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A NOTE ON LINEAR PRESERVERS OF SEMIPOSITIVE AND MINIMALLY SEMIPOSITIVE MATRICES*

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Abstract. Semipositive matrices (matrices that map at least one nonnegative vector to a positive vector) and minimally semipositive matrices (semipositive matrices whose no column-deleted submatrix is semipositive) are well studied in matrix theory. In this short note, the structure of linear maps which preserve the set of all semipositive/minimally semipositive matrices is studied. An open problem is solved, and some ambiguities in the article [J. Dorsey, T. Gannon, N. Jacobson, C.R. Johnson and M. Turnansky. Linear preservers of semi-positive matrices. *Linear and Multilinear Algebra*, 64:1853–1862, 2016.] are clarified.

Key words. Linear preserver, Semipositive matrix, Minimally semipositive matrix, Monomial matrix.

AMS subject classifications. 15A86, 15B48.

1. Introductory remarks. Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the real numbers. For an $m \times n$ matrix $A = (a_{ij})$; $A \geq 0$ means $a_{ij} \geq 0$. Similarly, $A > 0$ means all the entries of A are positive. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be *semipositive*, if there exists a vector $x \geq 0$ such that $Ax > 0$. By a simple perturbation argument, it follows that A is semipositive if, and only if, there exists a vector $x \in \mathbb{R}^n$ with $x > 0$ such that $Ax > 0$ [5, Lemma 2.1]. Such a vector x is called a *semipositivity vector* of A. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be *minimally semipositive* if it is semipositive and no column-deleted submatrix of A is semipositive. It is known that a square matrix A is minimally semipositive if, and only if, A^{-1} exists and $A^{-1} \geq 0$ [5, Theorem 3.4]. More generally, an $m \times n$ $(m \geq n)$ matrix A is minimally semipositive if, and only if, A is semipositive and has a nonnegative left inverse (see [5, Theorem 3.6] and [7, Theorem 2.3]). A matrix $A \in \mathbb{R}^{n\times n}$ is called *row positive* if $A \geq 0$ and each row of A contains a non-zero entry. A matrix $A \in \mathbb{R}^{n\times n}$ is called *inverse nonnegative* if A^{-1} exists and $A^{-1} \geq 0$. A square matrix A is called *monomial*, if $A \geq 0$ and every column and every row of A contains exactly one non-zero entry. Let us also recall a result that characterizes monomial matrices. An $n \times n$ matrix A is monomial if, and only if, A is row positive and inverse nonnegative [3, Lemma 2.6].

A linear operator $L : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ is said to be an *into linear preserver* of some set S of matrices, if $L(S) \subseteq S$. A linear operator $L : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ is said to be an *onto linear preserver* of S, if $L(S) = S$. There is a vast literature on linear preserver problems. For more details, we refer to [4, 6].

In [3], the authors considered linear preserver problems for the set of all semipositive and minimally semipositive matrices. Consider the map $L : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ defined by $L(A) = XAY$ for some $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{n\times n}$.

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1. Then L is an into linear preserver of semipositive matrices if, and only if, either X is row positive and Y is inverse nonnegative, or $-X$ is row positive and $-Y$ is inverse nonnegative [3, Theorem 2.4]. L is an onto linear preserver of semipositive matrices if, and only if, either X and Y are monomial, or $-X$ and $-Y$ are monomial [3, Corollary 2.7].
2. If $m > n$, then L is an into linear preserver of minimally semipositive matrices if, and only if, either X is monomial and Y is inverse nonnegative, or $-X$ is monomial and $-Y$ is inverse nonnegative. L is an onto linear preserver of minimally semipositive matrices if, and only if, either X and Y are monomial, or $-X$ and $-Y$ are monomial [3, Theorem 2.11].

