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CONSISTENCY OF QUATERNION MATRIX EQUATIONS $AX^\star - XB = C$ AND $X - AX^\star B = C^*$

XIN LIU†, QING-WEN WANG‡, AND YANG ZHANG§

Abstract. For a given ordered units triple $\{q_1, q_2, q_3\}$, the solutions to the quaternion matrix equations $AX^\star - XB = C$ and $X - AX^\star B = C$, $X^\star \in \{X, X^\eta, X^*, X^{\eta *}\}$, where X^* is the conjugate transpose of X, $X^\eta = -\eta X \eta$ and $X^{\eta *} = -\eta X^* \eta$, $\eta \in \{q_1, q_2, q_3\}$, are discussed. Some new real representations of quaternion matrices are used, which enable one to convert η-conjugate (transpose) matrix equations into some real matrix equations. By using this idea, conditions for the existence and uniqueness of solutions to the above quaternion matrix equations are derived. Also, methods to construct the solutions from some related real matrix equations are presented.

Key words. Quaternion matrix equations, Real representations, η-conjugates, η-conjugate transposes, Ordered units triple.

AMS subject classifications. 15A24, 15A33, 15B57.

1. Introduction.

The real quaternion $\mathbb{H}$ is a four-dimensional non-commutative associative algebra over real numbers $\mathbb{R}$, that is,

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

It was discovered by Hamilton in 1843, and has many applications in recent research areas such as computer graphics, control theory, and signal processing (see, e.g., [2, 6, 7, 8, 17, 18, 21, 22]). One of the research directions in this area is solving matrix equations over $\mathbb{H}$. In the literature, the generalized Sylvester matrix equation $AX^\star - XB = C$ and Stein matrix equation $X - AX^\star B = C$ have been well studied over real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ due to many applications in control theory (see, e.g., [1, 3, 4, 5, 13, 16, 20, 25, 27, 30]). Because of the non-commutative algebraic structure of $\mathbb{H}$, solving matrix equations over $\mathbb{H}$ appears more challenging and has attracted more and more attentions recently. Some methods corresponding to non-commutativity have been explored (see, e.g., [9, 11, 12, 15, 23, 24, 28, 29, 31]). For instance, Futorny et al. [9] discussed the criteria for the matrix equations $AX - \hat{X}B = C$ and $X - A\hat{X}B = C$ over $\mathbb{H}$ to be consistent with any fixed involution automorphism $q \to \hat{q}$. Using the particular structure of the real representations of quaternion matrices as well as the Kronecker product of matrices, Zhang et al. [31] explored several kinds of solutions to the quaternion matrix equation $AXB + CXD = E$.

In this paper, we focus on solving two types of quaternion matrix equations. We first introduce some definitions and notations. Throughout this paper, we denote the set of all $m \times n$ matrices over the real number field $\mathbb{R}$, the complex number field $\mathbb{C}$ and the quaternion $\mathbb{H}$ by $\mathbb{R}^{m\times n}, \mathbb{C}^{m\times n}$ and $\mathbb{H}^{m\times n}$, respectively. Let the symbols $I_n, 0, A^T$ stand for the $n \times n$ identity matrix, the zero matrix with appropriate size, the

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transpose of a matrix A, respectively. In Rodman's book [19], an ordered units triple is introduced, which can be used to treat many questions more efficiently.

Definition 1.1. (Section 2.4, [19]) An ordered triple of quaternions $\{q_1, q_2, q_3\}$ is said to be a *units triple* if

$$q_1^2 = q_2^2 = q_3^2 = -1, \quad q_1q_2 = -q_2q_1 = q_3, \quad q_2q_3 = -q_3q_2 = q_1, \quad q_3q_1 = -q_1q_3 = q_2.$$

The well-known units triples are $\{i, j, k\}, \{j, k, i\}, \{k, i, j\}$. Let $\{q_1, q_2, q_3\}$ be a fixed units triple. Then any $A \in \mathbb{H}^{m\times n}$ can be uniquely expressed as $A = A_0 + A_1q_1 + A_2q_2 + A_3q_3$, where $A_0, \ldots, A_3 \in \mathbb{R}^{m\times n}$, and the conjugate transpose is defined as $A^* = {A_0}^T - {A_1}^T q_1 - {A_2}^T q_2 - {A_3}^T q_3$. For each $q_i, i = 1, 2, 3$, we define the corresponding *η-conjugate* as $A^{q_i} = -q_iAq_i$, and the *η-conjugate transpose* of A as $A^{q_i*} = -q_iA^*q_i$.

In the past years, several types of quaternion matrix equations have been studied. For example, the quaternion j-conjugate matrix equations $AX^j - XB = C$ and $X - AX^jB = C$ were discussed by using the real/complex representation method (see, e.g., [11, 12, 23]). They established the necessary and sufficient conditions for the matrix equations to be consistent, and constructed the solutions from some related real matrix equations. Futorny et al. [9] discussed the quaternion matrix equations $AX^\star - XB = C$ and $X - AX^\star B = C$, where $X^\star \in \{X, X^i, X^j, X^k\}$. They gave the conditions for the existence of solutions. But they did not provide a way to derive the solutions.

In this paper, we consider more generalized quaternion matrix equations $AX^\star - XB = C$ and $X - AX^\star B = C$, where $X^\star \in \{X, X^{q_i}, X^*, X^{q_i*}\}$ for a given units triple $\{q_i\}_{i=1}^3$. Three new real representations are defined by $\{q_1, q_2, q_3\}$. Depending on those real representations, we convert these quaternion matrix equations into the types of $\mathcal{A}\mathcal{Y} - \mathcal{Y}\mathcal{B} = \mathcal{C}$, $\mathcal{Y} - \mathcal{A}\mathcal{Y}\mathcal{B} = \mathcal{C}$ or $\mathcal{A}\mathcal{Y}^T - \mathcal{Y}\mathcal{B} = \mathcal{C}$, $\mathcal{Y} - \mathcal{A}\mathcal{Y}^T\mathcal{B} = \mathcal{C}$ over $\mathbb{R}$. By the relations between the matrix equations over $\mathbb{H}$ and $\mathbb{R}$, we derive the criterions for the existence and uniqueness of solutions of the above quaternion matrix equations, and we also construct the solutions from those of some classic real matrix equations.

