Electronic Journal of Linear Algebra

Volume 35 *Volume 35* Article 5

2019

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Recommended Citation

Soto, Ricardo L.; Julio, Ana I.; and Collao, Macarena A.. (2019), "Brauer's theorem and nonnegative matrices with prescribed diagonal entries", *Electronic Journal of Linear Algebra*, Volume 35, pp. 53-64.
DOI: https://doi.org/10.13001/1081-3810.3886

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
http://repository.uwyo.edu/ela

BRAUER'S THEOREM AND NONNEGATIVE MATRICES WITH PRESCRIBED DIAGONAL ENTRIES*

RICARDO L. SOTO†, ANA I. JULIO†, AND MACARENA COLLAO†

Abstract. The problem of the existence and construction of nonnegative matrices with prescribed eigenvalues and diagonal entries is an important inverse problem, interesting by itself, but also necessary to apply a perturbation result, which has played an important role in the study of certain nonnegative inverse spectral problems. A number of partial results about the problem have been published by several authors, mainly by H. Šmigoc. In this paper, the relevance of a Brauer's result, and its implication for the nonnegative inverse eigenvalue problem with prescribed diagonal entries is emphasized. As a consequence, given a list of complex numbers of Šmigoc type, or a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\operatorname{Re}\lambda_i \leq 0$, $\lambda_1 \geq -\sum_{i=2}^{n} \lambda_i$, and $\left\{-\sum_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\right\}$ being realizable; and given a list of nonnegative real numbers $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, the remarkably simple condition $\gamma_1 + \cdots + \gamma_n = \lambda_1 + \cdots + \lambda_n$ is necessary and sufficient for the existence and construction of a realizing matrix with diagonal entries Γ. Conditions for more general lists of complex numbers are also given.

Key words. Inverse eigenvalue problem, Nonnegative matrix, Prescribed diagonal entries.

AMS subject classifications. 15A18, 15A29.

1. Introduction. The problem of the existence and construction of nonnegative matrices with prescribed eigenvalues and diagonal entries is an important inverse problem, interesting by itself but also necessary to apply a perturbation result, due to R. Rado and published by H. Perfect [12], which has played an important role in the study of the nonnegative inverse eigenvalue problem *(NIEP),* and the nonnegative universal realizability problem *(NURP)*, which is the problem of finding necessary and sufficient conditions for the existence of a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ for any possible Jordan canonical form *(JCF)* allowed by Λ. A first motivation for this work was a result due to Fillmore [7] and extended by Borobia in [2]. In 1969, Fillmore [7] proved that if A is an $n \times n$ non-scalar matrix over a field $\mathbb{F}$, and $\gamma_1, \ldots, \gamma_n \in \mathbb{F}$ with $\sum_{i=1}^{n} \gamma_i = trA$, then there exists a matrix B similar to A having diagonal entries $\gamma_1, \ldots, \gamma_n$. The proof in [7] is by induction on n, and it provides, implicity, an algorithm in $(n-1)$ steps, to construct a matrix B. Borobia [2] develops an explicit and simple algorithm, in only two steps, to compute the matrix B, and then extends Fillmore Theorem to matrices with integer entries. In this work, we present a different and very simple way to compute a matrix B similar to A with arbitrarily prescribed diagonal entries $\gamma_1, \ldots, \gamma_n$ (except for the trace property).

Results of Fillmore and Borobia do not consider the nonnegativity hypothesis. Our main motivation is to consider the nonnegativity hypothesis and to give conditions under which a given list of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, is realizable by a nonnegative matrix with arbitrarily prescribed diagonal entries. In particular, we show that given a list of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, which is the spectrum of a

*Received by the editors on September 10, 2018. Accepted for publication on January 21, 2019. Handling Editor: Michael Tsatsomeros. Corresponding Author: Ricardo L. Soto.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
http://repository.uwyo.edu/ela

nonnegative matrix, with λ_1 being the Perron eigenvalue, and λ_i, $i = 2, \ldots, n$, belonging to

$$i)\ \mathcal{F} = \{\lambda_i \in \mathbb{C} : Re\lambda_i \leq 0,\ \ |Re\lambda_i| \geq |Im\lambda_i|\}, \tag{1.1}$$

or

$$ii)\ \mathcal{G} = \left\{\lambda_i \in \mathbb{C} : Re\lambda_i \leq 0,\ \ \left|\sqrt{3}Re\lambda_i\right| \geq |Im\lambda_i|\right\}, \tag{1.2}$$

and given real nonnegative numbers $\gamma_1, \ldots, \gamma_n$ such that

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i, \tag{1.3}$$

then there exists an $n \times n$ nonnegative matrix A with spectrum Λ and diagonal entries $\gamma_1, \ldots, \gamma_n$ if and only if (1.3) holds.

