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Mashael M. Al Baidani
Washington State University, mashaelalbaidani@gmail.com

Judi J. McDonald
Washington State University, judijmcdonald@gmail.com

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ON THE BLOCK STRUCTURE AND FROBENIUS NORMAL FORM OF POWERS OF MATRICES*

MASHAEL M. ALBAIDANI† AND JUDI J. MCDONALD‡

Abstract. The Frobenius normal form of a matrix is an important tool in analyzing its properties. When a matrix is powered up, the Frobenius normal form of the original matrix and that of its powers need not be the same. In this article, conditions on a matrix A and the power q are provided so that for any invertible matrix S, if $S^{-1}A^qS$ is block upper triangular, then so is $S^{-1}AS$ when partitioned conformably. The result is established for general matrices over any field. It is also observed that the contributions of the index of cyclicity to the spectral properties of a matrix hold over any field. The article concludes by applying the block upper triangular powers result to the cone Frobenius normal form of powers of a eventually cone nonnegative matrix.

Key words. Fields, Frobenius normal form, Block upper triangular matrices, Cones, Eventually nonnegative matrices.

AMS subject classifications. 15A18, 15A21, 15B48.

1. Introduction. Many interesting properties of matrices, particularly nonnegative matrices, can be gleaned from looking at their irreducible classes and Frobenius normal form. In particular, the Perron-Frobenius theorem identifies several spectral properties tied to the combinatorial structure of a nonnegative irreducible matrix. The study of spectral properties of reducible nonnegative matrices is summarized by Schneider in [17]. Properties of the Jordan form of the entire peripheral spectrum are presented in [12].

Extending ideas from Perron-Fronbenius theory to matrices whose powers become and remain nonnegative (eventually nonnegative matrices) was first discussed by Friedland in [7]. In [23] and [22], Zaslavsky develops properties of irreducible eventually nonnegative matrices with Tam and then expands on the reducible case with McDonald. In Section 3 of this paper, we generalize the results in [5] by showing that for an arbitrary matrix A over an arbitrary field $\mathbb{F}$, if A is nonsingular or all the Jordan blocks for zero are 1×1, then whenever q is chosen so that for all distinct eigenvalues μ and λ it follows that $\lambda^q \neq \mu^q$, then for any invertible matrix S, if $S^{-1}A^qS$ is block upper triangular, so is $S^{-1}AS$ when partitioned conformably, and the transitive closure of the (generalized) reduced graphs of $S^{-1}A^qS$ and $S^{-1}AS$ are equal. We also observe that the influence of the index of cyclicity on the spectrum of matrices is true over any field.

Additional work has been done on extending ideas from nonnegative matrix theory to more general contexts. See for example [6], [8], [9], [10], [11], [13], [14], [15], [16], [18], [19], [20], and [21]. In this paper, we consider cone nonnegative matrices (see Chapter 1 in [4]), taking a matrix theoretic approach similar to Barker [1], [2], [3]. In Section 4 of this paper, we provide a nonpolyhedral cone counter-example to Theorem 7 in [2], and use our results from Section 3 to expand upon the combinatorial structure of eventually cone nonnegative matrices.

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†Department of Mathematics, Washington State University, Pullman, WA 99164-1113, USA (mashaelalbaidani@gmail.com, judijmcdonald@gmail.com).

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2. Definitions and notation. Throughout Section 3, we will work over an arbitrary field $\mathbb{F}$ with algebraic closure $\mathbb{E}$. For Section 4, we will restrict our fields to $\mathbb{R}$ with algebraic closure $\mathbb{C}$. We begin with some standard definitions and notation that are appropriate for working over an arbitrary field.

Let $M_n(\mathbb{F})$ represent the set of $n \times n$ matrices with entries in $\mathbb{F}$.

We will write $\langle n\rangle$ for $\{1, \ldots, n\}$.