This paper is organized as follows. In Section 2, we introduce some useful real representations of quaternion matrices. In Section 3, we discuss the equation $AX^\star - XB = C$. Another type $X - AX^\star B = C$ is studied in Section 4. The uniqueness of solutions are presented in Section 5. Finally, we provide some numerical examples in Section 6.

Throughout this paper, we assume that $\{q_1, q_2, q_3\}$ is an ordered units triple.

2. Real representations.

One important technical research method for quaternion matrices is to use their real representations. For a given ordered units triple $\{q_1, q_2, q_3\}$, we define three real representations as follows:

Definition 2.1. Let $X = X_0 + X_1q_1 + X_2q_2 + X_3q_3 \in \mathbb{H}^{m\times n}$, $X_0, \ldots, X_3 \in \mathbb{R}^{m\times n}$. Three real representations are defined as:

$$X^\sigma = \begin{bmatrix} X_0 & -X_1 & X_2 & X_3 \\ X_1 & X_0 & X_3 & -X_2 \\ X_2 & X_3 & -X_0 & X_1 \\ X_3 & -X_2 & -X_1 & -X_0 \end{bmatrix}, \quad X^\tau = \begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{bmatrix},$$

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$$X^{\psi}=\begin{bmatrix} -X_1 & -X_0 & X_3 & -X_2 \\ X_0 & -X_1 & -X_2 & -X_3 \\ -X_3 & X_2 & -X_1 & -X_0 \\ X_2 & X_3 & X_0 & -X_1 \end{bmatrix}.$$

We will use these real representations in Sections 3, 4 and 5. To discuss their properties, we need to use the following special matrices:

$$Q_n=\begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, \quad R_n=\begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ -I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix},$$

$$S_n=\begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}, \quad W_n=\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \end{bmatrix},$$

$$G_n=\begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \end{bmatrix}.$$

It is easy to show that above special matrices have the following properties:

Lemma 2.2. *Q_n, R_n, S_n, W_n, G_n are all orthogonal matrices.*

Next, we summarize the properties of the above real representations in the following proposition, which will be used in Sections 3–6. The results either can be verified directly or can be proved similar to $\{i, j, k\}$ case (see, e.g., [14], [26]).

Proposition 2.3. *Let $A, B \in \mathbb{H}^{m\times n},\ C \in \mathbb{H}^{n\times s},\ a \in \mathbb{R}$. Then*

(a) *(i)* $(A+B)^{\sigma}=A^{\sigma}+B^{\sigma},\ (aA)^{\sigma}=aA^{\sigma}$;
(ii) $(A+B)^{\tau}=A^{\tau}+B^{\tau},\ (aA)^{\tau}=aA^{\tau}$;
(iii) $(A+B)^{\psi}=A^{\psi}+B^{\psi},\ (aA)^{\psi}=aA^{\psi}$;
(b) $(AC)^{\sigma}=A^{\sigma}W_nC^{\sigma}, (AC)^{\tau}=A^{\tau}C^{\tau},\ (AC)^{\psi}=-A^{\psi}G_nC^{\psi}$;
(c) *(i)* ${Q_m}^TA^{\sigma}Q_n=A^{\sigma},\ {R_m}^TA^{\sigma}R_n=-A^{\sigma},\ {S_m}^TA^{\sigma}S_n=-A^{\sigma}$;
(ii) ${Q_m}^TA^{\tau}Q_n=A^{\tau},\ {R_m}^TA^{\tau}R_n=A^{\tau},\ {S_m}^TA^{\tau}S_n=A^{\tau}$;
(iii) ${Q_m}^TA^{\psi}Q_n=A^{\psi},\ {R_m}^TA^{\psi}R_n=A^{\psi},\ {S_m}^TA^{\psi}S_n=A^{\psi}$;
(d) $(A^{q_1})^{\sigma}=W_mA^{\sigma}W_n,\ (A^{q_1})^{\psi}=-G_mA^{\psi}G_n$;
(e) $A^{\tau}=-G_mA^{\psi}$;
(f) $(A^*)^{\tau}=(A^{\tau})^T,\ (A^{q_1*})^{\psi}=-(A^{\psi})^T$.

3. Existence of solutions to $AX^{\star}-XB=C$. In this section, we use the real representations defined in Section 2 to discuss the consistency of the quaternion matrix equation $AX^{\star}-XB=C$, where $X^{\star}\in\{X, X^{q_i}, X^*, X^{q_i*}\}$. We first begin with some useful lemmas.

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Lemma 3.1. (Roth's theorem, [20]) *Let* $A \in \mathbb{F}^{m\times m},\ B \in \mathbb{F}^{n\times n},\ C \in \mathbb{F}^{m\times n}$, *where* $\mathbb{F}$ *is an arbitrary field. Then the matrix equation* $AX - XB = C$ *has a solution if and only if there exists a nonsingular* $P \in \mathbb{F}^{(m+n)\times(m+n)}$ *such that*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} P^{-1}.$$

Lemma 3.2. (Theorem 2.3, [25]) *Let* $A \in \mathbb{F}^{m\times n},\ B \in \mathbb{F}^{n\times m},\ C \in \mathbb{F}^{m\times m}$, *where* $\mathbb{F}$ *is a field of characteristic different from two. If* $\mathbb{F} = \mathbb{C}$, $()^\star$ *denotes the transpose or the conjugate transpose of a matrix, otherwise, it denotes the transpose of a matrix. There is some* $X \in \mathbb{F}^{m\times n}$ *such that* $AX^\star + XB = C$ *if and only if there exists a nonsingular* $P \in \mathbb{F}^{(m+n)\times(m+n)}$ *such that*

$$\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} = P \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} P^\star.$$

To derive the necessary and sufficient conditions for $AX^{q_i} - XB = C, AX - XB = C$ to be consistent, we apply Lemma 3.1 and real representations σ, τ to these equations, and obtain the following theorem.

Theorem 3.3. *Let* $A \in \mathbb{H}^{m\times m},\ B \in \mathbb{H}^{n\times n},\ C \in \mathbb{H}^{m\times n}$. *Then the quaternion matrix equation*

$$AX^\star - XB = C, \quad X^\star \in \{X^{q_1}, X\} \tag{3.1}$$

has a solution $X \in \mathbb{H}^{m\times n}$ *if and only if one of the following equivalent statements hold:*

(a) The real matrix equation

$$\mathcal{A}\mathcal{Y} - \mathcal{Y}\mathcal{B} = \mathcal{C} \tag{3.2}$$

has a solution $\mathcal{Y} \in \mathbb{R}^{4m\times 4n}$, *where*

(i) in case of $X^\star = X^{q_1}$, $\mathcal{A} = A^\sigma$, $\mathcal{B} = B^\sigma$, $\mathcal{C} = C^\sigma$,

(ii) in case of $X^\star = X$, $\mathcal{A} = A^\tau$, $\mathcal{B} = B^\tau$, $\mathcal{C} = C^\tau$.