Lists $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ in the left half plane, with λ_i, $i = 2, \ldots, n$, belonging to

$$iii)\ \mathcal{H} = \{\lambda_i \in \mathbb{C} : Re\lambda_i \leq 0\},\ \text{with } \lambda_1 \geq -\sum_{i=2}^{n} \lambda_i,\ \text{and}$$

$$\Lambda' = \left\{-\sum_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\right\}\ \text{ being realizable,}$$

are also realizable with spectrum Λ and with diagonal entries $\gamma_1, \ldots, \gamma_n$ if and only if (1.3) holds. It is well known that under condition (1.3), lists with the property $i)$ (lists of Suleimanova type) and lists with the property $ii)$ (lists of Šmigoc type) are realizable. The novelty here is that, under the remarkably simple condition (1.3), Λ is not only realizable, but it is realizable with arbitrary prescribed diagonal entries. This surprising simple result simplifies, significantly, for lists $i)$, $ii)$ and $iii)$, the procedure proposed by Ellard and Šmigoc in [6]. We also consider more general lists of complex numbers, and we prove that they are the spectrum of a nonnegative matrix with arbitrarily prescribed diagonal entries if and only if the trace property holds. As a by product of these results, we also consider the nonnegative structured case and the universal realizability of lists with prescribed diagonal entries.

The set of all matrices with constant row sums equal to α will be denoted by $\mathcal{CS}_\alpha$. It is clear that $\mathbf{e}^T = [1, 1, \ldots, 1]$ is an eigenvector of any matrix $A \in \mathcal{CS}_\alpha$, corresponding to the eigenvalue α. The importance of matrices with constant row sums is due to the well known fact that if λ_1 is the desired Perron eigenvalue, then the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum Λ. We shall say that the list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is realizable if there is a nonnegative matrix A with spectrum Λ. In this case, we say that A is the realizing matrix. Our main tools will be two perturbation results, due to Brauer [3] and Rado [12], and a lemma, which tell us how is the JCF of the Brauer's perturbation.

Theorem 1.1. [3] *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{v}^T = [v_1, v_2, \ldots, v_n]$ be an eigenvector of A associated with the eigenvalue λ_k and let $\mathbf{q}$ be any $n-$dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$, $k = 1, 2, \ldots, n$.*

Lemma 1.2. [14] *Let $A \in \mathcal{CS}_{\lambda_1}$ be a matrix with Jordan canonical form $J(A)$. Let $\mathbf{q}^T = [q_1, \ldots, q_n]$ be an arbitrary $n-$dimensional vector such that $\lambda_1 + \sum\limits_{i=1}^{n} q_i \neq \lambda_i$, $i = 2, \ldots, n$. Then the matrix $A + \mathbf{e}\mathbf{q}^T$ has Jordan form $J(A) + \left(\sum\limits_{i=1}^{n} q_i\right) E_{11}$. In particular, if $\sum\limits_{i=1}^{n} q_i = 0$, then A and $A + \mathbf{e}\mathbf{q}^T$ are similar.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
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THEOREM 1.3. [12] *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let $\Omega = diag\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ for some $r < n$. Let X be an $n \times r$ matrix with rank r such that its columns $\mathbf{x}_1, \ldots, \mathbf{x}_r$ satisfy $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, 2, \ldots, r$. Let C be an $r \times n$ arbitrary matrix. Then the matrix $A + XC$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \mu_2, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + CX$.*

The paper is organized as follows: In Section 2, we consider the general case, that is, we introduce a very simple way to compute a matrix B with arbitrarily prescribed diagonal entries, which is similar to a given non-scalar matrix A. In Section 3, we consider the nonnegative case and we show a nice and useful consequence of Brauer's result, Theorem 1.1, which allow us to give necessary and sufficient conditions for the existence and construction of nonnegative matrices with prescribed eigenvalues and diagonal entries. In particular, we show that realizable lists of Suleimanova and Šmigoc type, are realizable by a nonnegative matrix with arbitrarily prescribed diagonal entries if and only if the trace property is satisfied. We also show that a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers in the left half plane, with $\lambda_i \in \mathcal{H}$, $i = 2, \ldots n$, $\lambda_1 \geq -\sum\limits_{i=2}^{n} \lambda_i$, and $\left\{-\sum\limits_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\right\}$ being realizable, is in particular realizable by a nonnegative matrix with prescribed diagonal entries $\gamma_1, \ldots, \gamma_n$ if and only if $\sum\limits_{i=1}^{n} \gamma_i = \sum\limits_{i=1}^{n} \lambda_i$. In Section 4, we consider structured matrices and give sufficient conditions for the existence of symmetric, normal, and persymmetric matrices with prescribed eigenvalues and diagonal entries. Finally, in Section 5, we study some lists universally realizable with prescribed diagonal entries. We provided some examples to illustrate the results.

2. The general case. In this section, we introduce a very simple way to compute a matrix B similar to a given non-scalar matrix A, with B having arbitrarily prescribed diagonal entries $\gamma_1, \ldots, \gamma_n$. Of course, we need that $\sum\limits_{i=1}^{n} \gamma_i = trA$. Then we have:

THEOREM 2.1. *Let $A = (a_{ij})$ be an $n \times n$ non-scalar complex matrix and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a given list of complex numbers. Let $\mathbf{x}^T = [x_1, \ldots, x_n]$ be an eigenvector of A having all its entries nonzero. Then there exists a matrix B similar to A with diagonal entries $\gamma_1, \ldots, \gamma_n$, if and only if $\sum\limits_{i=1}^{n} \gamma_i = trA$.*

Proof. It is clear that the condition is necessary. Let $\sum\limits_{i=1}^{n} \gamma_i = trA$. First, let A be an $n \times n$ non-diagonal matrix, and let $D = diag\{x_1, \ldots, x_n\}$. Then D is a nonsingular matrix and $D^{-1}AD \in \mathcal{CS}_\lambda$, where $A\mathbf{x} = \lambda \mathbf{x}$. Let

$$\mathbf{q}^T = [q_1, \ldots, q_n] \text{ with } q_i = \gamma_i - a_{ii},\ i = 1, \ldots, n.$$

