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INEQUALITIES BETWEEN $\mid A \mid + \mid B \mid$ AND $\mid A^* \mid + \mid B^* \mid^*$

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Abstract. Let A and B be complex square matrices. Some inequalities between $\mid A \mid + \mid B \mid$ and $\mid A^* \mid + \mid B^* \mid$ are established. Applications of these inequalities are also given. For example, in the Frobenius norm,

$$\parallel A + B \parallel_F \leq \sqrt[4]{2} \parallel \mid A \mid + \mid B \mid \parallel_F .$$

Key words. Unitarily invariant norms, Frobenius norm, Singular values.

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1. Introduction. We denote by M_n the vector space of all complex $n \times n$ matrices with the inner product $\langle X, Y \rangle = \mathrm{tr}(Y^*X)$, where $\mathrm{tr}\, X$ denotes the trace of X and Y^* is the conjugate transpose of Y. Let the eigenvalues of $A \in M_n$ be $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. We denote $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ and $|\lambda(A)| = (|\lambda_1|, \ldots, |\lambda_n|)$. The *singular values* of $A \in M_n$ are the nonnegative square roots of the eigenvalues of A^*A. The absolute value of $A \in M_n$ is $\mid A \mid = (A^*A)^{\frac{1}{2}}$. Thus, the singular values of A are the eigenvalues of $|A|$. We denote the singular values of $A \in M_n$ by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ and denote $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$. The operator norm on M_n induced by the Euclidean norm $\| \cdot \|$ on $\mathbb{C}^n$ is the *spectral norm*:

$$\parallel A \parallel_\infty = \max\{\parallel Ax \parallel : \parallel x \parallel = 1,\ x \in \mathbb{C}_n\}.$$

The Euclidean norm on M_n is the *Frobenius norm*:

$$\parallel A \parallel_F := \left(\sum_{i,j} \mid a_{ij} \mid^2 \right)^{\frac{1}{2}} = (\mathrm{tr}\,(A^*A))^{\frac{1}{2}} = \left(\sum_{i=1}^n s_i^2(A) \right)^{\frac{1}{2}}, \quad A = (a_{ij}) \in M_n.$$

A norm on M_n is *unitarily invariant* if $\parallel UAV \parallel = \parallel A \parallel$ for any $A \in M_n$ and any unitary $U, V \in M_n$. The spectral norm and the Frobenius norm are unitarily invariant.

Our work is motivated by the following inequalities due to Lee in [5]. Let $A, B \in M_n$. Then for every unitarily norm,

$$\parallel A + B \parallel \leq \parallel \mid A \mid + \mid B \mid \parallel^{\frac{1}{2}} \parallel \mid A^* \mid + \mid B^* \mid \parallel^{\frac{1}{2}} \tag{1.1}$$

$$\leq \max\{\parallel \mid A \mid + \mid B \mid \parallel, \parallel \mid A^* \mid + \mid B^* \mid \parallel\}. \tag{1.2}$$

For the topic of norm inequalities and singular value inequalities, see [3, 6].

In this note, we focus on establishing inequalities between $|A| + |B|$ and $|A^*| + |B^*|$. Applications of these inequalities are also given.

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2. Auxilliary results and proofs. We now list some lemmas that are used in our proofs.

LEMMA 2.1. *If $A, B \in M_n$ and $s_i(A) \leq s_i(B)$ for all $i = 1, \ldots, n$, then for every unitarily invariant norm, $\|A\| \leq \|B\|$.*

LEMMA 2.2. [7, Theorem 1.27] *Let A, $B \in M_n$. Then AB and BA have the same eigenvalues (multiplicities counted).*

LEMMA 2.3. (Fan, [2]) *Let $A, B \in M_n$, $1 \leq i, j \leq n$, $i + j - 1 \leq n$. Then*

$$s_{i+j-1}(AB) \leq s_i(A)s_j(B).$$

In particular, $s_j(AB) \leq s_1(A)s_j(B)$, $s_j(AB) \leq s_1(B)s_j(A)$.

LEMMA 2.4. [7, p. 101] *Let $A, B \in M_n$ be positive semidefinite. Then*

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| \leq \| A + B \|$$

for all unitarily invariant norms.