(b) There exists a nonsingular $P \in \mathbb{R}^{4(m+n)\times 4(m+n)}$ *such that*

$$\begin{bmatrix} \mathcal{A} & \mathcal{C} \\ 0 & \mathcal{B} \end{bmatrix} = P \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix} P^{-1}.$$

Moreover, if $\mathcal{Y}$ *is a solution to (3.2), then*

$$X = \frac{1}{16} \begin{bmatrix} I_m & q_1 I_m & q_2 I_m & q_3 I_m \end{bmatrix} \hat{\mathcal{Y}} \begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n \end{bmatrix}$$

is a solution to (3.1), where

$$\hat{\mathcal{Y}} = \mathcal{Y} + Q_m \mathcal{Y} {Q_n}^T + R_m \mathcal{Y} {R_n}^T + S_m \mathcal{Y} {S_n}^T.$$

Proof. Only the argument of the case $X^\star = X^{q_1}$ is provided. Suppose that (3.1) has a solution X over $\mathbb{H}^{m\times n}$. Applying (b) of Proposition 2.3 to (3.1) yields $A^\sigma W_m (X^{q_1})^\sigma - X^\sigma W_n B^\sigma = C^\sigma$. By (d) of Proposition 2.3, we have $A^\sigma X^\sigma W_n - X^\sigma W_n B^\sigma = C^\sigma$. Then it implies that $X^\sigma W_n$ is a solution of (3.2).

Conversely, suppose that (3.2) has a solution Y, i.e., $A^\sigma Y - Y B^\sigma = C^\sigma$. Applying (i) in (c) of Proposition 2.3 to (3.2) gives $({Q_m}^T A^\sigma Q_m)Y - Y({Q_n}^T B^\sigma Q_n) = {Q_m}^T C^\sigma Q_n$, $(-{R_m}^T A^\sigma R_m)Y - Y(-{R_n}^T B^\sigma R_n) =$

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$-{R_m}^T C^\sigma R_n$, $(-{S_m}^T A^\sigma S_m)Y - Y(-{S_n}^T B^\sigma S_n) = -{S_m}^T C^\sigma S_n$. Thus, $A^\sigma Q_m Y {Q_n}^T - Q_m Y {Q_n}^T B^\sigma = C^\sigma$, $A^\sigma R_m Y {R_n}^T - R_m Y {R_n}^T B^\sigma = C^\sigma$, $A^\sigma S_m Y {S_n}^T - S_m Y {S_n}^T B^\sigma = C^\sigma$, which imply that $Q_m Y {Q_n}^T$, $R_m Y {R_n}^T$ and $S_m Y {S_n}^T$ are solutions of (3.2). Then

$$\mathcal{Y} = \frac{1}{4}(Y + Q_m Y {Q_n}^T + R_m Y {R_n}^T + S_m Y {S_n}^T)$$

is also a solution of (3.2). Setting $Y = (Y_{ij})_{4\times 4}$. By direct computation, we have

$$\mathcal{Y} = \begin{bmatrix} Z_0 & -Z_1 & -Z_2 & -Z_3 \\ Z_1 & Z_0 & -Z_3 & Z_2 \\ Z_2 & Z_3 & Z_0 & -Z_1 \\ Z_3 & -Z_2 & Z_1 & Z_0 \end{bmatrix},$$

where

$$Z_0 = \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \quad Z_1 = \frac{1}{4}(Y_{21} - Y_{12} + Y_{43} - Y_{34}),$$
$$Z_2 = \frac{1}{4}(Y_{31} - Y_{42} - Y_{13} + Y_{24}), \quad Z_3 = \frac{1}{4}(Y_{41} + Y_{32} - Y_{23} - Y_{14}).$$

Now, we construct a quaternion matrix

$$X = Z_0 + Z_1 q_1 + Z_2 q_2 + Z_3 q_3 = \frac{1}{4}\begin{bmatrix} I_m & q_1 I_m & q_2 I_m & q_3 I_m \end{bmatrix} \mathcal{Y} \begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n \end{bmatrix}.$$

Then $X^\sigma W_n = \mathcal{Y}$. Therefore, X satisfies $A^\sigma X^\sigma W_n - X^\sigma W_n B^\sigma = C^\sigma$, and thus, $A^\sigma W_m (X^{q_1})^\sigma - X^\sigma W_n B^\sigma = C^\sigma$. By (b) of Proposition 2.3, we have $AX^{q_1} - XB = C$, which implies that X is a solution of (3.1). Now, we proved (i) of (a) and construct our solution if (3.1) is consistent. (b) follows from Lemma 3.1. ∎

The method in Theorem 3.3 can also be adapted to prove the following result about the case of $X^\star \in \{X^{q_1 *}, X^*\}$.

Theorem 3.4. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{n\times m}$, $C \in \mathbb{H}^{m\times m}$. Then the quaternion matrix equation*

(3.3) $$AX^\star - XB = C, \quad X^\star \in \{X^{q_1 *}, X^*\}$$

has a solution $X \in \mathbb{H}^{m\times n}$ if and only if one of the following equivalent statements hold:

(a) The real matrix equation

(3.4) $$\mathcal{A}\mathcal{Y}^T - \mathcal{Y}\mathcal{B} = \mathcal{C}$$

has a solution $\mathcal{Y} \in \mathbb{R}^{4m\times 4n}$, where
(i) in case of $X^\star = X^{q_1 *}$, $\mathcal{A} = A^\psi$, $\mathcal{B} = B^\psi$, $\mathcal{C} = -C^\psi$,
(ii) in case of $X^\star = X^*$, $\mathcal{A} = A^\tau$, $\mathcal{B} = B^\tau$, $\mathcal{C} = C^\tau$.