Then, $B = D^{-1}AD + \mathbf{e}\mathbf{q}^T$ has diagonal entries $\gamma_1, \ldots, \gamma_n$ and from Lemma 1.2, B is similar to A. Second, let A be diagonal. Consider

$$S = \begin{bmatrix} 1 & \mathbf{0}^T \\ -\mathbf{e} & I_{n-1} \end{bmatrix} \text{ and } S^{-1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{e} & I_{n-1} \end{bmatrix}.$$

Then $S^{-1}AS \in \mathcal{CS}_\lambda$, where λ is an eigenvalue of A (a diagonal entry of A), and

$$B = S^{-1}AS + \mathbf{e}\mathbf{q}^T, \text{ with } q_i = \gamma_i - a_{ii},$$

is similar to A and has diagonal entries $\gamma_1, \ldots, \gamma_n$. ∎

Let us consider [2, Example 6]:

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
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Example 1. We want to construct a matrix with diagonal entries $3, 5, -2, 6, -1$ and similar to

$$A = \begin{bmatrix} 4 & 0 & 4 & -3 & 5 \\ 2 & 3 & 0 & 2 & 3 \\ 0 & -2 & 2 & 5 & 4 \\ 7 & 1 & 3 & 4 & 0 \\ 2 & 5 & 3 & 0 & -2 \end{bmatrix}.$$

A has not constant row sums, but it has an eigenvector with all nonzero entries

$$\mathbf{x}^T = [x_i]_{i=1}^5 = \begin{bmatrix} \frac{497}{601} & \frac{1259}{1033} & \frac{335}{241} & \frac{1698}{899} & 1 \end{bmatrix}.$$

Let $D = diag\{x_1, \ldots, x_5\}$. Then

$$D^{-1}AD = \begin{bmatrix} 4 & 0 & \frac{3893}{579} & -\frac{1713}{250} & \frac{2352}{389} \\ \frac{802}{591} & 3 & 0 & \frac{1652}{533} & \frac{3131}{1272} \\ 0 & -\frac{2078}{1185} & 2 & \frac{2901}{427} & \frac{964}{335} \\ \frac{2081}{679} & \frac{533}{826} & \frac{1742}{789} & 4 & 0 \\ \frac{994}{601} & \frac{1298}{213} & \frac{1005}{241} & 0 & -2 \end{bmatrix} \in \mathcal{CS}_{\frac{5197}{524}}.$$

Now, from Theorem 2.1 we have that

$$B = D^{-1}AD + \mathbf{e}\,[-1, 2, -4, 2, 1]$$

has the required diagonal entries $3, 5, -2, 6, -1$. Moreover, from Lemma 1.2 B is similar to A.

The next result shows that if A is an integer matrix, we may construct new integer matrices with prescribed integer diagonal entries.

Corollary 2.2. *Let $A \in \mathcal{CS}_\lambda$ be an $n \times n$ non-scalar real matrix with integer entries, and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a given list of integer real numbers. Then, there exists an integer matrix B similar to A, with diagonal entries $\gamma_1, \ldots, \gamma_n$, if and only if $\sum\limits_{i=1}^{n} \gamma_i = trA$.*

3. The nonnegative case. Brauer's result, Theorem 1.1, has played an important role in the study of the *NIEP* and the *NURP*. Another important issue of Theorem 1.1, is that it can be efficiently applied to the *NIEP* with prescribed diagonal entries, which is the problem we study in this paper. In fact, a straightforward consequence of Theorem 1.1 is the following result:

Lemma 3.1. *Let $\Lambda' = \left\{-\sum\limits_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\right\}$ be a realizable list of complex numbers, with Perron eigenvalue $-\sum\limits_{i=2}^{n} \lambda_i$. Then for any $\lambda_1 \geq -\sum\limits_{i=2}^{n} \lambda_i$, the list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative matrix with prescribed diagonal entries $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, if and only if $\sum\limits_{i=1}^{n} \gamma_i = \sum\limits_{i=1}^{n} \lambda_i$.*

Proof. Let $\sum\limits_{i=1}^{n} \gamma_i = \sum\limits_{i=1}^{n} \lambda_i$. Since Λ' is realizable, there exists a nonnegative matrix $B \in \mathcal{CS}_\beta$, where $\beta = -\sum\limits_{i=2}^{n} \lambda_i$, with spectrum Λ' and $tr(B) = 0$. Let $\gamma^T = [\gamma_1, \ldots, \gamma_n]$. Then, from Theorem 1.1, $A = B + \mathbf{e}\gamma^T$ is a nonnegative matrix with spectrum Λ and with diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$. It is clear that the condition is necessary. $\square$

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A publication of the International Linear Algebra Society
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We point out that Lemma 3.1 can be used for more general lists Λ', not necessarily realizable lists with zero trace. However, in this case, we cannot obtain all possible values of diagonal entries γ_i.

