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POTENTIALLY EVENTUALLY POSITIVE 2-GENERALIZED STAR SIGN PATTERNS*

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Abstract. A sign pattern is a matrix whose entries belong to the set $\{+,-,0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to be potentially eventually positive if there exists at least one real matrix A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. An n-by-n sign pattern $\mathcal{A}$ is said to be potentially eventually exponentially positive if there exists at least one real matrix A with the same sign pattern as $\mathcal{A}$ and a nonnegative integer t_0 such that $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} > 0$ for all $t \geq t_0$. Identifying necessary and sufficient conditions for an n-by-n sign pattern to be potentially eventually positive (respectively, potentially eventually exponentially positive), and classifying these sign patterns are open problems. In this article, the potential eventual positivity of the 2-generalized star sign patterns is investigated. All the minimal potentially eventually positive 2-generalized star sign patterns are identified. Consequently, all the potentially eventually positive 2-generalized star sign patterns are classified. As an application, all the minimal potentially eventually exponentially positive 2-generalized star sign patterns are identified. Consequently, all the potentially eventually exponentially positive 2-generalized star sign patterns are classified.

Key words. Eventually positive matrix, Eventually exponentially positive matrix, 2-generalized star sign pattern, Checkerboard block sign pattern.

AMS subject classifications. 15A48, 15A18, 05C50.

1. Introduction.

In the literature, the research on combinatorial and qualitative information based on the signs of the entries of a matrix has attracted much attention. A *sign pattern* is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in the set $\{+,-,0\}$. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a *realization* of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the *qualitative class* of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A *subpattern* of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a *proper subpattern* of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is said to be a *superpattern* of $\mathcal{B}$. A pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{pmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is *irreducible* if it is not reducible; see, e.g. [4] and [5] for more details.

A sign pattern matrix $\mathcal{A}$ is said to *require* a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and *allow* P or be *potentially* P if there is some $A \in Q(\mathcal{A})$ that has the property P.

Recall that an n-by-n real matrix A is said to be *eventually positive* if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [9]. Eventually positive matrices have applications to dynamical

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systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter; see e.g., [10]. An n-by-n sign pattern $\mathcal{A}$ is said to *allow an eventually positive matrix* or be *potentially eventually positive* (*PEP* for short) if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [3] and the references therein. An n-by-n sign pattern $\mathcal{A}$ is said to be a *minimal potentially eventually positive sign pattern* (*MPEP* for short) if $\mathcal{A}$ is PEP and there is no proper subpattern of $\mathcal{A}$ that is PEP; see, e.g., [15] and the references therein.

PEP sign patterns were studied first in [3], where some sufficient conditions and some necessary conditions for a sign pattern to be PEP were established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be PEP remains open. Also open is the classification of PEP sign patterns. Recently, PEP sign patterns with reducible positive part were constructed in [2]. PEP double star sign patterns and PEP tridiagonal sign patterns were respectively characterized in [15] and [14]. Some relations between the minimal PEP sign patterns and the minimal power-positive sign patterns were established in [6] and [12]. PEP star sign patterns were classified in [13]. More recently, potential eventual positivity of sign patterns with the underlying broom graph was investigated in [11].

An n-by-n sign pattern $\mathcal{A}$ is said to *allow an eventually exponentially positive matrix* or be *potentially eventually exponentially positive* (*PEEP* for short) if there exists some $A \in Q(\mathcal{A})$ such that A is eventually exponentially positive; see, e.g., [1]. Similarly, an n-by-n sign pattern $\mathcal{A}$ is said to be a *minimal potentially eventually exponentially positive sign pattern* (*MPEEP* for short) if $\mathcal{A}$ is PEEP and there is no proper subpattern of $\mathcal{A}$ that is PEEP. Sign patterns that allow an eventually exponentially positive matrix have been studied first in [1], where some sufficient conditions and some necessary conditions for an n-by-n sign pattern to be PEEP have been established. However, there are many open questions about the study of PEEP sign patterns. For example, identifying the necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be PEEP remains open. Also open is the classification of sign patterns that are PEEP.

In this article, we focus on the eventual positivity of the $(2n-1)$-by-$(2n-1)$ 2-generalized star sign patterns $\mathcal{GS}_{2n-1}$ whose underlying graph $G(\mathcal{GS}_{2n-1})$ is a 2-generalized star, which consists of a star S_n with n vertices, together with $(n-1)$ pendent vertices adjacent to the $n-1$ leaf vertices of S_n. Our work is organized as follows. In Section 2, some preliminary results for a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern to be PEP are established, all the MPEP 2-generalized star sign patterns of order $2n-1$ are identified. Our results indicate that there are exactly three MPEP 2-generalized star sign patterns. Then, all the PEP 2-generalized star sign patterns are classified. In Section 3, as a byproduct, we identify the exactly one MPEEP sign pattern $\mathcal{GS}^o_{2n-1}$, and consequently, classify all PEEP 2-generalized star sign patterns as a superpattern of $\mathcal{GS}^o_{2n-1}$.

2. PEP 2-generalized star sign patterns.

We begin this section with introducing some necessary graph theoretical concepts which can be seen from [4], [9] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is *combinatorially symmetric* if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices $1, 2, \ldots, n$ and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the *graph* of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a *star* (respectively, 2-*generalized star*) *sign pattern* if $G(\mathcal{A})$ is a star (respectively, 2-generalized star) graph.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has *signed digraph* $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \ldots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) *simple cycle of*

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length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ such that the vertices $i_1, \ldots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its simple cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $v_i \in V$, $v_j \in V$, there is a walk of length k from v_i to v_j. A nonnegative sign pattern $\mathcal{A}$ is *primitive* if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [3] for more details.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *positive part* of $\mathcal{A}$ is defined to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The *negative part* of $\mathcal{A}$ can be defined similarly. Following [1], let $\mathcal{A}_{D(+)}$ (respectively, $\mathcal{A}_{D(0)}$ and $\mathcal{A}_{D(-)}$) be the sign pattern obtained from sign pattern $\mathcal{A}$ by changing all diagonal entries to $+$ (respectively, 0 and $-$). Note that $\hat{A}$ is used for $\mathcal{A}_{D(+)}$ in [3]. Below,we cite some necessary conditions and some sufficient conditions for an n-by-n sign pattern to be PEP or PEEP in order to state our work clearly.

