstrate that $\sigma(n) \leq 3n - 7$ for those values of n.

EXAMPLE 7.6. The 8×8 companion matrix

$$A = \left[\begin{array}{cccccc|cc}
-1 & -3 & 4 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 2 & 1 & -1 & 1 & 0 & 0 \\
-1 & -2 & 3 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 3 & 2 & -4 & 1 & 1 & 0 \\
-1 & -2 & 3 & 2 & -3 & 1 & 0 & 0 \\
-1 & -3 & 2 & 2 & 1 & 1 & -1 & 0 \\
\hline
-a_7 & 0 & -a_5 & 0 & -a_3 & 0 & -a_1 & 1 \\
-a_8 & 0 & -a_6 & 0 & -a_4 & 0 & -a_2 & 0
\end{array}\right]$$

is obtained from the construction of Theorem 7.3, and shows that $\sigma(8) \leq 17$.

EXAMPLE 7.7. The 10×10 companion matrix

$$A = \left[\begin{array}{cccccccc|cc}
-1 & -3 & 4 & -3 & 3 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 2 & -3 & 4 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 3 & -3 & 3 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 3 & -2 & 1 & 1 & -1 & 1 & 0 & 0 \\
-1 & -2 & 3 & -2 & 2 & 1 & -2 & 1 & 0 & 0 \\
-1 & -2 & 3 & -2 & 2 & 2 & -4 & 1 & 1 & 0 \\
-1 & -2 & 3 & -2 & 2 & 2 & -3 & 1 & 0 & 0 \\
-1 & -3 & 2 & -2 & 2 & 2 & 1 & 1 & -1 & 0 \\
\hline
-a_9 & 0 & -a_7 & 0 & -a_5 & 0 & -a_3 & 0 & -a_1 & 1 \\
-a_{10} & 0 & -a_8 & 0 & -a_6 & 0 & -a_4 & 0 & -a_2 & 0
\end{array}\right]$$

is obtained from the construction of Theorem 7.3, and shows that $\sigma(10) \leq 23$.

THEOREM 7.8. *If n is even, then $\frac{3}{2}n - 2 \leq \sigma(n) \leq \frac{5}{2}n - 4$.*

Proof. The stated lower bound is Theorem 7.1. Note that $\sigma(2) = 1$ and $\sigma(4) = 6$ satisfy the given upper bound. To establish the claimed upper bound, we need to improve upon the construction given by Theorem 7.3 when $n \geq 6$. To this end, let A be an $n \times n$ companion matrix as given by Theorem 7.3, for some fixed even $n \geq 6$.

Observe that except for entry $N_{r,r-1}$, the matrix N defined in (7.20) has a full striped sign pattern, as can been seen in the proof of Lemma 7.4. In particular, every row other than row r has the same sign pattern. Further, the sign of column $r - 2$ is opposite that of column $2i$ for $1 \leq i \leq \frac{r-4}{2}$. This means that if we add row $r - 2$ of A to row $2i$ for $1 \leq i \leq \frac{r-4}{2}$, and perform the corresponding column operations, we obtain a similar companion matrix, still with only nonzero entries in the upper $r \times r$ submatrix, but with more nonzero entries in column $r + 1$. In particular, column $r + 1$ will have additional nonzero entries in row $2i$ for $1 \leq i \leq \frac{r-4}{2} = \frac{n-6}{2}$. By Lemma 7.4, A as constructed has $3n - 7$ zero entries. Hence, the new matrix shows that $\sigma(n) \leq (3n - 7) - \frac{n-6}{2} = \frac{5}{2}n - 4$. ∎

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EXAMPLE 7.9. Applied to the 10×10 matrix given in Example 7.7, the row operations indicated in the proof of Theorem 7.8 give the companion matrix

$$\left[\begin{array}{cccccccc|cc} -1 & -3 & 4 & -3 & 3 & 1 & -2 & 1 & 0 & 0 \\ -2 & -4 & 5 & -5 & 6 & 12 & -6 & 2 & 1 & 0 \\ -1 & -2 & 3 & -3 & 3 & 6 & -2 & 1 & 0 & 0 \\ -2 & -4 & 6 & -4 & 3 & 11 & -5 & 2 & 1 & 0 \\ -1 & -2 & 3 & -2 & 2 & 5 & -2 & 1 & 0 & 0 \\ -1 & -2 & 3 & -2 & 2 & 6 & -4 & 1 & 1 & 0 \\ -1 & -2 & 3 & -2 & 2 & 6 & -3 & 1 & 0 & 0 \\ -1 & -3 & 2 & -2 & 2 & 7 & 1 & 1 & -1 & 0 \\ \hline -a_9 & 0 & -a_7 & 0 & -a_5 & 0 & -a_3 & 0 & -a_1 & 1 \\ -a_{10} & 0 & -a_8 & 0 & -a_6 & 0 & -a_4 & 0 & -a_2 & 0 \end{array}\right]$$

with $\frac{5}{2}(10) - 4 = 21$ zero entries, showing that $\sigma(10) \le 21$.

8. Concluding comments. In this paper, we have described properties of companion matrices and developed constructions of new companion matrices. While the sparse companion matrices are characterized in [4], we do not yet have a characterization for non-sparse companion matrices. Every companion matrix that we have observed is equivalent to a matrix having a submatrix R that contains all of the variable entries, in which the lower left corner entry of R is $-a_n$, the upper right corner entry of R is $-a_1$, and in general, $-a_k$ appears on the kth diagonal of R. This description is known to apply to all sparse companion matrices, as noted in Theorem 1.2. It remains an open question, however, as to whether this description applies to all non-sparse companion matrices as well.

In Theorem 5.4, we showed that companion matrices that are proper superpatterns of Fiedler companion matrices must have some additional nonzero entries in the submatrix corresponding to the R of the Fiedler matrix. Every such superpattern we have found has had its additional nonzero entries *only* inside of this submatrix. We wonder if, in producing a companion matrix that is a superpattern of a Fiedler companion matrix, the extra nonzero entries are always restricted to the submatrix corresponding to the R block of the Fiedler companion matrix.

In Section 7, we developed some bounds for the minimum number of zero entries in an $n \times n$ companion matrix. More work could be done to improve the bounds, and to investigate bounds in the case that n is odd.

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