PROPOSITION 2.5. *Let $A, B \in M_n$. Then for $1 \leq j \leq n$,*

$$s_j(| A^* | + | B^* |) \leq 2s_j(| A | \oplus | B |). \tag{2.3}$$

These inequalities are sharp.

Proof. Let $A = U \mid A \mid$ and $B = V \mid B \mid$ be polar decompositions with U, V unitary. Then we have

$$| A^* |= U \mid A \mid U^*, \quad | B^* |= V \mid B \mid V^*.$$

Denote

$$P_0 = \begin{pmatrix} I & U^*V \\ V^*U & I \end{pmatrix}, \quad Q = \begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}.$$

Then $P_0^* = P_0 = \frac{1}{2}P_0^2$ and P_0 is positive semidefinite with $s_1(P_0) = 2$. Applying Lemma 2.2 and $P_0 = \frac{1}{2}P_0^2$, we have

$$\begin{aligned}
\lambda\left((| A^* | + | B^* |) \oplus 0\right) &= \lambda\left(\begin{pmatrix} U & V \\ 0 & 0 \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\begin{pmatrix} U^* & 0 \\ V^* & 0 \end{pmatrix}\right) \\
&= \lambda\left(\begin{pmatrix} U^* & 0 \\ V^* & 0 \end{pmatrix}\begin{pmatrix} U & V \\ 0 & 0 \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\right) \\
&= \lambda\left(\begin{pmatrix} I & U^*V \\ V^*U & I \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\right) \\
&= \lambda\left(P_0Q\right) \\
&= \lambda\left(\frac{1}{2}P_0^2Q\right) \\
&= \lambda\left(\frac{1}{2}P_0QP_0\right).
\end{aligned}$$

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Note that both $|A^*|+|B^*|$ and P_0QP_0 are positive semidefinite. Since for positive semidefinite matrices singular values and eigenvalues are the same, applying Lemma 2.3 we obtain for $1 \le j \le n$,

$$\begin{aligned} s_j\left(|A^*|+|B^*|\right)\oplus 0) &= s_j\left(|A^*|+|B^*|\right)) \\ &= s_j\left(\frac{1}{2}P_0QP_0\right) \\ &\le s_j\left(QP_0\right) \\ &\le 2s_j\left(Q\right) \\ &= 2s_j\left(|A|\oplus|B|\right). \end{aligned}$$

Next we show that equality is possible in the inequalities (2.3) for some nonzero square matrices A and B. Consider

$$A=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

where I is the identity matrix of order n. A calculation indicates that

$$|A|\oplus|B|=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\oplus\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad |A^*|+|B^*|=\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $s_j(|A^*|+|B^*|)=2s_j\left(|A|\oplus|B|\right)=2$, for $1\le j\le n$. □

Applying Proposition 2.5 and Lemmas 2.1 and 2.4 we deduce the following corollary.

Corollary 2.6. *Let $A, B \in M_n$. Then we have*

$$|||\, |A^*|+|B^*|\, ||| \le 2\, |||\, |A|+|B|\, ||| \tag{2.4}$$

for every unitarily invariant norm.

Remark 2.7. The inequality (2.4) is sharp for the spectral norm. Consider

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

Then

$$|A|+|B|=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |A^*|+|B^*|=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

and so,

$$|||\, |A^*|+|B^*|\, |||_\infty = 2\, |||\, |A|+|B|\, |||_\infty = 2.$$

Bourin and Uchiyama [1] proved the following triangle inequality: Let A, $B \in M_n$ be normal. Then for all unitarily invariant norms,

$$\|A+B\| \le |||\, |A|+|B|\, |||\,. \tag{2.6}$$

In the general case, Lee [5] proved for all $A, B \in M_n$ and all unitarily invariant norms,

$$\|A+B\| \le \sqrt{2}\, |||\, |A|+|B|\, |||\,. \tag{2.7}$$

The following result interpolates Lee's inequality (2.7).