Lemma 2.1. ([3, Theorem 2.1]). *Let $\mathcal{A}$ be a sign pattern such that $\mathcal{A}^+$ is primitive. Then $\mathcal{A}$ is PEP.*

Lemma 2.2. ([3, Theorem 3.1]). *If $\mathcal{A}$ is a PEP sign pattern, then every superpattern of $\mathcal{A}$ is also PEP.*

Lemma 2.3. ([3, Theorem 3.3]). *If $\mathcal{A}$ is a PEP sign pattern, then $\mathcal{A}_{D(+)}$ is also PEP.*

Lemma 2.4. ([1, Observation 1.4]). *If an n-by-n sign pattern $\mathcal{A}$ is PEEP, then $\mathcal{A}_{D(+)}$ is PEP.*

Lemma 2.5. ([3, Lemma 4.3]). *If an n-by-n sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix $A \in Q(\mathcal{A})$ such that*

(1) $\rho(A) = 1$;

(2) $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones vector;

(3) if $n \geq 2$, the sum of all the off-diagonal entries of A is positive.

We denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$). Let $[+]_i$ be a square block sign pattern of order i consisting entirely of positive entries. For block sign patterns, we have the following lemma which is shown in [1] and [3].

Lemma 2.6. ([1, Remark 2.13] and ([3, Proposition 5.3]). *Let $\mathcal{A}$ be the checkerboard block sign pattern*

$$\begin{pmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with square diagonal blocks. If $\mathcal{A}$ has a negative entry, then

(1) $-\mathcal{A}$ is not PEP, but is PEEP;

(2) $\mathcal{A}$ is not PEP and is not PEEP.

Now we turn to the 2-generalized star sign patterns $\mathcal{GS}_{2n-1}$ ($n \geq 1$). As shown in Figure 1, the underlying graph $G(\mathcal{GS}_{2n-1})$ consists of a star S_n with n vertices, together with $(n-1)$ pendant vertices adjacent to each leaf vertex of S_n, respectively; see [8] for the symmetric case.

Note that $\mathcal{GS}_{2n-1}$ is a tridiagonal sign pattern for $n < 4$. The PEP tridiagonal sign patterns have been investigated in [14]. Throughout the paper, we assume that $n \geq 4$. Since sign pattern $\mathcal{A}$ is PEP if and only if

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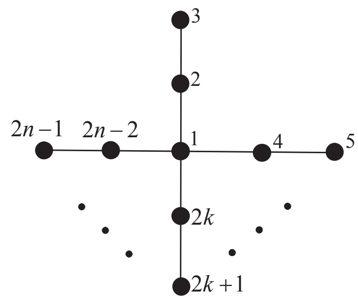

FIGURE 1. *The graph of* 2*-generalized star sign patterns.*

$\mathcal{A}^T$ or $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is PEP, for any permutation pattern $\mathcal{P}$, without loss of generality, let the $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ be of the following form

$$\begin{pmatrix} ? & * & & * & & \cdots & * & \\ * & ? & * & & & & & \\ & * & ? & & & & & \\ * & & & ? & * & & & \\ & & & * & ? & & & \\ \vdots & & & & & \ddots & & \\ * & & & & & & ? & * \\ & & & & & & * & ? \end{pmatrix},$$

where ? denotes an entry from $\{+,-,0\}$, $*$ denotes a nonzero entry, and the unspecified entries are all zeros.

The following theorem is necessary for a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ to be PEP.

THEOREM 2.7. *Let $\mathcal{GS}_{2n-1} = ((gs)_{i,j})$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. If $\mathcal{GS}_{2n-1}$ is PEP, then $\mathcal{GS}_{2n-1}$ is symmetric.*

Proof. Since the 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ is PEP, there is an eventually positive matrix $A = (a_{i,j}) \in Q(\mathcal{GS}_{2n-1})$ such that $\rho(A) = 1$ and $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones vector. Let $W = (w_1, w_2, \dots, w_{2n-1})^T$ be the positive left eigenvector. Then by $W^TA = W^T$, we obtain that for all $k = 1, 2, \dots, n-1$,

$$w_{2k}a_{2k,2k+1} + w_{2k+1}(1 - a_{2k+1,2k}) = w_{2k+1} \tag{2.1}$$

and

$$w_1a_{1,2k} + w_{2k}(1 - a_{2k,1} - a_{2k,2k+1}) + w_{2k+1}a_{2k+1,2k} = w_{2k}. \tag{2.2}$$

By equality (2.1), we deduce that

$$w_{2k}a_{2k,2k+1} = w_{2k+1}a_{2k+1,2k},$$

and by adding equality (2.1) to equality (2.2), we obtain

$$w_1a_{1,2k} = w_{2k}a_{2k,1}.$$

Thus, $\frac{a_{2k,2k+1}}{a_{2k+1,2k}} = \frac{w_{2k+1}}{w_{2k}} > 0$ and $\frac{a_{1,2k}}{a_{2k,1}} = \frac{w_{2k}}{w_1} > 0$. It follows that $(gs)_{2k,2k+1} = (gs)_{2k+1,2k}$ and $(gs)_{1,2k} = (gs)_{2k,1}$ for all $k = 1, 2, \dots, n-1$. Therefore, $\mathcal{GS}_{2n-1}$ is symmetric. ∎

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It is known that if an n-by-n ($n \geq 2$) sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix realization A such that the sum of all nonzero off-diagonal entries of A is positive. Interestingly enough, the following theorem indicates that all nonzero off-diagonal entries of every matrix realization $A \in Q(\mathcal{GS}_{2n-1})$ are positive, for a $(2n-1)$-by-$(2n-1)$ PEP 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$.