(b) There exists a nonsingular $P \in \mathbb{R}^{4(m+n)\times 4(m+n)}$ such that

$$\begin{bmatrix} \mathcal{C} & \mathcal{A} \\ -\mathcal{B} & 0 \end{bmatrix} = P \begin{bmatrix} 0 & \mathcal{A} \\ -\mathcal{B} & 0 \end{bmatrix} P^T.$$

Moreover, if $\mathcal{Y}$ is a solution to (3.4), then

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(i) in case of $X^\star = X^{q_1*}$,

$$X = \frac{1}{16}\begin{bmatrix} q_1 I_m \\ -I_m \\ q_3 I_m \\ -q_2 I_m \end{bmatrix}^T (\mathcal{Y} + Q_m \mathcal{Y} {Q_n}^T + R_m \mathcal{Y} {R_n}^T + S_m \mathcal{Y} {S_n}^T) \begin{bmatrix} q_1 I_n \\ I_n \\ q_3 I_n \\ -q_2 I_n \end{bmatrix},$$

(ii) in case of $X^\star = X^*$,

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ q_1 I_m \\ q_2 I_m \\ q_3 I_m \end{bmatrix}^T (\mathcal{Y} + Q_m \mathcal{Y} {Q_n}^T + R_m \mathcal{Y} {R_n}^T + S_m \mathcal{Y} {S_n}^T) \begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n \end{bmatrix}$$

is a solution to (3.3).

Proof. We only prove the case $X^\star = X^{q_1*}$. Suppose that (3.3) has a solution X over $\mathbb{H}^{m\times n}$. It follows from (b) of Proposition 2.3 that $-A^\psi G_n (X^{q_1*})^\psi + X^\psi G_n B^\psi = C^\psi$. Since $(X^{q_1*})^\psi = -(X^\psi)^T$, $G_n = -{G_n}^T$, we obtain $A^\psi (X^\psi G_n)^T - X^\psi G_n B^\psi = -C^\psi$. Then, it implies that $X^\psi G_n$ is a solution of (3.4).

Conversely, suppose that (3.4) has a solution Y, i.e., $A^\psi Y^T - Y B^\psi = -C^\psi$. Applying (iii) in (c) of Proposition 2.3 to the above equation gives $({Q_m}^T A^\psi Q_n) Y^T - Y({Q_n}^T B^\psi Q_m) = -{Q_m}^T C^\psi Q_m$, $({R_m}^T A^\psi R_n) Y^T - Y({R_n}^T B^\psi R_m) = -{R_m}^T C^\psi R_m$, $({S_m}^T A^\psi S_n) Y^T - Y({S_n}^T B^\psi S_m) = -{S_m}^T C^\psi S_m$. Hence,

$$A^\psi (Q_m Y {Q_n}^T)^T - Q_m Y {Q_n}^T B^\psi = -C^\psi,$$
$$A^\psi (R_m Y {R_n}^T)^T - R_m Y {R_n}^T B^\psi = -C^\psi,$$
$$A^\psi (S_m Y {S_n}^T)^T - S_m Y {S_n}^T B^\psi = -C^\psi,$$

which imply that $Q_m Y {Q_n}^T$, $R_m Y {R_n}^T$ and $S_m Y {S_n}^T$ are solutions of (3.4). Then

$$\mathcal{Y} = \frac{1}{4}(Y + Q_m Y {Q_n}^T + R_m Y {R_n}^T + S_m Y {S_n}^T)$$

is also a solution of (3.4). Setting $Y = (Y_{ij})_{4\times 4}$. By direct computation, we have

$$\mathcal{Y} = \begin{bmatrix} -Z_0 & Z_1 & -Z_2 & -Z_3 \\ -Z_1 & -Z_0 & -Z_3 & Z_2 \\ Z_2 & Z_3 & -Z_0 & Z_1 \\ Z_3 & -Z_2 & -Z_1 & -Z_0 \end{bmatrix},$$

where

$$Z_0 = -\frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \quad Z_1 = -\frac{1}{4}(Y_{21} - Y_{12} + Y_{43} - Y_{34}),$$
$$Z_2 = \frac{1}{4}(Y_{31} - Y_{42} - Y_{13} + Y_{24}), \quad Z_3 = \frac{1}{4}(Y_{41} + Y_{32} - Y_{23} - Y_{14}).$$

Next, we construct a quaternion matrix

$$X = Z_0 + Z_1 q_1 + Z_2 q_2 + Z_3 q_3 = \frac{1}{4}\begin{bmatrix} q_1 I_m & -I_m & q_3 I_m & -q_2 I_m \end{bmatrix} \mathcal{Y} \begin{bmatrix} q_1 I_n \\ I_n \\ q_3 I_n \\ -q_2 I_n \end{bmatrix}.$$

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Obviously, $X^{\psi}G_n = \mathcal{Y}$. Therefore, X satisfies $A^{\psi}(X^{\psi}G_n)^T - X^{\psi}G_nB^{\psi} = -C^{\psi}$. Then, according to the previous arguments, $-A^{\psi}G_n(X^{q_1*})^{\psi}+X^{\psi}G_nB^{\psi} = C^{\psi}$. By (b) of Proposition 2.3, we have $AX^{q_1*}-XB = C$, which implies that X is a solution of (3.3). That is, (i) of (a) is proved. And by Lemma 3.2, (b) follows. □

4. Existence of solutions to $X - AX^{\star}B = C$. In this section, we discuss the existences of solutions to the quaternion matrix equation $X - AX^{\star}B = C$. We first introduce a well known result.

LEMMA 4.1. ([27]) *Let $A \in \mathbb{F}^{m\times m}, B \in \mathbb{F}^{n\times n}, C \in \mathbb{F}^{m\times n}$, where $\mathbb{F}$ is a field. Then the matrix equation $X - AXB = C$ has a solution $X \in \mathbb{F}^{m\times n}$ if and only if there exist nonsingular $R, S \in \mathbb{F}^{(m+n)\times(m+n)}$ such that*

$$\begin{bmatrix} A & C \\ 0 & I_n \end{bmatrix} R = S \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}, \quad \begin{bmatrix} I_m & 0 \\ 0 & B \end{bmatrix} R = S \begin{bmatrix} I_m & 0 \\ 0 & B \end{bmatrix}.$$

By Lemma 4.1, we obtain the following statements.