EXAMPLE 2. The lists $\Lambda_1 = \{7, 5, 1, -3, -4, -6\}$, $\Lambda_2 = \{6, 1, 1, -4, -4\}$, and $\Lambda_3 = \{4, -2, -1 + 2i, -1 - 2i\}$, satisfy conditions of Lemma 3.1. Then, if $t \geq 0$, the lists $\Lambda_{1,t} = \{7 + t, 5, 1, -3, -4, -6\}$, $\Lambda_{2,t} = \{6 + t, 1, 1, -4, -4\}$, and $\Lambda_{3,t} = \{4 + t, -2, -1 + 2i, -1 - 2i$, are realizable by a nonnegative matrix with diagonal entries $\gamma_1, \ldots, \gamma_n$, if and only if $\sum\limits_{i=1}^{n} \gamma_i = t$. In particular, if $\Lambda = \{5, 4, 0, -3, -3, -3\}$, we may compute a realizing matrix for Λ (see [15, Example 3.3]), that is,

$$B = \begin{bmatrix} 0 & 3 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 2 & 0 \end{bmatrix}.$$

Then $A = B + \mathbf{e}\gamma^T$, with $\gamma^T = [\gamma_1, \gamma_2, \ldots, \gamma_6] \geq 0$, $\sum\limits_{i=1}^{6} \gamma_i = t$, is nonnegative with spectrum $\Lambda_t = \{5 + t, 4, 0, -3, -3, -3\}$ and diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_6$.

Rado's result, Theorem 1.3, is a perturbation result, due to R. Rado and published by Perfect in [12]. It says that we may simultaneously change r eigenvalues of an $n \times n$ matrix, $r < n$, without changing the $n - r$ remaining eigenvalues. Rado's result has also played an important role in the determination of sufficient conditions for a solution to both problems, the *NIEP* and the *NURP.* Since Rado's result is a generalization of Brauer's result, it gives in general, better information about the realizability of a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. For instance, the lists $\Lambda_2 = \{6, 1, 1, -4, -4\}$, $\Lambda_4 = \{6, 3, 3, -5, -5\}$, and $\Lambda_5 = \{5, 4, 0, -3, -3, -3\}$, are not Brauer realizable, but they are Rado realizable. Observe that to apply Theorem 1.3 we need to guarantee the existence of an $r \times r$ matrix, $r < n$, with prescribed eigenvalues and diagonal entries. This is precisely the problem that interests us in this paper.

We also show that if the trace condition (1.3) is satisfied, then lists $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with only one positive eigenvalue, of Šmigoc type in (1.2), are always realizable by a nonnegative matrix with arbitrarily prescribed diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$. The following results are a consequence of Lemma 3.1:

COROLLARY 3.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a realizable list of complex numbers of Šmigoc type, that is, with $\lambda_i \in \mathcal{G}$, $i = 2, \ldots, n$. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a list of nonnegative real numbers. Then there exists a nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$ with spectrum Λ and diagonal entries $\gamma_1, \ldots, \gamma_n$, if and only if $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i$.*

Proof. The condition is necessary. Let $\Lambda' = \{-\sum_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\}$. Then Λ' is Šmigoc realizable, that is, there exists a nonnegative matrix B with spectrum Λ' and $tr(B) = 0$. Observe that B can be taken with constant row sums equal to $-\sum_{i=2}^{n} \lambda_i$. Let $\gamma^T = [\gamma_1, \gamma_2, \ldots, \gamma_n]$ with $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i$. Then, $A = B + \mathbf{e}\gamma^T$ is nonnegative with spectrum Λ and diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$. ∎

Since lists in (1.2) contains lists in (1.1), it is clear that lists of Suleimanova type also satisfies Corollary 3.2.

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A publication of the International Linear Algebra Society
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EXAMPLE 3. Consider the list

$$\Lambda = \{16, -1, -2, -2+2i, -2-2i, -2+3i, -2-3i\}.$$

We want to compute a nonnegative matrix with spectrum Λ and diagonal entries $\{0, 1, 2, 0, 2, 0, 0\}$. Then we take the list

$$\Lambda' = \{11, -1, -2, -2+2i, -2-2i, -2+3i, -2-3i\},$$

which is Šmigoc realizable by the nonnegative matrix

$$B = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 & 7 \\ 2 & 0 & 2 & 0 & 0 & 0 & 7 \\ 0 & 4 & 0 & 0 & 0 & 0 & 7 \\ \frac{30}{11} & 0 & 0 & 0 & 0 & 4 & \frac{47}{11} \\ \frac{30}{11} & 0 & 0 & \frac{13}{4} & 0 & \frac{3}{4} & \frac{47}{11} \\ \frac{30}{11} & 0 & 0 & 0 & 4 & 0 & \frac{47}{11} \\ 0 & 0 & 0 & 11 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{CS}_{11}.$$

Hence, $A = B + \mathbf{e}\gamma^T$, with $\gamma^T = [0, 1, 2, 0, 2, 0, 0]$, is nonnegative with spectrum Λ and desired diagonal entries.

In [11, Theorem 3], Laffey and Šmigoc solved the *NIEP* in the left half plane. They proved that if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a list of complex numbers with $\operatorname{Re}\lambda_i \leq 0$, $\Lambda = \overline{\Lambda}$, $\lambda_1 \geq |\lambda_i|$, $i = 2, \ldots, n$, then Λ is realizable if and only if

$$s_1 = \sum_{i=1}^{n} \lambda_i \geq 0, \quad s_2 = \sum_{i=1}^{n} \lambda_i^2 \geq 0, \quad s_1^2 \leq n s_2. \tag{3.4}$$

Now, we show that lists in the left half plane, $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with $\lambda_i \in \mathcal{H}$, $i = 2, \ldots, n$, can also be realizable with prescribed diagonal entries.