Theorem 2.8. *Let $\mathcal{GS}_{2n-1} = ((gs)_{i,j})$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. If $\mathcal{GS}_{2n-1}$ is PEP, then $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$.*

Proof. Since $\mathcal{GS}_{2n-1}$ is PEP, $\mathcal{GS}_{2n-1}$ is symmetric by Theorem 2.7. To complete the proof, it suffices to show that there is no $k \in \{1, 2, \ldots, n-1\}$ such that $(gs)_{1,2k} = (gs)_{2k,1} = -$ or $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = -$. To state clearly, let s be the number of i such that $(gs)_{1,2i} = (gs)_{2i,1} = -$ and $(gs)_{2i+1,2i} = (gs)_{2i,2i+1} = -$, t be the number of i such that $(gs)_{1,2i} = (gs)_{2i,1} = -$ and $(gs)_{2i+1,2i} = (gs)_{2i,2i+1} = +$, p be the number of i such that $(gs)_{1,2i} = (gs)_{2i,1} = +$ and $(gs)_{2i+1,2i} = (gs)_{2i,2i+1} = -$. Then $0 \leq s \leq n-1$, $0 \leq t \leq n-1$, $0 \leq p \leq n-1$ and $0 \leq s+t+p \leq n-1$. To complete the proof, it suffices to show that $s = 0$, $t = 0$ and $p = 0$. By a way of contradiction, we consider the following seven cases: (1) $s = t = 0$ and $p > 0$; (2) $s = p = 0$ and $t > 0$; (3) $t = p = 0$ and $s > 0$; (4) $s = 0$, $t > 0$ and $p > 0$; (5) $t = 0$, $s > 0$ and $p > 0$; (6) $p = 0$, $t > 0$ and $s > 0$; (7) $s > 0$, $t > 0$ and $p > 0$.

Case 1. $s = t = 0$ and $p > 0$.

Up to equivalence,

$$\mathcal{GS}_{2n-1} = \begin{pmatrix} \mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,p+1} & \mathcal{GS}_{1,p+2} \\ (\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & \\ \vdots & & \ddots & & \\ (\mathcal{GS}_{1,p+1})^T & & & \mathcal{GS}_{p+1,p+1} & \\ (\mathcal{GS}_{1,p+2})^T & & & & \mathcal{GS}_{p+2,p+2} \end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$, $\mathcal{GS}_{1,k} = (+, 0)$ for all $k = 2, 3, \ldots, p+1$, the 1-by-$2(n-p-1)$ matrix $\mathcal{GS}_{1,p+2} = (+, 0, +, 0, \ldots, +, 0)$ and

$$\mathcal{GS}_{p+2,p+2} = \begin{pmatrix} ? & + & & & & & \\ + & ? & & & & & \\ & & ? & + & & & \\ & & + & ? & & & \\ & & & & \ddots & & \\ & & & & & ? & + \\ & & & & & + & ? \end{pmatrix}$$

is of order $2(n-p-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern

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$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} = \begin{pmatrix} [+]_2 & [-] & [+] & \cdots & [-] & [+] \\ [-] & [+]_1 & [-] & \cdots & [+] & [-] \\ [+] & [-] & [+]_1 & \cdots & [-] & [+] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [-] & [+] & [-] & \cdots & [+]_1 & [-] \\ [+] & [-] & [+] & \cdots & [-] & [+]_{2(n-p)-1} \end{pmatrix}$$

with $2p-2$ diagonal blocks $[+]_1$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Case 2. $s = p = 0$ and $t > 0$.

Up to equivalence,

$$\mathcal{GS}_{2n-1} = \begin{pmatrix} \mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,t+1} & \mathcal{GS}_{1,t+2} \\ (\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & \\ \vdots & & \ddots & & \\ (\mathcal{GS}_{1,t+1})^T & & & \mathcal{GS}_{t+1,t+1} & \\ (\mathcal{GS}_{1,t+2})^T & & & & \mathcal{GS}_{t+2,t+2} \end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & + \\ + & ? \end{pmatrix}$, $\mathcal{GS}_{1,k} = (-, 0)$ for all $k = 2, 3, \ldots, t+1$, the 1-by-$2(n-t-1)$ matrix $\mathcal{GS}_{1,t+2} = (+, 0, +, 0, \ldots, +, 0)$ and

$$\mathcal{GS}_{t+2,t+2} = \begin{pmatrix} ? & + & & & & & \\ + & ? & & & & & \\ & & ? & + & & & \\ & & + & ? & & & \\ & & & & \ddots & & \\ & & & & & ? & + \\ & & & & & + & ? \end{pmatrix}$$

is of order $2(n-t-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern

$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} = \begin{pmatrix} [+]_1 & [-] & [+] \\ [-] & [+]_{2t} & [-] \\ [+] & [-] & [+]_{2(n-t-1)} \end{pmatrix}$$

is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Case 3. $t = p = 0$ and $s > 0$.