THEOREM 4.2. *Let $A \in \mathbb{H}^{m\times m}, B \in \mathbb{H}^{n\times n}, C \in \mathbb{H}^{m\times n}$. Then the quaternion matrix equation*

$$X - AX^{\star}B = C, \quad X^{\star} \in \{X^{q_1}, X\} \tag{4.5}$$

has a solution $X \in \mathbb{H}^{m\times n}$ if and only if one of the following equivalent statements hold:

(a) The real matrix equation

$$\mathcal{Y} - \mathcal{A}\mathcal{Y}\mathcal{B} = \mathcal{C} \tag{4.6}$$

has a solution $\mathcal{Y} \in \mathbb{R}^{4m\times 4n}$, where
(i) in case of $X^{\star} = X^{q_1}$, $\mathcal{A} = A^{\sigma}$, $\mathcal{B} = B^{\sigma}$, $\mathcal{C} = C^{\sigma}$,
(ii) in case of $X^{\star} = X$, $\mathcal{A} = A^{\tau}$, $\mathcal{B} = B^{\tau}$, $\mathcal{C} = C^{\tau}$.

(b) There exist nonsingular $R, S \in \mathbb{R}^{4(m+n)\times 4(m+n)}$ such that

$$\begin{bmatrix} \mathcal{A} & \mathcal{C} \\ 0 & I_{4n} \end{bmatrix} R = S \begin{bmatrix} \mathcal{A} & 0 \\ 0 & I_{4n} \end{bmatrix}, \quad \begin{bmatrix} I_{4m} & 0 \\ 0 & \mathcal{B} \end{bmatrix} R = S \begin{bmatrix} I_{4m} & 0 \\ 0 & \mathcal{B} \end{bmatrix}.$$

Moreover, if Y is a solution to (4.6), then

(i) in case of $X^{\star} = X^{q_1}$,

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ q_1 I_m \\ q_2 I_m \\ q_3 I_m \end{bmatrix}^T (Y + Q_mYQ_n{}^T - R_mYR_n{}^T - S_mYS_n{}^T) \begin{bmatrix} I_n \\ -q_1 I_n \\ q_2 I_n \\ q_3 I_n \end{bmatrix},$$

(ii) in case of $X^{\star} = X$,

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ q_1 I_m \\ q_2 I_m \\ q_3 I_m \end{bmatrix}^T (Y + Q_mYQ_n{}^T + R_mYR_n{}^T + S_mYS_n{}^T) \begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n \end{bmatrix}$$

is a solution to (4.5).

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Proof. We only prove the case $X^{\star} = X^{q_1}$. Suppose that (4.5) has a solution X over $\mathbb{H}^{m\times n}$. Applying (b) of Proposition 2.3 to (4.5) yields

$$X^{\sigma} - A^{\sigma}W_m(X^{q_1})^{\sigma}W_nB^{\sigma} = C^{\sigma}.$$

By (d) of Proposition 2.3, we have $X^{\sigma} - A^{\sigma}X^{\sigma}B^{\sigma} = C^{\sigma}$, which implies that X^{σ} is a solution of (4.6).

Conversely, suppose that (4.6) has a solution Y, i.e., $Y - A^{\sigma}YB^{\sigma} = C^{\sigma}$. Applying (i) in (c) of Proposition 2.3 to (4.6) gives

$$\begin{array}{rcl} Y - ({Q_m}^TA^{\sigma}Q_m)Y({Q_n}^TB^{\sigma}Q_n) &=& {Q_m}^TC^{\sigma}Q_n, \\ Y - (-{R_m}^TA^{\sigma}R_m)Y(-{R_n}^TB^{\sigma}R_n) &=& -{R_m}^TC^{\sigma}R_n, \\ Y - (-{S_m}^TA^{\sigma}S_m)Y(-{S_n}^TB^{\sigma}S_n) &=& -{S_m}^TC^{\sigma}S_n. \end{array}$$

Hence,

$$\begin{array}{rcl} (Q_mY{Q_n}^T) - A^{\sigma}(Q_mY{Q_n}^T)B^{\sigma} &=& C^{\sigma}, \\ (-R_mY{R_n}^T) - A^{\sigma}(-R_mY{R_n}^T)B^{\sigma} &=& C^{\sigma}, \\ (-S_mY{S_n}^T) - A^{\sigma}(-S_mY{S_n}^T)B^{\sigma} &=& C^{\sigma}, \end{array}$$

that is, $Q_mY{Q_n}^T$, $-R_mY{R_n}^T$, and $-S_mY{S_n}^T$ are solutions of (4.6). Then

$$\mathcal{Y} = \frac{1}{4}(Y + Q_mY{Q_n}^T - R_mY{R_n}^T - S_mY{S_n}^T)$$

is also a solution of (4.6). Setting $Y = (Y_{ij})_{4\times 4}$. By direct computation, we have

$$\mathcal{Y} = \begin{bmatrix} Z_0 & -Z_1 & Z_2 & Z_3 \\ Z_1 & Z_0 & Z_3 & -Z_2 \\ Z_2 & Z_3 & -Z_0 & Z_1 \\ Z_3 & -Z_2 & -Z_1 & -Z_0 \end{bmatrix},$$

where

$$Z_0 = \frac{1}{4}(Y_{11} + Y_{22} - Y_{33} - Y_{44}), \quad Z_1 = \frac{1}{4}(Y_{21} - Y_{12} - Y_{43} + Y_{34}),$$

$$Z_2 = \frac{1}{4}(Y_{31} - Y_{42} + Y_{13} - Y_{24}), \quad Z_3 = \frac{1}{4}(Y_{41} + Y_{32} + Y_{23} + Y_{14}).$$

Now, we construct a quaternion matrix

$$X = Z_0 + Z_1q_1 + Z_2q_2 + Z_3q_3 = \frac{1}{4}\begin{bmatrix} I_m & q_1I_m & q_2I_m & q_3I_m \end{bmatrix}\mathcal{Y}\begin{bmatrix} I_n \\ -q_1I_n \\ q_2I_n \\ q_3I_n \end{bmatrix}.$$

It is easy to verify that $X^{\sigma} = \mathcal{Y}$, and thus, $X^{\sigma} - A^{\sigma}X^{\sigma}B^{\sigma} = C^{\sigma}$. Hence,

$$X^{\sigma} - A^{\sigma}W_m(X^{q_1})^{\sigma}W_nB^{\sigma} = C^{\sigma}.$$

By (d) of Proposition 2.3, $X - AX^{q_1}B = C$, that is, X is a solution of (4.5). Now, (i) of (a) is proved and the solution of (4.5) is given. By Lemma 4.1, then (b) follows. ∎