The multiset

$$\sigma(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of } A\ \},$$

where each eigenvalue is listed the number of times it occurs as a root of the characteristic polynomial, is referred to as the *spectrum* of A. An eigenvalue $\lambda \in \sigma(A)$ is called *simple* if its algebraic multiplicity is one and we will use the $\text{index}_\lambda(A)$ to denote the degree of λ as a root of the minimal polynomial of A. The $\text{index}_\lambda(A)$ can also be thought of as the size of the largest Jordan block corresponding to A with eigenvalue λ.

Let $\alpha, \beta \subseteq \langle n\rangle$. We will write $A_{\alpha\beta}$ to represent the submatrix of A whose rows are indexed from α and whose columns are indexed from β. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is an ordered partition of $\langle n\rangle$, we write

$$A_\alpha = \begin{bmatrix} A_{\alpha_1\alpha_1} & A_{\alpha_1\alpha_2} & \cdots & A_{\alpha_1\alpha_k} \\ A_{\alpha_2\alpha_1} & A_{\alpha_2\alpha_2} & \cdots & A_{\alpha_2\alpha_k} \\ \vdots & \vdots & & \vdots \\ A_{\alpha_k\alpha_1} & A_{\alpha_k\alpha_2} & \cdots & A_{\alpha_k\alpha_k} \end{bmatrix}.$$

We say A_α is *block upper triangular* if $A_{\alpha_i\alpha_j} = 0$ whenever $i > j$. A matrix A is said to be *reducible* if there is a nontrivial partition α of $\langle n\rangle$ so that A_α is block upper triangular. We say it is *irreducible* otherwise. If A_α is block upper triangular and each $A_{\alpha_i\alpha_i}$ is irreducible, we say that A_α is in *Frobenius normal form.* For any matrix $A \in M_n(\mathbb{R})$, it is well known that there is an ordered partition α of $\langle n\rangle$ so that A_α is in Frobenius normal form. The same proof works over any field.

We write $J_k(\lambda)$ to represent a $k \times k$ upper triangular matrix whose diagonal elements are λ, whose first superdiagonal elements are 1, and all other elements are zero. We will refer to such a matrix as a *Jordan block with eigenvalue* λ.

A nonzero vector x is a *generalized eigenvector* of A corresponding to the eigenvalue λ provided $(\lambda I - A)^m x = 0$ for some positive integer m. Note that when $A \in M_n(\mathbb{F})$, we may need to consider generalized eigenvectors in $\mathbb{E}^n$. The *generalized eigenspace* of the matrix A corresponding to the eigenvalue λ, denoted $E_\lambda(A)$, is the nullspace of $(A - \lambda I)^r$, where $r = index_\lambda(A)$. A *Jordan chain* corresponding to $E_\lambda(A)$ is a set of nonzero vectors $\{x, (\lambda I - A)x, \ldots, (\lambda I - A)^{r-1}x\}$, where $(\lambda I - A)^r x = 0$. A *Jordan basis* for the generalized eigenspace of A is a basis of $E_\lambda(A)$ consisting of the union of Jordan chains.

Let $\mathcal{G} = (V, E)$ be a graph. We say $p = (j_1, j_2), \ldots, (j_{k-1}, j_k)$ is a path in $\mathcal{G}$ provided each ordered pair $(j_i, j_{i+1}) \in E$. If in addition, $j_k = j_1$, we say the path is a cycle. The *index of cyclicity* of a graph is the greatest common divisor of the lengths of the cycles in $\mathcal{G}$. We say a vertex l has access to a vertex j if there is a path from l to j in $\mathcal{G}$. We define the *transitive closure* of $\mathcal{G}$ by $\overline{\mathcal{G}} = (V, \overline{E})$ where $\overline{E} = \{(j, l) | j \text{ has access to } l\ in\ \mathcal{G}\}$. If l has access to j and j has access to l, we say j and l *communicate.* The communication relation defines an equivalence relation on $\mathcal{G}$ that partitions the vertices of $\mathcal{G}$ into irreducible classes.