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Up to equivalence,

$$\mathcal{GS}_{2n-1} = \begin{pmatrix} \mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,s+1} & \mathcal{GS}_{1,s+2} \\ (\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & \\ \vdots & & \ddots & & \\ (\mathcal{GS}_{1,s+1})^T & & & \mathcal{GS}_{s+1,s+1} & \\ (\mathcal{GS}_{1,s+2})^T & & & & \mathcal{GS}_{s+2,s+2} \end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$, $\mathcal{GS}_{1,k} = (-, 0)$ for all $k = 2, 3, \ldots, s+1$, the 1-by-$2(n-s-1)$ matrix $\mathcal{GS}_{1,p+2} = (+, 0, +, 0, \ldots, +, 0)$ and

$$\mathcal{GS}_{s+2,s+2} = \begin{pmatrix} ? & + & & & & & \\ + & ? & & & & & \\ & & ? & + & & & \\ & & + & ? & & & \\ & & & & \ddots & & \\ & & & & & ? & + \\ & & & & & + & ? \end{pmatrix}$$

is of order $2(n-s-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern

$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} = \begin{pmatrix} [+]_1 & [-] & [+] & \cdots & [-] & [+] \\ [-] & [+]_1 & [-] & \cdots & [+] & [-] \\ [+] & [-] & [+]_1 & \cdots & [-] & [+] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [-] & [+] & [-] & \cdots & [+]_1 & [-] \\ [+] & [-] & [+] & \cdots & [-] & [+]_{2(n-s)-1} \end{pmatrix}$$

with $2s$ diagonal blocks $[+]_1$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Case 4. $s = 0$, $t > 0$ and $p > 0$.

Up to equivalence, $\mathcal{GS}_{2n-1} =$

$$\begin{pmatrix} \mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,t+1} & \mathcal{GS}_{1,t+2} \\ (\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & \\ \vdots & & \ddots & & \\ (\mathcal{GS}_{1,t+1})^T & & & \mathcal{GS}_{t+1,t+1} & \\ (\mathcal{GS}_{1,t+2})^T & & & & \mathcal{GS}_{t+2,t+2} \\ \vdots & & & & \\ (\mathcal{GS}_{1,t+p+1})^T & & & & \\ (\mathcal{GS}_{1,t+p+2})^T & & & & \end{pmatrix}$$

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$$\begin{matrix} \cdots & \mathcal{GS}_{1,t+p+1} & \mathcal{GS}_{1,t+p+2} \\ & & \\ & & \\ \ddots & & \\ & \mathcal{GS}_{t+p+1,t+p+1} & \\ & & \mathcal{GS}_{t+p+2,t+p+2} \end{matrix} \Bigg) ,$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & + \\ + & ? \end{pmatrix}$, $\mathcal{GS}_{1,k} = (-,0)$ for all $k = 2,3,\ldots,t+1$, and $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$, $\mathcal{GS}_{1,k} = (+,0)$ for all $k = t+2, t+3, \ldots, t+p+1$, the 1-by-$2(n-t-p-1)$ matrix $\mathcal{GS}_{1,t+p+2} = (+,0,+,0,\ldots,+,0)$ and

$$\mathcal{GS}_{t+p+2,t+p+2} = \begin{pmatrix} ? & + & & & & & \\ + & ? & & & & & \\ & & ? & + & & & \\ & & + & ? & & & \\ & & & & \ddots & & \\ & & & & & ? & + \\ & & & & & + & ? \end{pmatrix}$$

is of order $2(n-t-p-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern

$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} = \begin{pmatrix} [+]_1 & [-] & [+] & \cdots & [-] & [+] \\ [-] & [+]_{2t} & [-] & \cdots & [+] & [-] \\ [+] & [-] & [+]_1 & \cdots & [-] & [+] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [-] & [+] & [-] & \cdots & [+]_1 & [-] \\ [+] & [-] & [+] & \cdots & [-] & [+]_{2(n-t-p)-1} \end{pmatrix}$$

with $2p$ diagonal blocks $[+]_1$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Case 5. $t = 0$, $s > 0$ and $p > 0$.

Up to equivalence, $\mathcal{GS}_{2n-1} =$

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$$\begin{pmatrix}
\mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,s+1} & \mathcal{GS}_{1,s+2} & \cdots & \mathcal{GS}_{1,s+p+1} & \mathcal{GS}_{1,s+p+2} \\
(\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & & & & \\
\vdots & & \ddots & & & & & \\
(\mathcal{GS}_{1,s+1})^T & & & \mathcal{GS}_{s+1,s+1} & & & & \\
(\mathcal{GS}_{1,s+2})^T & & & & \mathcal{GS}_{s+2,s+2} & & & \\
\vdots & & & & & \ddots & & \\
(\mathcal{GS}_{1,s+p+1})^T & & & & & & \mathcal{GS}_{s+p+1,s+p+1} & \\
(\mathcal{GS}_{1,s+p+2})^T & & & & & & & \mathcal{GS}_{s+p+2,s+p+2}
\end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$, for all $k = 2, 3, \ldots, s+p+1$, $\mathcal{GS}_{1,k} = (-, 0)$ for all $k = 2, 3, \ldots, s+1$, and $(+, 0)$ for all $k = s+2, s+3, \ldots, s+p+1$, the 1-by-$2(n-s-p-1)$ matrix $\mathcal{GS}_{1,s+p+2} = (+, 0, +, 0, \ldots, +, 0)$ and

$$\mathcal{GS}_{s+p+2,s+p+2} = \begin{pmatrix}
? & + & & & & & \\
+ & ? & & & & & \\
& & ? & + & & & \\
& & + & ? & & & \\
& & & & \ddots & & \\
& & & & & ? & + \\
& & & & & + & ?
\end{pmatrix}$$