COROLLARY 3.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i \in \mathcal{H}$, $i = 2, \ldots, n$, be a list of complex numbers such that $\lambda_1 \geq -\sum_{i=2}^{n} \lambda_i$ and $\Lambda' = \{-\sum_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n\}$ is realizable. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a list of nonnegative real numbers. Then there exists a nonnegative matrix A with spectrum Λ and diagonal entries $\gamma_1, \ldots, \gamma_n$ if and only if $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i$.*

Proof. The condition is necessary. Let $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i$. Since Λ' is realizable, there exists a nonnegative matrix B with constant row sums equal to $-\sum_{i=2}^{n} \lambda_i$, having the spectrum Λ' and $tr(B) = 0$. Let $\gamma^T = [\gamma_1, \ldots, \gamma_n]$. Then, $A = B + \mathbf{e}\gamma^T$ is nonnegative with the spectrum Λ and diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$. ∎

EXAMPLE 4. Consider, in the left half plane, the list

$$\Lambda = \{15, -1, -2, -3+5i, -3-5i, -2+5i, -2-5i\},$$

with

$$\Lambda' = \{13, -1, -2, -3+5i, -3-5i, -2+5i, -2-5i\}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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Λ' satisfies conditions (3.4). Then it is realizable by the companion matrix

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 25\,636 & 44\,542 & 23\,592 & 5593 & 956 & 50 & 0 \end{bmatrix}.$$

$\mathcal{C}$ has a positive eigenvector $\mathbf{v}^T = [v_i]_{i=1}^7 = \left[1, a, a^2, a^3, a^4, a^5, a^6\right]$ with $a = 13$. Then if $D = diag\{1, a, \ldots, a^6\}$, $B = D^{-1}\mathcal{C}D \in \mathcal{CS}_{13}$ is nonnegative with spectrum Λ' and $tr(B) = 0$. Hence, if $\gamma^T = [\gamma_1, \ldots, \gamma_n]$ is nonnegative with $\sum_{i=1}^n \gamma_i = 2$, the matrix $A = B + \mathbf{e}\gamma^T$ is nonnegative with the spectrum Λ and diagonal entries $\gamma_1, \ldots, \gamma_n$.

4. The nonnegative structured case. We recall the following result in [1, Theorem 2.5], which shows how Theorem 1.3 is applied to compute a solution to the problem of finding a nonnegative matrix with prescribed spectrum and diagonal entries:

THEOREM 4.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a list of nonnegative real numbers with $\displaystyle\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \gamma_i$. Suppose there exist partitions*

$$\begin{aligned} &\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{p_0} \text{ and } \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{p_0}, \text{ with} \\ &\Lambda_0 = \{\lambda_{01}, \ldots, \lambda_{0p_0}\}, \quad \lambda_{0j} \in \Lambda, \ j = 1, \ldots, p_0, \ \lambda_{01} = \lambda_1, \\ &\Lambda_k = \{\lambda_{k1}, \ldots, \lambda_{kp_k}\}, \ \Gamma_k = \{\gamma_{k1}, \ldots, \gamma_{kp_k}, \gamma_{k(p_k+1)}\}, \ \lambda_{ki} \in \Lambda, \ \gamma_{kj} \in \Gamma, \\ &\ k = 1, 2, \ldots, p_0, \quad i = 1, \ldots, p_k, \quad j = 1, \ldots, p_k + 1, \end{aligned}$$

where some lists Λ_k, $k = 1, 2, \ldots, p_0$, can be empty. Suppose that the following conditions hold:

i) For each $k = 1, 2, \ldots, p_0$, there exists a nonnegative matrix, with spectrum $\mathcal{U}_k = \{\mu_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}$ and diagonal entries Γ_k.

ii) There exists a $p_0 \times p_0$ nonnegative matrix with spectrum Λ_0 and diagonal entries $\mu_1, \mu_2, \ldots, \mu_{p_0}$.

Then there exists a nonnegative matrix with spectrum Λ and diagonal entries Γ.

Theorem 4.1 can be easily applied to obtain structured nonnegative matrices with prescribed spectrum and diagonal entries. Here, we consider symmetric, normal, and persymmetric nonnegative matrices.

4.1. Symmetric nonnegative matrices with prescribed diagonal entries. In [13, Theorem 2.6], the authors prove a symmetric version of the Rado's result, Theorem 1.3. Based on this result and Theorem 4.1, we give sufficient conditions for the existence and construction of a symmetric nonnegative matrix with prescribed spectrum and diagonal entries:

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A publication of the International Linear Algebra Society
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Theorem 4.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be lists of real numbers, with $\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}\gamma_i$, and $\gamma_i \geq 0$, $i = 1, \ldots, n$. Suppose there exist partitions*

$$\begin{aligned}
&\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{p_0} \text{ and } \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{p_0}, \text{ with} \\
&\Lambda_0 = \{\lambda_{01}, \ldots, \lambda_{0p_0}\}, \quad \lambda_{0j} \in \Lambda, \ j = 1, \ldots, p_0, \ \lambda_{01} = \lambda_1, \\
&\Lambda_k = \{\lambda_{k1}, \ldots, \lambda_{kp_k}\}, \ \Gamma_k = \{\gamma_{k1}, \ldots, \gamma_{kp_k}, \gamma_{k(p_k+1)}\}, \\
&\quad k = 1, 2, \ldots, p_0, \quad \lambda_{ki} \in \Lambda, \ \gamma_{kj} \in \Gamma, \ i = 1, \ldots, p_k, \quad j = 1, \ldots, p_k + 1,
\end{aligned}$$

where some lists Λ_k, $k = 1, 2, \ldots, p_0$, can be empty. Suppose that the following conditions hold:

i) For each $k = 1, 2, \ldots, p_0$, there exists a symmetric nonnegative matrix, with spectrum $\mathcal{U}_k = \{\mu_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}$ and diagonal entries Γ_k.