In this short note, we study the structure of linear maps which preserve the set of all semipositive/minimally semipositive matrices. We solve an open problem and clarify some ambiguities in the work [3]. First, we show that if a vector $v \in \mathbb{R}^n$ contains both positive and negative entries and $0 \neq w \in \mathbb{R}^n$, then there exists a nonnegative invertible matrix B such that $Bv = w$ (Theorem 2.2). A similar type of result for $v \geq 0$ and $w > 0$ is proved in Theorem 2.4. In view of the statement (2), a natural question arises: can we obtain a necessary and sufficient condition for L to be an into linear preserver of minimally semipositive matrices, when $m = n$. We answer this question in Theorem 2.10. In Theorem 2.13, we give a necessary and sufficient condition for L to be an onto linear preserver of minimally semipositive matrices, when $m = n$. In the process, we point to an error in [3] and provide a correct proof.

2. Main results. The first result characterizes onto linear preservers of a set S of matrices. In [3], it is mentioned that this result is presented in [2]. For the sake of completeness, we include a proof here.

Lemma 2.1. *If S contains a basis for $\mathbb{R}^{m\times n}$, then L is an onto linear preserver of S if, and only if, L and L^{-1} are into linear preservers on S.*

Proof. If L is an onto linear preserver on S, then $L(S)$ contains a basis for $\mathbb{R}^{m\times n}$. Thus, L is invertible, $L(S) \subseteq S$ and $L^{-1}(S) \subseteq S$. Conversely, if $L(S) \subseteq S$ and $L^{-1}(S) \subseteq S$, then it is clear that $L(S) = S$. ∎

Theorem 2.2. *Let $v, w \in \mathbb{R}^n$ such that v contains both positive and negative entries and $w \neq 0$. Then there exists a nonnegative invertible matrix B such that $Bv = w$.*

Proof. Without loss of generality assume that, $v_1 > 0, v_n < 0$ and $w_1 \neq 0$. Now, we construct the matrix B as follows: Let b^i denote the i^{th} row of B.

Step 1: The first row of the matrix B is constructed as follows:

(a) If $w_1 > 0$, then $b^1 := \left(\frac{w_1}{v_1}, 0, \ldots, 0\right)$.
(b) If $w_1 < 0$, then $b^1 := \left(0, \ldots, 0, \frac{w_1}{v_n}\right)$.

From the construction, it is clear that the first row of B is nonnegative.

Step 2: Now, we construct the i^{th} row for $1 < i < n$ as follows:

(a) If $w_i > 0$ and $v_i > 0$, then $b^i := \left(\frac{w_i}{2v_1}, 0, \ldots, \frac{w_i}{2v_i}, \ldots, 0\right)$.
(b) If $w_i > 0$ and $v_i < 0$, then $b^i := \left(\frac{w_i+1}{v_1}, 0, \ldots, \frac{-1}{v_i}, \ldots, 0\right)$.
(c) If $w_i > 0$ and $v_i = 0$, then $b^i := \left(\frac{w_i}{v_1}, 0, \ldots, 1, \ldots, 0\right)$.

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(d) If $w_i < 0$ and $v_i > 0$, then $b^i := \left(0, \ldots, \frac{1}{v_i}, \ldots, \frac{w_i - 1}{v_n}\right)$.
(e) If $w_i < 0$ and $v_i < 0$, then $b^i := \left(0, \ldots, \frac{w_i}{2v_i}, \ldots, \frac{w_i}{2v_n}\right)$.
(f) If $w_i < 0$ and $v_i = 0$, then $b^i := \left(0, \ldots, 1, \ldots, \frac{w_i}{v_n}\right)$.
(g) If $w_i = 0$ and $v_i > 0$, then $b^i := \left(0, \ldots, \frac{1}{v_i}, \ldots, \frac{-1}{v_n}\right)$.
(h) If $w_i = 0$ and $v_i < 0$, then $b^i := \left(\frac{1}{v_1}, 0, \ldots, \frac{-1}{v_i}, \ldots, 0\right)$.
(i) If $w_i = 0$ and $v_i = 0$, then $b^i := (0, \ldots, 1, \ldots, 0)$.

In the above construction, for $2 \leq i \leq n-1$, the i^{th} entry is non-zero only in the i^{th} row. This establishes the linear independence of the rows from 1 to $n-1$.