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For $A \in M_n(\mathbb{F})$, we define $\mathcal{D}(A) = (V, E)$ to be the *digraph* of A, where $V = \langle n \rangle$ and $(l, j) \in E$ whenever $a_{lj} \neq 0$. The matrix A is irreducible precisely when $\mathcal{D}(A)$ has exactly one irreducible class. The *index of cyclicity of* A is defined to be the index of cyclicity of $\mathcal{D}(A)$. We define the notion of the *reduced graph* of A to include partitions $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $\langle n \rangle$ where A_α is upper triangular, but the diagonal blocks $A_{\alpha_i\alpha_i}$ need not be irreducible as is the case in earlier papers. Thus, $\mathcal{R}_\alpha(A) = (V_\alpha, E_\alpha)$, where $V_\alpha = \{\alpha_1, \ldots, \alpha_k\}$ and $E_\alpha = \{(\alpha_j, \alpha_l)|$ there is an edge in $\mathcal{D}(A)$ from a vertex in α_j to a vertex in $\alpha_l\}$.

The following definitions require the field to have additional properties such as an ordering, an absolute value, and a topology, hence, we will define these properties over $\mathbb{R}$ with algebraic closure $\mathbb{C}$.

For $A \in M_n(\mathbb{R})$, we call $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$, the *spectral radius* of A, and the multiset $\pi(A) = \{\lambda \in \sigma(A)| \ \ |\lambda| = \rho(A)\}$, the *peripheral spectrum* of A.

Let $\mathcal{K}$ be a nonempty subset of $\mathbb{R}^n$. The set $\mathcal{K}$ is a *cone* provided for every $x \in \mathcal{K}$ and for every $\alpha > 0$, the element $\alpha x \in \mathcal{K}$. A cone $\mathcal{K}$ is said to be *convex* proviced $\alpha x + \beta y \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and for all nonnegative α and β in $\mathbb{R}$. For $S \subset \mathbb{R}^n$, we denote the convex cone generated by S as S^G. Note that S^G consists of all finite nonnegative linear combinations of elements from S. If $K \cap (-K) = \{0\}$, then the cone $\mathcal{K}$ is said to be *pointed.* A cone $\mathcal{K}$ is said to *solid* if its topological interior is nonempty. A topologically closed, solid, pointed, convex cone is referred to as a *proper cone.*

Let $\mathcal{K}$ and $\mathcal{F} \subseteq \mathcal{K}$ be pointed closed cones. Then $\mathcal{F}$ is a *face* of $\mathcal{K}$ provided $x \in \mathcal{F}$, $y \in \mathcal{K}$, and $x - y \in \mathcal{K}$ implies $y \in \mathcal{F}$. A face $\mathcal{F}$ of $\mathcal{K}$ is a trivial face if $\mathcal{F} = \{0\}$ or $\mathcal{F} = \mathcal{K}$.

Let $\mathcal{K}$ be proper cone in $\mathbb{R}^n$ and $A \in M_n(\mathbb{R})$. We say A is *nonnegative* if $a_{jk} \geq 0$ for all $j, k \in < n >$. We say A is $\mathcal{K}$-*nonnegative* provided $A\mathcal{K} \subseteq \mathcal{K}$. The matrix A is said to be *eventually nonnegative* if there exists a positive integer N such that for all integers $m \geq N$, $A^m \geq 0$ and A is said to be $\mathcal{K}$-*eventually nonnegative* if there exists a positive integer N such that for all integers $m \geq N$, $A^m\mathcal{K} \subseteq \mathcal{K}$.

If A is $\mathcal{K}$-nonnegative, then a face $\mathcal{F}$ of $\mathcal{K}$ is said to be A-invariant if $A\mathcal{F} \subseteq \mathcal{F}$. If A is $\mathcal{K}$-nonnegative with the only A-invariant faces being the trivial faces, we say A is $\mathcal{K}$-irreducible.

Next we formalize the idea of cone Frobenius normal form discussed by Barker in [2]. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be an ordered partition of $\langle n \rangle$. Let n_j represent the number of elements in α_j. Set $\tilde{I}_j = I_{\alpha_j\langle n\rangle}$, which are the rows of the identity corresponding to α_j.