ii) There exists a $p_0 \times p_0$ symmetric nonnegative matrix with spectrum Λ_0 and diagonal entries $\mu_1, \mu_2, \ldots, \mu_{p_0}$.

Then there exists a symmetric nonnegative matrix with spectrum Λ and diagonal entries Γ.

Proof. From *i)* let A_k be a symmetric nonnegative matrix with spectrum $\mathcal{U}_k$ and diagonal entries Γ_k. Then

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{p_0} \end{bmatrix}$$

is symmetric nonnegative with spectrum $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{p_0}$ and diagonal entries $\Gamma_1 \cup \cdots \cup \Gamma_{p_0}$. From *ii)* let B a symmetric nonnegative matrix with spectrum Λ_0 and diagonal entries $\mu_1, \mu_2, \ldots, \mu_{p_0}$. Then if X is the matrix of the normalized eigenvectors of A associated with the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{p_0}$, and $C = B - \Omega$, where $\Omega = diag\{\mu_1, \mu_2, \ldots, \mu_{p_0}\}$, then

$$M = A + XCX^T$$

is symmetric nonnegative with spectrum Λ and diagonal entries Γ. ∎

Remark 1. As an application of Theorem 4.2, it is easy to show by induction, that lists of real numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_1 > 0$, $\lambda_j \leq 0$, $j = 2, \ldots n$ (lists of real Suleimanova type), are realizable by a symmetric nonnegative matrix with diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$, if and only if $\sum_{i=1}^{n}\gamma_i = \sum_{i=1}^{n}\lambda_i$. For the proof it is enough to consider the partition

$$\begin{aligned}
&\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \text{ with} \\
&\Lambda_0 = \{\lambda_1, \lambda_2, \lambda_3\}, \ \Lambda_1 = \Lambda_3 = \emptyset, \ \Lambda_2 = \Lambda - \Lambda_0
\end{aligned}$$

with the realizable associated lists

$$\mathcal{U}_1 = \{\gamma_1\}, \ \mathcal{U}_3 = \{\gamma_n\}, \ \mathcal{U}_2 = \{\lambda_1 + \lambda_2 + \lambda_3 - \gamma_1 - \gamma_n, \lambda_4, \ldots, \lambda_n\}.$$

Example 5. We want to compute a symmetric nonnegative matrix with spectrum $\Lambda = \{10, 5, 1, -3, -4, -6\}$ and diagonal entries $\Gamma = \{1, 1, 1, 0, 0, 0\}$. We take the partition

$$\Lambda_0 = \{10, 5, 1\}, \ \Lambda_1 = \{-6\}, \ \Lambda_2 = \{-4\}, \ \Lambda_3 = \{-3\}$$

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A publication of the International Linear Algebra Society
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with realizable associated lits

$$\mathcal{U}_1 = \{8, -6\},\ \mathcal{U}_2 = \{5, -4\},\ \mathcal{U}_3 = \{3, -3\}$$

with diagonal entries

$$\Gamma_1 = \{1, 1\},\ \Gamma_2 = \{1, 0\},\ \Gamma_3 = \{0, 0\},\ \text{respectively.}$$

The symmetric nonnegative matrices

$$A_1 = \begin{bmatrix} 1 & 7 \\ 7 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 2\sqrt{5} \\ 2\sqrt{5} & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

have the required spectra and diagonal entries. From [13, Remark 3.3] we compute a symmetric nonnegative matrix B with spectrum Λ_0, and diagonal entries $8, 5, 3$. That is,

$$B = \begin{bmatrix} 8 & 2 & \sqrt{2} \\ 2 & 5 & 2\sqrt{2} \\ \sqrt{2} & 2\sqrt{2} & 3 \end{bmatrix},\quad \text{with}\quad C = \begin{bmatrix} 0 & 2 & \sqrt{2} \\ 2 & 0 & 2\sqrt{2} \\ \sqrt{2} & 2\sqrt{2} & 0 \end{bmatrix}.$$

Then, for

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix},\ X^T = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\sqrt{5} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix},$$

$M = A + XCX^T$ is symmetric nonnegative with spectrum Λ and diagonal entries Γ.