Step 3: Construct the n^{th} row of the matrix B as follows:

(a) If $w_n > 0$ and $w_1 < 0$, then $b^n := \left(\frac{w_n}{v_1}, 0, \ldots, 0\right)$.
(b) If $w_n > 0$ and $w_1 > 0$, then $b^n := \left(\frac{w_n + 1}{v_1}, 0, \ldots, \frac{-1}{v_n}\right)$.
(c) If $w_n < 0$ and $w_1 < 0$, then $b^n := \left(\frac{1}{v_1}, 0, \ldots, \frac{w_n - 1}{v_n}\right)$.
(d) If $w_n < 0$ and $w_1 > 0$, then $b^n := \left(0, \ldots, \frac{w_n}{v_n}\right)$.
(e) If $w_n = 0$, then $b^n := \left(\frac{1}{v_1}, 0, \ldots, \frac{-1}{v_n}\right)$.

From the construction, it is clear that the rows of the matrix B are linear independent, nonnegative and $Bv = w$. ☐

Let us illustrate the construction of the previous result, by an example.

EXAMPLE 2.3. Let $v = (1, 0, -5, -1)^T$ and $w = (3, 2, -10, 0)^T$. Then v contains both positive and negative entries and $w \neq 0$. Since $v_1 = 1 > 0$ and $w_1 = 3 > 0$, the first row of B is $(3, 0, 0, 0)$. Now $v_2 = 0$ and $w_2 = 2 > 0$, so the second row of B is $(2, 1, 0, 0)$. Again $v_3 = -5 < 0$ and $w_3 = -10 < 0$, so the third row of B is $(0, 0, 1, 5)$. Finally $v_4 = -1 < 0$ and $w_4 = 0$, so the last row of B is $(1, 0, 0, 1)$. The matrix

$$B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is nonnegative, invertible and $Bv = w$.

Let e^i denote the vector in $\mathbb{R}^n$ whose i^{th} entry is 1 and all other entries are zero.

THEOREM 2.4. *Let $v, w \in \mathbb{R}^n$ be non-zero vectors such that $v \geq 0$ and $w > 0$. Then there exists a nonnegative invertible matrix B such that $Bv = w$.*

Proof. Without loss of generality, assume that the first k entries of the vector v are positive. Now, we

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construct the matrix B as follows:

$$\begin{aligned}
Be^1 &= \left(\frac{w_1}{v_1}, \frac{w_2}{2v_1}, \dots, \frac{w_k}{2v_1}, \frac{w_{k+1}}{v_1}, \frac{w_{k+2}}{v_1}, \dots, \frac{w_n}{v_1}\right)^T \\
Be^2 &= \frac{w_2}{2v_2} e^2 \\
&\vdots \\
Be^k &= \frac{w_k}{2v_k} e^k \\
Be^{k+1} &= e^{k+1} \\
&\vdots \\
Be^n &= e^n.
\end{aligned}$$

Thus, B is a nonnegative lower triangular matrix whose diagonal entries are non-zero. So, B is invertible and $Bv = w$. ∎

Here is an example to illustrate Theorem 2.4.

Example 2.5. Let $v = (v_1, v_2, v_3, v_4)^T = (1, 3, 2, 0)^T \geq 0$ and $w = (1, 6, 4, 2)^T > 0$. Since v_1, v_2 and v_3 are positive, the first column of B is $(1, 3, 2, 2)^T$, the second column of B is $(0, 1, 0, 0)^T$, the third column of B is $(0, 0, 1, 0)^T$ and the last column of B is $(0, 0, 0, 1)^T$. Thus, B is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix B is nonnegative, invertible and one may verify that $Bv = w$.

Using Theorem 2.2 and Theorem 2.4, we present an alternative and simpler proof of Lemma 2.10 of [3], in the next result.