Theorem 4.3. *Let $A, B, C \in \mathbb{H}^{m\times n}$. Then the quaternion matrix equation*

$$X - AX^{\star}B = C, \quad X^{\star} \in \{X^{q_1*}, X^{*}\} \tag{4.7}$$

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has a solution $X \in \mathbb{H}^{m\times n}$ *if and only if the real matrix equation*

$$\mathcal{Y} - \mathcal{A}\mathcal{Y}^T\mathcal{B} = \mathcal{C} \tag{4.8}$$

has a solution $\mathcal{Y} \in \mathbb{R}^{4m\times 4n}$, *where*

(i) *in case of* $X^\star = X^{q_1*}$, $\mathcal{A} = -(A^{q_1})^\psi$, $\mathcal{B} = B^\psi$, $\mathcal{C} = C^\tau$,
(ii) *in case of* $X^\star = X^*$, $\mathcal{A} = A^\tau$, $\mathcal{B} = B^\tau$, $\mathcal{C} = C^\tau$.

Moreover, if $\mathcal{Y}$ *is a solution of (4.8), then*

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ q_1 I_m \\ q_2 I_m \\ q_3 I_m\end{bmatrix}^T (\mathcal{Y} + Q_m\mathcal{Y}{Q_n}^T + R_m\mathcal{Y}{R_n}^T + S_m\mathcal{Y}{S_n}^T)\begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n\end{bmatrix}.$$

is a solution to (4.7).

Proof. For the case of $X^\star = X^{q_1*}$. Suppose that (4.7) has a solution X. Applying (b) of Proposition 2.3 to (4.7) yields $X^\psi - A^\psi G_n (X^{q_1*})^\psi G_m B^\psi = C^\psi$. Noting that $(X^{q_1*})^\psi = -(X^\psi)^T, G_n = -{G_n}^T$ and pre-multiplying the both sides of matrix equation by G_m. Then we have $G_m X^\psi - G_m A^\psi G_n (G_m X^\psi)^T B^\psi = G_m C^\psi$. By (d) and (e) of Proposition 2.3, we obtain $X^\tau + (A^{q_1})^\psi (X^\tau)^T B^\psi = C^\tau$, which implies that X^τ is a solution of (4.8).

Conversely, suppose that (4.8) has a solution Y, i.e., $Y + (A^{q_1})^\psi Y^T B^\psi = C^\tau$. Applying (iii) of (c) of Propositions 2.3 to (4.8) gives

$$Y + ({Q_m}^T (A^{q_1})^\psi Q_n) Y^T ({Q_m}^T B^\psi Q_n) = {Q_m}^T C^\tau Q_n,$$

$$Y + ({R_m}^T (A^{q_1})^\psi R_n) Y^T ({R_m}^T B^\psi R_n) = {R_m}^T C^\tau R_n,$$

$$Y + ({S_m}^T (A^{q_1})^\psi S_n) Y^T ({S_m}^T B^\psi S_n) = {S_m}^T C^\tau S_n.$$

Hence,

$$Q_m Y Q_n^T + (A^{q_1})^\psi (Q_m Y Q_n^T)^T B^\psi = C^\tau,$$

$$R_m Y R_n^T + (A^{q_1})^\psi (R_m Y R_n^T)^T B^\psi = C^\tau,$$

$$S_m Y S_n^T + (A^{q_1})^\psi (S_m Y S_n^T)^T B^\psi = C^\tau,$$

which imply that $Q_m Y {Q_n}^T$, $R_m Y {R_n}^T$ and $S_m Y {S_n}^T$ are solutions of (4.8). Then $\mathcal{Y}$ is also a solution of (4.8), where $\mathcal{Y} = \frac{1}{4}(Y + Q_m Y {Q_n}^T + R_m Y {R_n}^T + S_m Y {S_n}^T)$. By direct computations, we have

$$\mathcal{Y} = \begin{bmatrix} Z_0 & -Z_1 & -Z_2 & -Z_3 \\ Z_1 & Z_0 & -Z_3 & Z_2 \\ Z_2 & Z_3 & Z_0 & -Z_1 \\ Z_3 & -Z_2 & Z_1 & Z_0 \end{bmatrix},$$

where

$$Z_0 = \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \quad Z_1 = \frac{1}{4}(Y_{21} - Y_{12} + Y_{43} - Y_{34}),$$

$$Z_2 = \frac{1}{4}(Y_{31} - Y_{42} - Y_{13} + Y_{24}), \quad Z_3 = \frac{1}{4}(Y_{41} + Y_{32} - Y_{23} - Y_{14}).$$

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Set a new quaternion matrix

$$X = Z_0 + Z_1 q_1 + Z_2 q_2 + Z_3 q_3 = \frac{1}{4} \begin{bmatrix} I_m & q_1 I_m & q_2 I_m & q_3 I_m \end{bmatrix} \mathcal{Y} \begin{bmatrix} I_n \\ -q_1 I_n \\ -q_2 I_n \\ -q_3 I_n \end{bmatrix}.$$

It is easy to prove that $X^\tau = \mathcal{Y}$, and thus, X is a solution of (4.7). ∎

5. Uniqueness of solutions of two kinds of matrix equations. In this section, we establish some conditions for the uniqueness of solutions. Recall that the matrix pencil $A - \lambda B$ is said to be regular if $\det(A - \lambda B) \neq 0$ for some $\lambda \in \mathbb{C}$. Let $\sigma(A, B)$ denote the spectrum of $A - \lambda B$, in particular, $\sigma(A) = \sigma(A, I)$. An eigenvalue of A is simple if its algebraic multiplicity is 1. For the finite subset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers, Λ is called T-reciprocal free if and only if $\lambda_i \neq \frac{1}{\lambda_j}, \lambda_i, \lambda_j \in \Lambda$. This definition also regards 0 and ∞ as reciprocal of each other.

We will use the following criterions for uniqueness of solutions to prove our results.

Lemma 5.1. ([16]) *Let $A \in \mathbb{C}^{m\times m},\ B \in \mathbb{C}^{n\times n},\ C \in \mathbb{C}^{m\times n}$.*

(i) The matrix equation $AX - XB = C$ has a unique solution if and only if $\sigma(A) \bigcap \sigma(B) = \emptyset$.
(ii) The matrix equation $X - AXB = C$ has a unique solution if and only if $\alpha_i \beta_j \neq 1$, for all eigenvalues α_i of A and β_j of B, respectively.