4.2. Normal nonnegative matrices with prescribed diagonal entries. In [8, Theorem 2.1], the authors prove a normal version of Theorem 1.3, and based on this result they give sufficient conditions for the existence of a normal nonnegative matrix with prescribed spectrum [8, Theorem 3.1] Then, we have:

THEOREM 4.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers, and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a list of nonnegative real numbers, such that $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \lambda_i$. Suppose there exist partitions*

$$\begin{aligned} &\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{p_0} \ \text{and}\ \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{p_0},\ \text{with} \\ &\Lambda_0 = \{\lambda_{01}, \ldots, \lambda_{0p_0}\},\quad \lambda_{0j} \in \Lambda,\ j = 1, \ldots, p_0,\ \lambda_{01} = \lambda_1, \\ &\Lambda_k = \{\lambda_{k1}, \ldots, \lambda_{kp_k}\},\ \Gamma_k = \{\gamma_{k1}, \ldots, \gamma_{kp_k}, \gamma_{k(p_k+1)}\}, \\ &\ k = 1, 2, \ldots, p_0,\quad \lambda_{ki} \in \Lambda,\ \gamma_{kj} \in \Gamma,\ i = 1, \ldots, p_k,\quad j = 1, \ldots, p_k + 1, \end{aligned}$$

where some lists Λ_k, $k = 1, 2, \ldots, p_0$, can be empty. Suppose that the following conditions hold:

i) For each $k = 1, 2, \ldots, p_0$, there exists a normal nonnegative matrix, with spectrum $\mathcal{U}_k = \{\mu_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}$ and diagonal entries Γ_k.

ii) There exists a $p_0 \times p_0$ normal nonnegative matrix with spectrum Λ_0 and diagonal entries $\mu_1, \mu_2, \ldots, \mu_{p_0}$.

Then there exists a normal nonnegative matrix with spectrum Λ and diagonal entries Γ.

Proof. The proof is similar to the proof of Theorem 4.2 (or [8, Theorem 3.1]). ∎

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A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
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4.3. Persymmetric nonnegative matrices with prescribed diagonal entries. In [9, Theorem 3.1], the authors prove a persymmetric version of Theorem 1.3, and based on this result they give sufficient conditions [9, Theorems 3.2 and 3.3], for the existence and construction of a persymmetric nonnegative matrix with prescribed spectrum. Then, from [9], we have for the case even p_0:

THEOREM 4.4. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a list of complex numbers, and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a list of nonnegative real numbers, such that $\sum\limits_{i=1}^{n}\gamma_i = \sum\limits_{i=1}^{n}\lambda_i$. Suppose there exist partitions*

$$\begin{aligned}
&\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{\frac{p_0}{2}} \cup \Lambda_{\frac{p_0}{2}} \cup \cdots \cup \Lambda_1 \ \textit{and}\\
&\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_{\frac{p_0}{2}} \cup \Gamma_{\frac{p_0}{2}} \cdots \cup \Gamma_1, \ \textit{with}\\
&\Lambda_0 = \{\lambda_{01}, \ldots, \lambda_{0p_0}\}, \quad \lambda_{0j} \in \Lambda, \ j = 1, \ldots, p_0, \ \lambda_{01} = \lambda_1,\\
&\Lambda_k = \{\lambda_{k1}, \ldots, \lambda_{kp_k}\}, \ \Gamma_k = \{\gamma_{k1}, \ldots, \gamma_{kp_k}, \gamma_{k(p_k+1)}\}, \ \lambda_{ki} \in \Lambda, \ \gamma_{kj} \in \Gamma,\\
&\quad k = 1, 2, \ldots, \frac{p_0}{2}, \quad i = 1, \ldots, p_k, \quad j = 1, \ldots, p_k + 1,
\end{aligned}$$

where some lists Λ_k, $k = 1, 2, \ldots, \frac{p_0}{2}$, can be empty. Suppose that the following conditions hold:

i) For each $k = 1, 2, \ldots, \frac{p_0}{2}$, there exists a nonnegative matrix, with spectrum $\mathcal{U}_k = \{\mu_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}$ and diagonal entries Γ_k.

ii) There exists a $p_0 \times p_0$ persymmetric nonnegative matrix with spectrum Λ_0 and diagonal entries $\mu_1, \mu_2, \ldots, \mu_{\frac{p_0}{2}}, \mu_{\frac{p_0}{2}}, \ldots, \mu_2, \mu_1$.

Then there exists a persymmetric nonnegative matrix with spectrum Λ and diagonal entries Γ.

Proof. The proof is similar to the proof in [9, Theorem 3.2]. The case odd p_0 is also similar to the proof in [9, Theorem 3.3]. ◻

REMARK 2. As an application of Theorem 4.4, it is easy to show by induction, that lists of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_1 > 0$, $\operatorname{Re}\lambda_j \leq 0$, $\left|\sqrt{3}\operatorname{Re}\lambda_j\right| \geq |\operatorname{Im}\lambda_j|$, $\ j = 2, \ldots n$ (lists of Šmigoc type), are realizable by a persymmetric nonnegative matrix with diagonal entries $\gamma_1, \gamma_2, \ldots, \gamma_n$, if and only if $\sum\limits_{i=1}^{n}\gamma_i = \sum\limits_{i=1}^{n}\lambda_i$. For the proof it is enough to consider the partition

$$\begin{aligned}
\Lambda &= \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_1 \text{ with}\\
\Lambda_0 &= \{\lambda_1, \lambda_i, \lambda_j\}, \ \Lambda_1 = \emptyset, \ \Lambda_2 = \Lambda - \Lambda_0,
\end{aligned}$$

where λ_i, λ_j are real or complex conjugate, with the realizable associated lists

$$\mathcal{U}_1 = \{\gamma_1\}, \ \mathcal{U}_2 = \{\lambda_1 + \lambda_i + \lambda_j - 2\gamma_1\} \cup \Lambda_2.$$