Corollary 2.6. *Let $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ be non-zero vectors with $n > m$. If v contains both positive and negative entries, then there exists a nonnegative full row rank matrix $B \in \mathbb{R}^{m\times n}$ such that $Bv = w$. The same holds if $v \geq 0$ and $w > 0$.*

Proof. Let $0 \neq x \in \mathbb{R}^{n-m}$. Then, the vector $z := (w^T, x^T)^T \in \mathbb{R}^n$ and $z \neq 0$. By Theorem 2.2, there exists a nonnegative invertible matrix F such that $Fv = z$. Let $F = (B^T, C^T)^T$, where $B \in \mathbb{R}^{m\times n}$. Then $B \geq 0$ and $Bv = w$. Since F is invertible, B has full row rank. The second part can be proved in a similar way, using Theorem 2.4. ∎

The next example demonstrates Corollary 2.6.

Example 2.7. Let $v = (1, 2, 0, -2)^T$ and $w = (1, 4, -6)^T$. The matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is nonnegative, has full row rank and it may be verified that $Bv = w$.

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Next, we consider linear preservers of minimally semipositive matrices of the form $L(A) = XAY$, for some fixed $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{n\times n}$. In [3, Theorem 2.11], the authors claim that, if $m > n$, then L is an into linear preserver of minimally semipositive matrices if, and only if, either X is monomial and Y is inverse nonnegative, or $-X$ is monomial and $-Y$ is inverse nonnegative. However, the converse in the above statement is not true and is illustrated next.

Example 2.8. Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = [1]$. Since 2×1 minimally semipositive matrices are only positive columns, it follows that $L(A) = XAY$ is an into linear preserver of minimally semipositive matrices. However, X is not monomial.

Next, we prove a property of a matrix X which is neither inverse nonnegative nor inverse nonpositive. This result is useful in studying properties of linear preservers of minimally semipositive matrices of the form $L(A) = XAY$.

Lemma 2.9. *Let $X \in \mathbb{R}^{n\times n}$ be invertible. Suppose that neither X nor $-X$ is inverse nonnegative. Then there exists $v \in \mathbb{R}^n$ such that $v \ngeq 0$, $v \nleq 0$ and $Xv \geq 0$.*

Proof. Since X^{-1} and $-X^{-1}$ are not nonnegative, there exist $u, w \in \mathbb{R}^n$ such that $Xu \geq 0$, $Xw \geq 0$, $u \ngeq 0$ and $w \nleq 0$. If either u or w contains both positive and negative entries, then we are done. Otherwise, $u \leq 0$ and $w \geq 0$. Also, u is not a multiple of w. Let $u = (u_1, \ldots, u_n)^T$ and $w = (w_1, \ldots, w_n)^T$. So, without loss of generality, let us assume that the matrix $D = \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix}$ is invertible. If $det(D) > 0$, then define $\alpha = \frac{w_1+w_2}{det(D)}$ and $\beta = -\frac{u_1+u_2}{det(D)}$. It is easy to see that both α and β are positive. Now, $D\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Setting $v = \alpha u + \beta w$, we see that v contains both positive and negative entries, and $Xv \geq 0$. If $det(D) < 0$, then a similar argument applies. ◻

In the next result, we establish a necessary and sufficient condition for $L(A)$ to be an into linear preserver of minimally semipositive matrices, when $m = n$. This answers an open problem mentioned in [3, Remark 2.12].

Theorem 2.10. *Let $X, Y \in \mathbb{R}^{n\times n}$. Let $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $L(A) = XAY$. Then L is an into linear preserver of minimally semipositive matrices if, and only if, X and Y are inverse nonnegative matrices, or $-X$ and $-Y$ are inverse nonnegative matrices.*

Proof. Suppose that A is minimally semipositive. Then A^{-1} exists and $A^{-1} \geq 0$. If X and Y are inverse nonnegative, then $(XAY)^{-1} = Y^{-1}A^{-1}X^{-1}$ is nonnegative. Thus, $L(A)$ is minimally semipositive.

Conversely, suppose that $L(A)$ is an into linear preserver of minimally semipositive matrices. First, note that X and Y must be invertible, due to the reason that if either X or Y is not invertible, then XAY is not invertible. Next, suppose that neither X nor $-X$ is inverse nonnegative. Then, by Lemma 2.9, there exists a vector v such that $v \ngeq 0, v \nleq 0$ and $Xv \geq 0$. Let $w \ngeq 0$ and consider Yw. By Theorem 2.2, there exists nonnegative invertible matrix B such that $Bv = Yw$. Let $A = B^{-1}$. Then A is minimally semipositive. Now $L(A)w = XAYw = XABv = Xv \geq 0$. But $w \ngeq 0$, a contradiction. Thus, X or $-X$ is inverse nonnegative.