Definition 1. Let $A \in M_n(\mathbb{R})$ and $\mathcal{K}$ a proper cone in $\mathbb{R}^n$. We say A is in *Frobenius normal form with respect to* $\mathcal{K}$ provided A is $\mathcal{K}$-nonnegative and

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{bmatrix},$$

where for $1 \leq j \leq k$, A_{jj} is $\mathcal{K}_j$-irreducible, with $\mathcal{K}_j = \tilde{I}_j\mathcal{K}$.

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3. The combinatorial structure of general matrices. We provide a proof for the following observation for completeness and to illustrate that it is not dependent on the structure of the chosen field $\mathbb{F}$.

Observation 2. *Let $A \in M_n(\mathbb{F})$. Let x be a generalized eigenvector of A with eigenvalue λ. Build the chain $x_0 = x$, and $x_j = (A - \lambda I)x_{j-1}$, as well as the chain $y_0 = x$, and $y_j = (A^q - \lambda^q I)y_{j-1}$. Choose h such that $x_h = 0$, but $x_{h-1} \neq 0$. Then for $0 \leq j \leq h-1$,*

$$\mathrm{span}\{x_j, x_{j+1}, \ldots, x_{h-1}\} = \mathrm{span}\{y_j, y_{j+1}, \ldots, y_{h-1}\}$$

Proof. Since by construction, $x_h = 0$, for ease of notation we consider $x_l = 0$ for all integers $l \geq h$. First notice that for $j = 0, \ldots, h-1$, $Ax_j = \lambda x_j + x_{j+1}$, thus

$$A^q x_j = \sum_{i=0}^{q} \begin{pmatrix} q \\ i \end{pmatrix} \lambda^{q-i} x_{j+i},$$

and hence, for $0 \leq j \leq h-1$,

$$(A^q - \lambda^q I)x_j = \sum_{i=1}^{q} \begin{pmatrix} q \\ i \end{pmatrix} \lambda^{q-i} x_{j+i}.$$

Thus,

$$y_1 = (A^q - \lambda^q I)y_0 = (A^q - \lambda^q I)x_0 = \sum_{i=1}^{q} \begin{pmatrix} q \\ i \end{pmatrix} \lambda^{q-i} x_i = \sum_{i=0}^{q-1} c_{i1} x_{1+i}$$

for scalars c_{i1} with $i = 1, \ldots, q-1$. Once it is established that there are scalars $c_{0j}, \ldots, c_{m_j j}$, where $m_j = j(q-1)$ such that

$$y_j = \sum_{i=0}^{j(q-1)} c_{ij} x_{j+i},$$

then

$$\begin{aligned} y_{j+1} = (A^q - \lambda^q I)y_j = (A^q - \lambda^q I)\left(\sum_{i=0}^{j(q-1)} c_{ij} x_{j+i}\right) &= \sum_{i=0}^{j(q-1)} c_{ij}(A^q - \lambda^q I)x_{j+i} \\ = \sum_{i=0}^{j(q-1)} c_{ij} \sum_{l=1}^{q} \begin{pmatrix} q \\ l \end{pmatrix} \lambda^{q-l} x_{j+i+l} &= \sum_{i=0}^{(j+1)(q-1)} c_{i(j+1)} x_{j+1+i} \end{aligned}$$

for scalars $c_{0(j+1)}, \ldots, c_{m_{j+1}(j+1)}$, where $m_{j+1} = (j+1)(q-1)$. Since $x_l = 0$ for $l \geq h$, $\mathrm{span}\{y_j, y_{j+1}, \ldots, y_{h-1}\}$ $\subseteq \mathrm{span}\{x_j, x_{j+1}, \ldots, x_{h-1}\}$. Moreover, both sets are linearly independent with the same number of vectors, thus $\mathrm{span}\{y_j, y_{j+1}, \ldots, y_{h-1}\} = \mathrm{span}\ \{x_j, x_{j+1}, \ldots, x_{h-1}\}$. ◻