5. Lists universally realizable with prescribed diagonal entries. A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is said to be universally realizable (UR) if there exists a nonnegative matrix with spectrum Λ and any possible JCF allowed by Λ. In this section, we study the nonnegative universal realizability problem, *NURP,* with prescribed diagonal entries, that is, we study lists which are UR with realizing matrices having prescribed diagonal entries. In [5, Theorem 2.2] the authors proved that a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers of Šmigoc type is UR if and only if $\sum\limits_{i=1}^{n}\lambda_i \geq 0$. Moreover, In [10, Theorem 3.1] the authors

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
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prove that if a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is *UR,* then $\Lambda_t = \{\lambda_1 + t, \lambda_2, \dots, \lambda_n\}$ is also *UR.* Then for complex lists of Šmigoc type we have:

Corollary 5.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a realizable list of complex numbers of Šmigoc type, that is, with $\lambda_i \in \mathcal{G}$, $i = 2, \dots, n$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a list of nonnegative real numbers. Then, Λ is UR with realizing matrices having diagonal entries $\gamma_1, \dots, \gamma_n$, if and only if $\sum\limits_{i=1}^{n} \gamma_i = \sum\limits_{i=1}^{n} \lambda_i$.*

Proof. The condition is necessary. Let $\sum\limits_{i=1}^{n} \gamma_i = \sum\limits_{i=1}^{n} \lambda_i$. From [5, Theorem 2.2] the list $\Lambda' = \Big\{ -\sum\limits_{i=2}^{n} \lambda_i, \lambda_2, \dots, \lambda_n \Big\}$, is *UR.* Then there exists a nonnegative matrix B with spectrum Λ' for each JCF allowed by Λ'. Let $\gamma^T = [\gamma_1, \dots, \gamma_n]$. Then $A = B + \mathbf{e}\gamma^T$ is nonnegative with the spectrum Λ and diagonal entries $\gamma_1, \dots, \gamma_n$. Moreover, since $-\sum\limits_{i=2}^{n} \lambda_i + \sum\limits_{i=1}^{n} \gamma_i = \lambda_1$, from [10, Theorem 3.1], Λ is *UR*. $\square$

Observe that, since the construction of Ellard and Šmigoc in [6] is of the form $C + D$, where C is a companion matrix of zero trace and D is a diagonal matrix, then it only allows to obtain nonnegative matrices with JCF with just one Jordan block for each different eigenvalue. Thus, this construction cannot lead to universal realizabilty.

In [4], the authors study the universal realizability of lists of real numbers with two positive eigenvalues. Here, we also consider spectra *UR* with two positive eigenvalues and with realizing matrices having prescribed diagonal entries. Then, next result is a corollary from [4, Theorem 3.2].

Corollary 5.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ be a list of real numbers with $\lambda_1 > \lambda_2 > 0$, $\lambda_j < 0$, $j = 3, \dots, n$, $\sum\limits_{i=1}^{n}\lambda_i \geq 0$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a list of nonnegative real numbers such that $\sum\limits_{i=1}^{n}\gamma_i = \sum\limits_{i=1}^{n}\lambda_i$. If there exists a partition*

$$F = \{\lambda_3, \lambda_4, \dots, \lambda_n\} = F_1 \cup F_2 \text{ with}$$
$$F_1 = \{\alpha_1, \dots, \alpha_s\},\ F_2 = \{\beta_1, \dots, \beta_{n-s-2}\},$$

where $\alpha_i, \beta_j \in F$, $i = 1, \dots, s$, $j = 1, \dots, n-s-2$, such that

$$\lambda_1 \geq \sum_{i=1}^{s+1}\gamma_i - \sum_{i=1}^{s}\alpha_i \geq \sum_{i=s+2}^{n}\gamma_i - \sum_{i=1}^{n-s-2}\beta_i,$$

then Λ is UR with a realizing matrix having diagonal entries $\gamma_1, \dots, \gamma_n$.

Proof. Let

$$\omega_1 = \sum_{i=1}^{s+1}\gamma_i - \sum_{i=1}^{s}\alpha_i \quad \text{and} \quad \omega_2 = \sum_{i=s+2}^{n}\gamma_i - \sum_{i=1}^{n-s-2}\beta_i.$$

Then it is clear that the lists

$$F_1' = \{\omega_1, \alpha_1, \dots, \alpha_s\} \quad \text{and} \quad F_2' = \{\omega_2, \beta_1, \dots, \beta_{n-s-2}\}$$

are realizable by nonnegative matrices A_1 and A_2, with diagonal entries $\{\gamma_1, \dots, \gamma_{s+1}\}$ and $\{\gamma_{s+2}, \dots, \gamma_n\}$, respectively. Now, in order to apply Theorem 4.1, we compute a nonnegative matrix

$$B = \begin{bmatrix} \omega_1 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_2 & \omega_2 \end{bmatrix},$$

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A publication of the International Linear Algebra Society
Volume 35, pp. 53-64, February 2019.
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with spectrum $\{\lambda_1, \lambda_2\}$ and diagonal entries ω_1, ω_2 (see [1, Theorem 2.5] and the paragraph just below Example 2 in Section 3). Then,

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} + XC,$$

where X and C are the matrices in Theorem 1.3, X being the $n \times 2$ matrix of eigenvectors of A and $C = B - diagB$, is nonnegative with diagonal entries $\gamma_1, \ldots, \gamma_n$, and from [4, Theorem 3.2], Λ is *UR.* □

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