Lemma 5.2. ([10]) *Let $A, B, C \in \mathbb{R}^{n\times n}$. Suppose $A - \lambda B^T$ is regular. Then the matrix equation $AX + X^T B = C$ has a unique solution if and only if $\sigma(A^T - \lambda B)\backslash\{1\}$ is T-reciprocal free and whenever $1 \in \sigma(A^T - \lambda B)$, 1 is simple.*

Lemma 5.3. ([5]) *Let $A, B, C \in \mathbb{R}^{n\times n}$. The matrix equation $X - AX^T B = C$ has a unique solution if and only if $\sigma(A^T B)\backslash\{-1\}$ is T-reciprocal free and whenever $-1 \in \sigma(A^T B)$, -1 is simple.*

By Lemmas 5.1–5.3 and theorems in Sections 3 and 4, we have the following statements about the uniqueness of solutions to $AX^\star - XB = C$ and $X - AX^\star B = C$.

Corollary 5.4. *Let $\mathcal{A} \in \mathbb{R}^{4m\times 4m},\ \mathcal{B} \in \mathbb{R}^{4n\times 4n},\ \mathcal{C} \in \mathbb{R}^{4m\times 4n}$ be given in Theorem 3.3. If $\sigma(\mathcal{A}) \bigcap \sigma(\mathcal{B}) = \emptyset$, then the real matrix equation (3.2) has a unique solution $Y = (Y_{ij})_{4\times 4},\ Y_{ij} \in \mathbb{R}^{m\times n}$, and thus, the quaternion matrix equation (3.1) also has a unique solution $X = Y_{11} - Y_{12} q_1 - Y_{13} q_2 - Y_{14} q_3$.*

Proof. Assume that (3.2) has a unique solution Y. If (3.1) has two different solutions $X_0, \hat{X}_0$. Then by the argument in the first part of the proof of Theorem 3.3, ${X_0}^\sigma W_n$ and $\hat{X}_0^\sigma W_n$ are two solutions of (3.2). Clearly, they are different, which contradicts our assumption. Therefore, if (3.2) is uniquely solvable, then so is (3.1). Then, our conditions for the uniqueness of solution to matrix equation (3.1) follows from Lemma 5.1.

If (3.2) has a unique solution Y then (3.1) also has a unique solution X. From the arguments in the proof of Theorem 3.3, we see that $AX^{q_1} - XB = C$ is equivalent to $A^\sigma X^\sigma W_n - X^\sigma W_n B^\sigma = C^\sigma$, and thus, $X^\sigma W_n = Y$. Denote $Y = (Y_{ij})_{4\times 4},\ Y_{ij} \in \mathbb{R}^{m\times n}$. Then, equating

$$X^\sigma W_n = \begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}$$

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gives $X_0 = Y_{11},\ X_1 = -Y_{12},\ X_2 = -Y_{13},\ X_3 = -Y_{14}$. We omit the similar argument for the case $X^\star = X$. ◻

Using a similar idea, we can prove the other cases. Here we only present the results without proofs.

Corollary 5.5. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{4n\times 4n}$ be given in Theorem 3.4. Suppose $\mathcal{A} + \lambda\mathcal{B}^T$ is regular. If $\sigma(\mathcal{A}^T + \lambda\mathcal{B})\backslash\{1\}$ is T-reciprocal free and whenever $1 \in \sigma(\mathcal{A}^T + \lambda\mathcal{B})$, 1 is simple, then the real matrix equation (3.4) has a unique solution $Y = (Y_{ij})_{4\times 4}, Y_{ij} \in \mathbb{R}^{m\times n}$. In this case, the quaternion equation (3.3) also has a unique solution X:*

(i) in case of $X^\star = X^{q_1 *}$, $X = -Y_{11} + Y_{12}q_1 - Y_{13}q_2 - Y_{14}q_3$.
(ii) in case of $X^\star = X^*$, $X = Y_{11} - Y_{12}q_1 - Y_{13}q_2 - Y_{14}q_3$.

Corollary 5.6. *Let $\mathcal{A} \in \mathbb{R}^{4m\times 4m}$, $\mathcal{B} \in \mathbb{R}^{4n\times 4n}$, $\mathcal{C} \in \mathbb{R}^{4m\times 4n}$ be given in Theorem 4.2. If $\alpha_i\beta_j \neq 1$, for all eigenvalues α_i of $\mathcal{A}$ and β_j of $\mathcal{B}$, then the real matrix equation (4.6) has a unique solution $Y = (Y_{ij})_{4\times 4},\ Y_{ij} \in \mathbb{R}^{m\times n}$. In this case, the quaternion matrix equation (4.5) also has a unique solution X:*

(i) in case of $X^\star = X^{q_1}$, $X = Y_{11} - Y_{12}q_1 + Y_{13}q_2 + Y_{14}q_3$.
(ii) in case of $X^\star = X$, $X = Y_{11} - Y_{12}q_1 - Y_{13}q_2 - Y_{14}q_3$.

Corollary 5.7. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{4n\times 4n}$ be given in Theorem 4.3. If $\alpha_i\beta_j \neq 1$. If $\sigma(\mathcal{A}^T\mathcal{B})\backslash\{-1\}$ is T-reciprocal free and whenever $-1 \in \sigma(\mathcal{A}^T\mathcal{B})$, -1 is simple, then the real matrix equation (4.8) has a unique solution $Y = (Y_{ij})_{4\times 4},\ Y_{ij} \in \mathbb{R}^{m\times n}$, and this quaternion matrix equation (4.7) also has a unique solution $X = Y_{11} - Y_{12}q_1 - Y_{13}q_2 - Y_{14}q_3$.*

Remark 5.8. When the quaternion matrix equations (3.1), (3.3), (4.5), (4.7) has a unique solution, respectively, it's still unclear whether the corresponding real matrix equations (3.2), (3.4), (4.6), (4.8) also respectively has a unique solution or not.

6. Numerical examples.

We present two numerical examples in this section. All computations are done by MATLAB.