is of order $2(n-s-p-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} =$

$$\begin{pmatrix}
[+]_1 & [-] & [+] & \cdots & [-] & [+] & [-] & \cdots & [+] & [-] \\
[-] & [+]_1 & [-] & \cdots & [+] & [-] & [+] & \cdots & [-] & [+] \\
[+] & [-] & [+]_1 & \cdots & [-] & [+] & [-] & \cdots & [+] & [-] \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[-] & [+] & [-] & \cdots & [+]_1 & [-] & [+] & \cdots & \vdots & \vdots \\
[+] & [-] & [+] & \cdots & [-] & [+]_2 & [-] & \cdots & \vdots & \vdots \\
[-] & [+] & [-] & \cdots & [+] & [-] & [+]_1 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
[+] & [-] & [+] & \cdots & \cdots & \cdots & \cdots & & [+]_1 & [-] \\
[-] & [+] & [-] & \cdots & \cdots & \cdots & \cdots & & [-] & [+]_{2(n-s-p)-1}
\end{pmatrix}$$

with $2s$ diagonal blocks $[+]_1$ before diagonal block $[+]_2$ and $2(p-1)$ diagonal blocks $[+]_1$ after diagonal block

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$[+]_2$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Case 6. $p = 0$, $t > 0$ and $s > 0$.

Up to equivalence, $\mathcal{GS}_{2n-1} =$

$$\begin{pmatrix} \mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,s+1} & \mathcal{GS}_{1,s+2} & \cdots & \mathcal{GS}_{1,s+t+1} & \mathcal{GS}_{1,s+t+2} \\ (\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & & & & \\ \vdots & & \ddots & & & & & \\ (\mathcal{GS}_{1,s+1})^T & & & \mathcal{GS}_{s+1,s+1} & & & & \\ (\mathcal{GS}_{1,s+2})^T & & & & \mathcal{GS}_{s+2,s+2} & & & \\ \vdots & & & & & \ddots & & \\ (\mathcal{GS}_{1,s+t+1})^T & & & & & & \mathcal{GS}_{s+t+1,s+t+1} & \\ (\mathcal{GS}_{1,s+t+2})^T & & & & & & & \mathcal{GS}_{s+t+2,s+t+2} \end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$ for all $k = 2, 3, \ldots, s+1$, and $\begin{pmatrix} ? & + \\ + & ? \end{pmatrix}$ for all $k = s+2, s+3, \ldots, s+t+1$, $\mathcal{GS}_{1,k} = (-, 0)$ for all $k = 2, 3, \ldots, s+t+1$, the 1-by-$2(n-s-t-1)$ matrix $\mathcal{GS}_{1,s+t+2} = (+, 0, +, 0, \ldots, +, 0)$ and $\mathcal{GS}_{s+t+2,s+t+2} = \begin{pmatrix} ? & + & & & & & \\ + & ? & & & & & \\ & & ? & + & & & \\ & & + & ? & & & \\ & & & & \ddots & & \\ & & & & & ? & + \\ & & & & & + & ? \end{pmatrix}$ is of order $2(n-s-t-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that the checkerboard block sign pattern

$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} = \begin{pmatrix} [+]_1 & [-] & [+] & \cdots & [-] & [+] \\ [-] & [+]_1 & [-] & \cdots & [+] & [-] \\ [+] & [-] & [+]_1 & \cdots & [-] & [+] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [-] & [+] & [-] & \cdots & [+]_1 & [-] \\ [+] & [-] & [+] & \cdots & [-] & [+]_{2(n-s-1)} \end{pmatrix}$$

with $2s+1$ diagonal blocks $[+]_1$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

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Case 7. $s > 0$, $t > 0$ and $p > 0$.

Up to equivalence, let $\mathcal{GS}_{2n-1} =$

$$\begin{pmatrix}
\mathcal{GS}_{1,1} & \mathcal{GS}_{1,2} & \cdots & \mathcal{GS}_{1,s+1} & \mathcal{GS}_{1,s+2} & \cdots & \mathcal{GS}_{1,s+t+1} & \mathcal{GS}_{1,s+t+2} & \cdots & \mathcal{GS}_{1,s+t+p+1} & \mathcal{GS}_{1,s+t+p+2} \\
(\mathcal{GS}_{1,2})^T & \mathcal{GS}_{2,2} & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & \\
(\mathcal{GS}_{1,s+1})^T & & & \mathcal{GS}_{s+1,s+1} & & & & & & & \\
(\mathcal{GS}_{1,s+2})^T & & & & \mathcal{GS}_{s+2,s+2} & & & & & & \\
\vdots & & & & & \ddots & & & & & \\
(\mathcal{GS}_{1,s+t+1})^T & & & & & & \mathcal{GS}_{s+t+1,s+t+1} & & & & \\
(\mathcal{GS}_{1,s+t+2})^T & & & & & & & \mathcal{GS}_{s+t+2,s+t+2} & & & \\
\vdots & & & & & & & & \ddots & & \\
(\mathcal{GS}_{1,s+t+p+1})^T & & & & & & & & & \mathcal{GS}_{s+t+p+1,s+t+p+1} & \\
(\mathcal{GS}_{1,s+t+p+2})^T & & & & & & & & & & \mathcal{GS}_{s+t+p+2,s+t+p+2}
\end{pmatrix},$$

where $\mathcal{GS}_{1,1} = (?)$, $\mathcal{GS}_{k,k} = \begin{pmatrix} ? & - \\ - & ? \end{pmatrix}$ for all $k = 2, 3, \ldots, s+1, s+t+2, s+t+3, \ldots, s+t+p+1$, and $\begin{pmatrix} ? & + \\ + & ? \end{pmatrix}$ for all $k = s+2, s+3, \ldots, s+t+1$, $\mathcal{GS}_{1,k} = (-, 0)$ for all $k = 2, 3, \ldots, s+t+1$, $(+, 0)$ for all $k = s+t+2, s+t+3, \ldots, s+t+p+1$, the 1-by-$2(n-s-t-p-1)$ matrix $\mathcal{GS}_{1,s+t+p+2} = (+, 0, +, 0, \ldots, +, 0)$ and

$$\mathcal{GS}_{s+t+p+2,s+t+p+2} = \begin{pmatrix}
? & + & & & & & \\
+ & ? & & & & & \\
& & ? & + & & & \\
& & + & ? & & & \\
& & & & \ddots & & \\
& & & & & ? & + \\
& & & & & + & ?
\end{pmatrix}$$

is of order $2(n-s-t-p-1)$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is also PEP by Lemma 2.3. Note that