Now, suppose that X is inverse nonnegative, and Y is not inverse nonnegative. Then there exists a vector w such that $w > 0$ and $u = Y^{-1}w \ngeq 0$. Let $v = X^{-1}w$. Then $v \geq 0$ and $Xv > 0$. By Theorem 2.4, there exists a nonnegative invertible matrix B such that $Bv = w$. Let $A = B^{-1}$. Then A is minimally semipositive. Now $L(A)u = XAY(Y^{-1}w) = XAw = Xv > 0$. Since $L(A)^{-1} \geq 0$, we have $u \geq 0$, a

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contradiction. Thus, Y must be inverse nonnegative. An entirely similar argument proves that if $-X$ is inverse nonnegative then $-Y$ is inverse nonnegative. $\square$

A matrix $A \in \mathbb{R}^{n\times n}$ is minimally semipositive if, and only if, A^T is minimally semipositive. This is the same as saying that the linear map $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $L(A) = A^T$ is a linear preserver of minimally semipositive matrices. In the next result, we characterize those matrices X and Y for which the linear map $L(A) = XA^TY$ is an into linear preserver of minimally semipositive matrices. Its proof is similar to that of Theorem 2.10, and hence, it is omitted.

Theorem 2.11. *Let $X, Y \in \mathbb{R}^{n\times n}$. Let $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $L(A) = XA^TY$. Then L is an into linear preserver of minimally semipositive matrices if, and only if, X and Y are inverse nonnegative matrices, or $-X$ and $-Y$ are inverse nonnegative matrices.*

Now, we present a necessary and sufficient condition for $L(A)$ to be an onto linear preserver of minimally semipositive matrices, when $m = n$. Let us recall a result that will be used in its proof.

Theorem 2.12. [1, Theorem 3.1] *For $m \geq n$, the set of $m \times n$ minimally semipositive matrices contains a basis for the set of all $m \times n$ matrices.*

Theorem 2.13. *For a given pair of matrices $X, Y \in \mathbb{R}^{n\times n}$, let $L(A) := XAY, A \in \mathbb{R}^{n\times n}$. Then $L(A)$ is an onto linear preserver of minimally semipositive matrices if, and only if, X and Y are monomial matrices, or $-X$ and $-Y$ are monomial matrices.*

Proof. Let L be an onto linear preserver of minimally semipositive matrices. Then L is an into linear preserver of minimally semipositive matrices. Since, by Theorem 2.12, the set of all minimally semipositive matrices contains a basis for the set of all $n \times n$ matrices, by Lemma 2.1, it follows that an onto preserver is the same as an invertible into preserver (of minimally semipositive matrices) whose inverse is also an into preserver (of minimally semipositive matrices). Also, it is easy to see that the inverse of the map L is given by $L^{-1}(A) = X^{-1}AY^{-1}$. Thus, by Theorem 2.10, one infers that the matrices X and Y are either both nonnegative and inverse nonnegative, or the matrices $-X$ and $-Y$ are both nonnegative and inverse nonnegative. Hence, X and Y are monomial matrices or $-X$ and $-Y$ are monomial matrices. The converse part is easily verified. $\square$

In the next result, we characterize matrices X and Y for which the linear map $L(A) = XA^TY$ is an onto linear preserver of minimally semipositive matrices. The proof of this result is similar to that of Theorem 2.13.

Theorem 2.14. *For a given pair of matrices $X, Y \in \mathbb{R}^{n\times n}$, let $L(A) := XA^TY, A \in \mathbb{R}^{n\times n}$. Then $L(A)$ is an onto linear preserver of minimally semipositive matrices if, and only if, X and Y are monomial matrices, or $-X$ and $-Y$ are monomial matrices.*

Next, we point out an error in the proof of Theorem 2.4 in [3]. Let $X \in \mathbb{R}^{n\times n}$. Suppose that neither X nor $-X$ is row positive. The authors in [3] claim that there exists a vector $v > 0$ such that Xv has some zero entry. The following example shows that this is incorrect.