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THEOREM 2.8. *Let $A, B \in M_n$. Then for all unitarily invariant norms,*

$$\| A + B \| \leq \sqrt{2} \,\||\, A \mid + \mid B \,\||^{\frac{1}{2}} \||\, A \mid \oplus \mid B \,\||^{\frac{1}{2}} \leq \sqrt{2} \,\||\, A \mid + \mid B \,\|| \,. \tag{2.8}$$

Proof. Use Lee's inequality (1.1) to compute

$$\begin{aligned} \| A + B \| &\leq \| \mid A \mid + \mid B \mid \|^{\frac{1}{2}} \| \mid A^* \mid + \mid B^* \mid \|^{\frac{1}{2}} \\ &\leq \sqrt{2} \,\||\, A \mid + \mid B \,\||^{\frac{1}{2}} \||\, A \mid \oplus \mid B \,\||^{\frac{1}{2}} \\ &\leq \sqrt{2} \,\||\, A \mid + \mid B \,\||, \end{aligned}$$

where the second inequality follows from Proposition 2.5 and Lemma 2.1 and the last inequality follows from Lemma 2.4. ∎

In the case of the Frobenius norm, we can improve the inequality (2.4).

PROPOSITION 2.9. *Let $A, B \in M_n$. Then*

$$\||\, A^* \mid + \mid B^* \,\||_F \leq \sqrt{2} \,\||\, A \mid + \mid B \,\||_F \tag{2.9}$$

and this inequality is sharp.

Proof. Let $A = U \mid A \mid$ and $B = V \mid B \mid$ be polar decompositions with U, V unitary. Then

$$\mid A^* \mid = U \mid A \mid U^*, \quad \mid B^* \mid = V \mid B \mid V^*.$$

Since $tr(XY) = tr(YX)$, we deduce that

$$tr(|A|) = tr(|A^*|), \quad tr(|B|) = tr(|B^*|)$$

and

$$tr(|A||B|) = tr(|B||A|) = tr(|A|^{\frac{1}{2}}|B||A|^{\frac{1}{2}}) \geq 0.$$

Compute

$$\begin{aligned} \||\, A^* \mid + \mid B^* \,\||_F^2 &= tr(\mid A^* \mid^2 + \mid B^* \mid^2) + 2tr(\mid A^* \mid\mid B^* \mid) \\ &\leq tr(\mid A^* \mid^2 + \mid B^* \mid^2) + 2tr(\mid A^* \mid^2)^{\frac{1}{2}}\; tr(\mid B^* \mid^2)^{\frac{1}{2}} \\ &\leq tr(\mid A^* \mid^2 + \mid B^* \mid^2) + tr(\mid A^* \mid^2) + tr(\mid B^* \mid^2) \\ &= 2tr(\mid A^* \mid^2 + \mid B^* \mid^2) \\ &= 2tr(\mid A \mid^2 + \mid B \mid^2) \\ &\leq 2tr[(\mid A \mid + \mid B \mid)^2] \\ &= 2 \,\||\, A \mid + \mid B \,\||_F^2 \,. \end{aligned}$$

For the matrices in (2.5), we have $\||\, A \mid + \mid B \,\||_F = \sqrt{2}$ and $\||\, A^* \mid + \mid B^* \,\||_F = 2$, which shows that equality is possible in (2.9). ∎

THEOREM 2.10. *Let $A, B \in M_n$. Then*

$$\||\, A + B \,\||_F \leq \| \sqrt[4]{2} \,\||\, A \mid + \mid B \,\||_F \tag{2.10}$$

Proof. Apply Lee's inequality (1.1) and Proposition 2.9 to obtain

$$\begin{aligned} \| A + B \|_F &\leq \||\, A \mid + \mid B \,\||_F^{\frac{1}{2}} \||\, A^* \mid + \mid B^* \,\||_F^{\frac{1}{2}} \\ &\leq \sqrt[4]{2} \,\||\, A \mid + \mid B \,\||_F \,. \end{aligned}$$

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E.Y. Lee [4, 5] conjectured that for the Frobenius norm, the inequality

$$\||| A+B |||_F \le \sqrt{\frac{1+\sqrt{2}}{2}}\, \||| A| + |B| |||_F$$

holds. Note that $\sqrt{\frac{1+\sqrt{2}}{2}} \approx 1.099$ and $\sqrt[4]{2} \approx 1.189$. Although the factor $\sqrt[4]{2}$ in Theorem 2.10 is close to $\sqrt{\frac{1+\sqrt{2}}{2}}$, Lee's conjecture is still open.

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