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the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)} =$

$$\begin{pmatrix}
[+]_1 & [-] & [+] & \cdots & [+] & [-] & [+] & \cdots & [-] & [+] \\
[-] & [+]_1 & [-] & \cdots & [-] & [+] & [-] & \cdots & [+] & [-] \\
[+] & [-] & [+]_1 & \cdots & [+] & [-] & [+] & \cdots & [-] & [+] \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[+] & [-] & [+] & \cdots & [+]_1 & [-] & [+] & \cdots & \vdots & \vdots \\
[-] & [+] & [-] & \cdots & [-] & [+]_{2t} & [-] & \cdots & \vdots & \vdots \\
[+] & [-] & [+] & \cdots & [+] & [-] & [+]_1 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
[-] & [+] & [-] & \cdots & \cdots & \cdots & \cdots & \cdots & [+]_1 & [-] \\
[+] & [-] & [+] & \cdots & \cdots & \cdots & \cdots & \cdots & [-] & [+]_{2(n-s-t-p-1)}
\end{pmatrix}$$

with $2s+1$ diagonal blocks $[+]_1$ before the diagonal block $[+]_{2t}$, and $2p$ diagonal blocks $[+]_1$ after the diagonal block $[+]_{2t}$ is a proper superpattern of $(\mathcal{GS}_{2n-1})_{D(+)}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ is PEP by Lemma 2.2. But by Lemma 2.6, the checkerboard block sign pattern $(\widetilde{\mathcal{GS}_{2n-1}})_{D(+)}$ cannot be PEP. It is a contradiction.

Based on the previous discussion, we have shown that if the 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ is PEP, then $s = t = p = 0$. And thus, there is no $k \in \{1, 2, \ldots, n-1\}$ such that $(gs)_{1,2k} = (gs)_{2k,1} = -$ or $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = -$. It follows that $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$. ∎

Next we proceed to identify all MPEP 2-generalized star sign patterns. Recall that an n-by-n sign pattern $\mathcal{A}$ is said to be MPEP if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP. To identify all the MPEP subpatterns of $\mathcal{A}$, it is necessary to discuss the number of positive diagonal entries of PEP sign patterns.

Proposition 2.9. *Let $\mathcal{GS}_{2n-1} = ((gs)_{i,j})$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. If $\mathcal{GS}_{2n-1}$ is PEP, then $\mathcal{GS}_{2n-1}$ has at least one positive diagonal entry. That is, there exists some $k \in \{1, 2, \ldots, 2n-1\}$ such that $(gs)_{k,k} = +$.*

Proof. By a way of contradiction, assume that $(gs)_{k,k} = -$ or 0 for all $k = 1, 2, \ldots, 2n-1$. Since $\mathcal{GS}_{2n-1}$ is PEP, $(\mathcal{GS}_{2n-1})_{D(-)}$ is a superpattern of $\mathcal{GS}_{2n-1}$ and thus is PEP by Lemma 2.2. By Theorem 2.8, $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$. It follows that all the nonzero off-diagonal entries of $(\mathcal{GS}_{2n-1})_{D(-)}$ are $+$. Note that the checkerboard block sign pattern

$$(\widetilde{\mathcal{GS}_{2n-1}})_{D(-)} = \begin{pmatrix}
[-]_1 & [+] & \cdots & [-] \\
[+] & [-]_1 & \cdots & [+] \\
\vdots & \vdots & \ddots & \vdots \\
[-] & [+] & \cdots & [-]_1
\end{pmatrix}$$

with $2n-1$ diagonal block patterns $[+]_1$ is a proper superpattern of $\mathcal{GS}_{2n-1}$. Thus, $(\widetilde{\mathcal{GS}_{2n-1}})_{D(-)}$ is PEP by Lemma 2.2. But $(\widetilde{\mathcal{GS}_{2n-1}})_{D(-)}$ is a checkerboard block sign pattern and is not PEP by Lemma 2.6; a contradiction. ∎

For the sake of convenience, let $\mathcal{GS}^{(i)}_{2n-1}$ be the $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ with all nonzero off-diagonal entries equal to $+$, $(gs)_{i,i} = +$ and $(gs)_{j,j} = 0$ for all $j \neq i$, $i \in \{1, 2, \ldots, 2n-1\}$.