Example 1. Consider the quaternion matrix equation $AX^i - XB = C$, where

$$A = \begin{bmatrix} 2+3i & -i-2j+k \\ -1+2i-2j-k & -2i+j+k \end{bmatrix}, \quad B = \begin{bmatrix} 2i-j+5k & 1-3i+j-k \\ 3i+3k & 2i+4j+4k \end{bmatrix},$$

$$C = \begin{bmatrix} -2+4j-k & 1+3i+2j \\ 3-2i+2j & 1+i+5j \end{bmatrix},$$

and its corresponding real matrix equation is $A^\sigma Y - YB^\sigma = C^\sigma$. (To save space, we don't give the exact formats of A^σ, B^σ and C^σ.) Using MATLAB, $\sigma(A^\sigma) = \{2.1813 + 2.4441i, 2.1813 - 2.4441i, 2.5102 + 1.4441i, 2.5102 - 1.4441i, -2.1813 + 2.4441i, -2.1813 - 2.4441i, -2.5102 + 1.4441i, -2.5102 - 1.4441i\}$, $\sigma(B^\sigma) = \{-8.3876 + 0.2310i, -8.3876 - 0.2310i, -2.4305 + 3.7690i, -2.4305 - 3.7690i, 2.4305 + 3.7690i, 2.4305 - 3.7690i, 8.3876 + 0.2310i, 8.3876 - 0.2310i\}$ have no common eigenvalues. Thus, by Corollary 5.4, the real matrix

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equation has a unique solution

$$Y = \begin{bmatrix} 1.9310 & -0.1088 & 0.5162 & -1.7551 & 0.2134 & -1.0000 & 1.7691 & 0.5413 \\ 0.8249 & -1.1633 & -0.3170 & -1.1419 & -0.6346 & -0.5256 & 0.4355 & 0.1259 \\ -0.5162 & 1.7551 & 1.9310 & -0.1088 & 1.7691 & 0.5413 & -0.2134 & 1.0000 \\ 0.3170 & 1.1419 & 0.8249 & -1.1633 & 0.4355 & 0.1259 & 0.6346 & 0.5256 \\ -0.2134 & 1.0000 & -1.7691 & -0.5413 & 1.9310 & -0.1088 & 0.5162 & -1.7551 \\ 0.6346 & 0.5256 & -0.4355 & -0.1259 & 0.8249 & -1.1633 & -0.3170 & -1.1419 \\ -1.7691 & -0.5413 & 0.2134 & -1.0000 & -0.5162 & 1.7551 & 1.9310 & -0.1088 \\ -0.4355 & -0.1259 & -0.6346 & -0.5256 & 0.3170 & 1.1419 & 0.8249 & -1.1633 \end{bmatrix}.$$

By the formula of X in Corollary 5.4, our quaternion matrix equation also has a unique solution

$$X = \begin{bmatrix} 1.9310 - 0.5162i - 0.2134j - 1.7692k & -0.1088 + 1.7551i + 1.0000j - 0.5413k \\ 0.8249 + 0.3170i + 0.6346j - 0.4355k & -1.1633 + 1.1419i + 0.5256j - 0.1259k \end{bmatrix}.$$

EXAMPLE 2. Consider the quaternion matrix equation $X - AX^{k*}B = C$, where

$$A = \begin{bmatrix} 3 - 4i - 2j + 2k & 4 + i + 3j + 9k \\ -1 + 3j - 3k & 2 + 5j + k \end{bmatrix}, \quad B = \begin{bmatrix} 1 - i + 3j - 3k & 2 + i + 9j + 2k \\ 7j & 3 - 2j \end{bmatrix}$$

$$C = \begin{bmatrix} 2 - 2i - 2j & -1 + 3i + 3j + k \\ -3i - 3j - k & 2 + i + j + k \end{bmatrix},$$

and its corresponding real matrix equation is $Y + (A^k)^\psi Y^T B^\psi = C^\tau$. Using MATLAB, $\sigma(-((A^k)^\psi)^T B^\psi) = 100 * \{0.0064 + 1.3697i, 0.0064 - 1.3697i, 0.0064 + 1.3697i, 0.0064 - 1.3697i, 0.0736 + 0.3066i, 0.0736 - 0.3066i, 0.0736 + 0.3066i, 0.0736 - 0.3066i\}$, which implies that $\sigma(((-A^k)^\psi)^T B^\psi) \backslash \{-1\}$ is T-reciprocal free. Then, by Corollary 5.7, the real matrix equation has a unique solution

$$Y = \begin{bmatrix} 0.0144 & 0.0151 & -0.0158 & 0.0279 & 0.0401 & -0.0187 & 0.0164 & -0.0078 \\ 0.0509 & -0.0384 & 0.0860 & -0.0612 & -0.0081 & -0.0275 & 0.0069 & 0.0314 \\ 0.0158 & -0.0279 & 0.0144 & 0.0151 & 0.0164 & -0.0078 & -0.0401 & 0.0187 \\ -0.0860 & 0.0612 & 0.0509 & -0.0384 & 0.0069 & 0.0314 & 0.0081 & 0.0275 \\ -0.0401 & 0.0187 & -0.0164 & 0.0078 & 0.0144 & 0.0151 & -0.0158 & 0.0279 \\ 0.0081 & 0.0275 & -0.0069 & -0.0314 & 0.0509 & -0.0384 & 0.0860 & -0.0612 \\ -0.0164 & 0.0078 & 0.0401 & -0.0187 & 0.0158 & -0.0279 & 0.0144 & 0.0151 \\ -0.0069 & -0.0314 & -0.0081 & -0.0275 & -0.0860 & 0.0612 & 0.0509 & -0.0384 \end{bmatrix}.$$

By the formula of X in Corollary 5.7, our quaternion matrix equation also has a unique solution

$$X = \begin{bmatrix} 0.0144 - 0.0401i - 0.0164j + 0.0158k & 0.0151 + 0.0187i + 0.0078j - 0.0279k \\ 0.0509 + 0.0081i - 0.0069j - 0.0860k & -0.0384 + 0.0275i - 0.0314j + 0.0612k \end{bmatrix}.$$

REMARK 6.1. Based on an arbitrary ordered units triple $\{q_i\}_{i=1}^3$, we present a full discussion of the solutions to $AX^\star - XB = C$ and $X - AX^\star B = C$, where $X^\star \in \{X, X^\eta, X^*, X^{\eta*}\}$ with $\eta \in \{q_1, q_2, q_3\}$. For the different types of quaternion matrix equations, we may choose suitable ordered triples and apply our results. For example, we can choose the ordered units triple $\{k, i, j\}$ and apply Theorem 4.3 to the equation $X - AX^{k*}B = C$.

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