Example 2.15. Let $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then neither X nor $-X$ is row positive and there does not exists any $v > 0$ such that Xv contains a zero entry.

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Let e denote the vector in $\mathbb{R}^n$ all of whose entries are 1. In the next result, we give a correct proof of Theorem 2.4 in [3].

THEOREM 2.16. *For fixed* $X \in \mathbb{R}^{m\times m}$ *and* $Y \in \mathbb{R}^{n\times n}$, *let* $L : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ *be defined by* $L(A) = XAY$, $A \in \mathbb{R}^{m\times n}$. *Then* L *is an into linear preserver of semipositive matrices if, and only if, either* X *is row positive and* Y *is inverse nonnegative, or* $-X$ *is row positive and* $-Y$ *is inverse nonnegative.*

Proof. Suppose that A is semipositive. Then there exists a semipositivity vector v such that $Av > 0$. If X is row positive and Y is inverse nonnegative, then $u = Y^{-1}v > 0$ and $XAYu > 0$. Thus, $L(A)$ is semipositive. If $-X$ is row positive and $-Y$ is inverse nonnegative, then $L(A) = (-X)A(-Y) = XAY$ is semipositive.

Conversely, suppose that L is an into linear preserver of semipositive matrices. Suppose that neither X nor $-X$ is row positive.

Case (i): X contains a zero row. Then XAY contains a zero row for any A and Y. Thus, XAY is not semipositive.

Case (ii): X contains a non-zero row i with both positive and negative entries. Then we can construct a positive vector v such that $(Xv)_i = 0$. Set $A = (v, \dots, v)$. Then A is semipositive and XAY contains a zero row, and hence, XAY is not semipositive.

Case (iii): X does not contain a zero row or a non-zero row which has both positive and negative entries. Then there exists a row which is nonpositive. Since, $-X$ is also not row positive, X contains a nonnegative row. Without loss of generality, assume that the first row of X is nonpositive and that the second row of X is nonnegative. Let $A = (e, 0, \dots, 0)$. Then A is semipositive and $XA = (Xe, 0, \dots, 0)$. Let $Xe = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$. Then $\alpha_1 < 0$ and $\alpha_2 > 0$. Let $u \in \mathbb{R}^n$ and $Yu = v$. Then $XAYu = (\alpha_1 v_1, \alpha_2 v_1, \dots, \alpha_n v_1)$. Thus, $XAYu$ cannot be positive and so XAY is not semipositive. Hence, either X or $-X$ is row positive.

Now, suppose that X is row positive. We first prove that Y is invertible. Let $v = (v_1, \dots, v_n)^T$ be such that $v^TY = 0$. Suppose that $v \neq 0$. With out loss of generality, we can assume that $v_1 > 0$. Let A be the matrix each of whose rows equals the row vector v^T. Since the first column of A is positive, the matrix A is semipositive. However, by definition, the matrix AY is zero and so XAY is not semipositive. This is a contradiction. Hence, Y is invertible. Next, let us show that $C = Y^{-1} \geq 0$. On the contrary, suppose that $c_{ij} < 0$ for some i, j. Let A be the matrix whose rows are the negative of the i^{th} row of C. Since the j^{th} column of A is positive, the matrix A is semipositive. However the matrix AY is nonpositive, and hence, XAY is not semipositive, a contradiction. Thus, $Y^{-1} \geq 0$. A similar argument shows that if $-X$ is row positive, then $-Y$ is inverse nonnegative. $\square$

Let us conclude with the following example, in which we construct a linear preserver of semipositive matrices over $\mathbb{R}^{2\times 2}$.

EXAMPLE 2.17. Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then, X is row positive and Y is inverse nonnegative. One may verify that the construction of Theorem 2.16 leads to the preserver map $L(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a+c & (b+d)-(a+c) \\ a & -a+b \end{pmatrix}$.

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