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For example,

$$\mathcal{GS}_{2n-1}^{(1)} = \begin{pmatrix} + & + & & + & & \cdots & + & \\ + & 0 & + & & & & & \\ & + & 0 & & & & & \\ + & & & 0 & + & & & \\ & & & + & 0 & & & \\ \vdots & & & & & \ddots & & \\ + & & & & & & 0 & + \\ & & & & & & + & 0 \end{pmatrix},$$

$$\mathcal{GS}_{2n-1}^{(2)} = \begin{pmatrix} 0 & + & & + & & \cdots & + & \\ + & + & + & & & & & \\ & + & 0 & & & & & \\ + & & & 0 & + & & & \\ & & & + & 0 & & & \\ \vdots & & & & & \ddots & & \\ + & & & & & & 0 & + \\ & & & & & & + & 0 \end{pmatrix}$$

and

$$\mathcal{GS}_{2n-1}^{(3)} = \begin{pmatrix} 0 & + & & + & & \cdots & + & \\ + & 0 & + & & & & & \\ & + & + & & & & & \\ + & & & 0 & + & & & \\ & & & + & 0 & & & \\ \vdots & & & & & \ddots & & \\ + & & & & & & 0 & + \\ & & & & & & + & 0 \end{pmatrix}.$$

Note that sign patterns $\mathcal{GS}_{2n-1}^{(2)}$, $\mathcal{GS}_{2n-1}^{(4)}$, $\ldots$, $\mathcal{GS}_{2n-1}^{(2(n-1))}$ are equivalent to each other. Also respectively equivalent are $\mathcal{GS}_{2n-1}^{(3)}$, $\mathcal{GS}_{2n-1}^{(5)}$, $\ldots$, $\mathcal{GS}_{2n-1}^{(2n-1)}$.

THEOREM 2.10. *$\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are MPEP 2-generalized star sign patterns.*

Proof. By Lemma 2.1, the 2-generalized star sign patterns $\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are PEP for their positive parts $(\mathcal{GS}_{2n-1}^{(1)})^+$, $(\mathcal{GS}_{2n-1}^{(2)})^+$ and $(\mathcal{GS}_{2n-1}^{(3)})^+$ are primitive, respectively. If the diagonal entries of $\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are changed to be 0, then the corresponding sign patterns are not PEP by Proposition 2.9. If some nonzero off-diagonal entries of $\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are changed to be 0, then the corresponding sign patterns are not irreducible, and thus are not PEP. It follows that no proper subpatterns of $\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are PEP. So $\mathcal{GS}_{2n-1}^{(1)}$, $\mathcal{GS}_{2n-1}^{(2)}$ and $\mathcal{GS}_{2n-1}^{(3)}$ are MPEP. ∎

The following proposition follows readily from Theorem 2.10 and Proposition 2.9.

PROPOSITION 2.11. *If a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ is MPEP, then $\mathcal{GS}_{2n-1}$ has exactly one positive diagonal entry and all other diagonal entries are 0.*

Now we identify all $(2n-1)$-by-$(2n-1)$ MPEP 2-generalized star sign patterns, which implies that up to equivalence there are only three MPEP 2-generalized star sign patterns, and then we classify all

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PEP 2-generalized star sign patterns to be the superpatterns of these specific MPEP 2-generalized star sign patterns.

THEOREM 2.12. *Let $\mathcal{GS}_{2n-1}$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. Then $\mathcal{GS}_{2n-1}$ is MPEP if and only if $\mathcal{GS}_{2n-1}$ is equivalent to one of sign patterns $\mathcal{GS}^{(1)}_{2n-1}$, $\mathcal{GS}^{(2)}_{2n-1}$ and $\mathcal{GS}^{(3)}_{2n-1}$.*

Proof. The sufficiency follows from Theorem 2.10. For the necessity, since $\mathcal{GS}_{2n-1}$ is MPEP, then all nonzero off-diagonal entries are $+$ by Theorem 2.7, and $\mathcal{GS}_{2n-1}$ has exactly one positive diagonal entry and all other diagonal entries are 0 by Proposition 2.11. Thus, up to equivalence, $\mathcal{GS}_{2n-1}$ is one of sign patterns $\mathcal{GS}^{(1)}_{2n-1}$, $\mathcal{GS}^{(2)}_{2n-1}$ and $\mathcal{GS}^{(3)}_{2n-1}$. □

The following corollary follows readily from Theorem 2.12 and classifies all the PEP 2-generalized star sign patterns.

COROLLARY 2.13. *Let $\mathcal{GS}_{2n-1}$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. Then $\mathcal{GS}_{2n-1}$ is PEP if and only if $\mathcal{GS}_{2n-1}$ is equivalent to a superpattern of one of sign patterns $\mathcal{GS}^{(1)}_{2n-1}$, $\mathcal{GS}^{(2)}_{2n-1}$ and $\mathcal{GS}^{(3)}_{2n-1}$.*

Recall that an arbitrary n-by-n sign pattern $\mathcal{A}$ is said to require eventual positivity if every matrix $A \in Q(\mathcal{A})$ is eventually positive; see e.g., [7]. It is obvious that if an arbitrary sign pattern $\mathcal{A}$ requires eventual positivity, then $\mathcal{A}$ is also PEP. But the converse is not true in general. We conclude this section by drawing some interesting results about MPEP 2-generalized star sign patterns.

PROPOSITION 2.14. *Let $\mathcal{GS}_{2n-1}$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern with exactly one nonzero diagonal entry. Then the following statements are equivalent:*

(1) $\mathcal{GS}_{2n-1}$ is MPEP;

(2) $\mathcal{GS}_{2n-1}$ is nonnegative and primitive;

(3) $\mathcal{GS}_{2n-1}$ requires eventual positivity.

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 2.12. (2) $\Longrightarrow$ (3) follows from Theorem 2.3 in [7]. (3) $\Longrightarrow$ (1) follows from Corollary 2.13 and Theorem 2.12. □

3. PEEP 2-generalized star sign patterns. In this section, we use the previous results on the PEP 2-generalized star sign patterns to identify all the MPEEP 2-generalized star sign patterns, and then classify the PEEP 2-generalized star sign patterns. Recall that an n-by-n sign pattern $\mathcal{A}$ is said to PEEP if there exists some $A \in Q(\mathcal{A})$ such that A is eventually exponentially positive, and $\mathcal{A}$ is said to be MPEEP if $\mathcal{A}$ is PEEP and no proper subpattern of $\mathcal{A}$ is PEEP. The following is necessary for a 2-generalized star sign pattern to be PEEP.

PROPOSITION 3.1. *Let $\mathcal{GS}_{2n-1} = ((gs)_{i,j})$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. If $\mathcal{GS}_{2n-1}$ is PEEP, then $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$.*

Proof. Since 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ is PEEP, $(\mathcal{GS}_{2n-1})_{D(+)}$ is PEP by Lemma 2.4. By Theorem 2.8, all nonzero off-diagonal entries of $(\mathcal{GS}_{2n-1})_{D(+)}$ must be $+$. It follows that $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$. □

Now we turn to identify all MPEEP 2-generalized star sign patterns.

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PROPOSITION 3.2. *The 2-generalized star sign pattern*

$$\mathcal{GS}^o_{2n-1} = \begin{pmatrix} 0 & + & & + & & \cdots & + & \\ + & 0 & + & & & & & \\ & + & 0 & & & & & \\ + & & & 0 & + & & & \\ & & & + & 0 & & & \\ \vdots & & & & & \ddots & & \\ + & & & & & & 0 & + \\ & & & & & & + & 0 \end{pmatrix}$$

is MPEEP.

Proof. Sign pattern $\mathcal{GS}^o_{2n-1}$ is PEEP for its positive part $(\mathcal{GS}^o_{2n-1})^+$ is irreducible. It is clear that each proper subpattern of $\mathcal{GS}^o_{2n-1}$ is not irreducible and thus is not PEEP. It follows that $\mathcal{GS}^o_{2n-1}$ is MPEEP. ☐

THEOREM 3.3. *Let $\mathcal{GS}_{2n-1} = ((gs)_{i,j})$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. Then $\mathcal{GS}_{2n-1}$ is MPEEP if and only if $\mathcal{GS}_{2n-1} = \mathcal{GS}^o_{2n-1}$, up to equivalence.*

Proof. Proposition 3.2 implies the sufficiency. For the necessity, assume that the $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern $\mathcal{GS}_{2n-1}$ is MPEEP. Then $\mathcal{GS}_{2n-1}$ is PEEP, and by Proposition 3.1, $(gs)_{1,2k} = (gs)_{2k,1} = +$, $(gs)_{2k,2k+1} = (gs)_{2k+1,2k} = +$ for all $k = 1, 2, \ldots, n-1$. Suppose that some diagonal entries of $\mathcal{GS}_{2n-1}$ are nonzero. Then $\mathcal{GS}_{2n-1}$ is a proper superpattern of $\mathcal{GS}^o_{2n-1}$. Since $\mathcal{GS}_{2n-1}$ is MPEEP, $\mathcal{GS}^o_{2n-1}$ is not PEEP. But $\mathcal{GS}^o_{2n-1}$ is MPEEP by Proposition 3.2. It is a contradiction. Consequently, all the diagonal entries of $\mathcal{GS}_{2n-1}$ must be 0. It follows that $\mathcal{GS}_{2n-1} = \mathcal{GS}^o_{2n-1}$, up to equivalence. ☐

Now we turn to classify all the $(2n-1)$-by-$(2n-1)$ PEEP 2-generalized star sign patterns.

COROLLARY 3.4. *Let $\mathcal{GS}_{2n-1}$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. Then $\mathcal{GS}_{2n-1}$ is PEEP if and only if $\mathcal{GS}_{2n-1}$ is one of the superpatterns of $\mathcal{GS}^o_{2n-1}$, up to equivalence.*

Proof. If $\mathcal{GS}_{2n-1}$ is MPEEP, then the conclusion follows from Theorem 3.3. If $\mathcal{GS}_{2n-1}$ is PEEP, but not MPEEP, then $\mathcal{GS}_{2n-1}$ is a proper superpattern of some MPEEP 2-generalized star sign patterns. By Theorem 3.3, up to equivalence, $\mathcal{GS}^o_{2n-1}$ is the exactly one MPEEP sign pattern in the class of 2-generalized star sign pattern. Thus, $\mathcal{GS}_{2n-1}$ is a proper superpattern of $\mathcal{GS}^o_{2n-1}$. ☐

Recall that an arbitrary n-by-n sign pattern $\mathcal{A}$ is said to require eventual exponential positivity if every matrix $A \in Q(\mathcal{A})$ is eventually exponentially positive; see e.g., [7]. It is obvious that if an arbitrary sign pattern $\mathcal{A}$ requires eventual exponential positivity, then $\mathcal{A}$ is also PEEP. But the converse is not true in general. We conclude this paper by establishing some interesting characterizations about PEEP sign patterns which follow readily from Theorem 2.8, Corollary 3.4 and Theorem 2.9 in [7].

THEOREM 3.5. *Let $\mathcal{GS}_{2n-1}$ be a $(2n-1)$-by-$(2n-1)$ 2-generalized star sign pattern. Then the following are equivalent:*

(1) $\mathcal{GS}_{2n-1}$ is PEEP;

(2) All arcs of simple 2-cycles (v_i, v_j), (v_j, v_i) of the signed digraph $\Gamma(\mathcal{GS}_{2n-1})$ are positive;

(3) All nonzero off-diagonal entries of $\mathcal{GS}_{2n-1}$ are $+$;

(4) $\mathcal{GS}_{2n-1}$ requires eventual exponential positivity;

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(5) $\mathcal{GS}_{2n-1}$ requires eventual positivity.

Note that (2) of Theorem 3.5 is a combinatorial characterization of the PEEP 2-generalized star sign pattern.

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