Let $A \in M_n(\mathbb{F})$ be a matrix that is either nonsingular or for which the Jordan blocks corresponding to the eigenvalue zero are all 1×1. Choose q to be a positive integer such that for all distinct eigenvalues λ and μ in $\sigma(A)$, it follows that $\lambda^q \neq \mu^q$. Choose an invertible matrix S so that $S^{-1}A^qS$ is block upper triangular. Applying the above observation repeatedly to the generalized eigenvectors generated from the generalized eigenvectors of each diagonal block of $S^{-1}A^qS$, we establish that $S^{-1}AS$ is also block upper triangular with the same partitioning. This generalizes Theorem 3.4 of [5] in several ways. First the matrices need not be nonnegative real matrices - the result holds for any matrix over any field. Second, we do not require that the

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diagonal blocks be irreducible in the traditional sense. Example 10 shows why this second generalization is important when we move to viewing matrices that leave a cone invariant.

Before proceeding with our theorem, we provide two examples that show that if either of the two conditions on the spectral properties of A are removed, Theorem 5 is no longer true.

EXAMPLE 3. Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

Notice that $\operatorname{index}_0(A) = 2$ and although A is irreducible, A^q is reducible for all $q > 1$, and hence, there are partitions for which A^q is block upper triangular when A is not. For example,

$$A^2 = \left[\begin{array}{cc|cc} 2 & 2 & 4 & 4 \\ 2 & 2 & 4 & 4 \\ \hline 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right].$$

EXAMPLE 4. Next, consider the matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which has distinct eigenvalues 1, -1, i, and $-i$. For any even number q, notice $(-1)^q = 1^q$, $(-i)^q = i^q$ and B^q is reducible, even though B is irreducible. For example,

$$B^2 = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

is block upper triangular with the indicated partition, whereas B is not.

THEOREM 5. *Let $A \in M_n(\mathbb{F})$ with $\operatorname{index}_0(A) \leq 1$. Choose q to a be positive integer such that for all distinct eigenvalues λ and μ in $\sigma(A)$, it follows that $\lambda^q \neq \mu^q$. Then for any invertible matrix S and any partition that makes $S^{-1}A^qS$ block upper triangular, the matrix $S^{-1}AS$ is block upper triangular with the same partition.*

Proof. Write

$$S^{-1}A^qS = B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ 0 & B_{22} & \cdots & B_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{kk} \end{bmatrix},$$

where each diagonal block B_{jj} is square. We proceed by induction on k.

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If $k = 1$, then B and A are square, and thus, $S^{-1}AS$ and B are in block upper triangular with the same partitioning. If $k > 1$, then B is reducible. Set

$$C_{12} = \begin{bmatrix} B_{12} & B_{13} & \cdots & B_{1k} \end{bmatrix} \quad \text{and} \quad C_{22} = \begin{bmatrix} B_{22} & B_{23} & \cdots & B_{2k} \\ 0 & B_{32} & \cdots & B_{3k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & B_{kk} \end{bmatrix}$$

so $B = \begin{bmatrix} B_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}$. Partition A and S conformably so $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, and write $S^{-1}AS = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$.

Assume B_{11} is $s \times s$. Choose a basis in $\mathbb{E}^n$ of Jordan chains for B_{11},

$$\beta = \{x_{10}, x_{11}, \ldots, x_{1k_1}, x_{20}, x_{21}, \ldots, x_{2k_2}, \ldots, x_{m0}, x_{m1}, \ldots, x_{mk_m}\}.$$

Then,

$$\gamma = \left\{ \begin{bmatrix} x_{10} \\ 0 \end{bmatrix}, \begin{bmatrix} x_{11} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} x_{mk_m} \\ 0 \end{bmatrix} \right\}$$

is a set of Jordan chains for B. By Observation 2,

$$S^{-1}AS \begin{bmatrix} x_{jl} \\ 0 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} x_{jl} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} x_{jk_j} \\ 0 \end{bmatrix} \right\},$$

and thus,

$$S^{-1}AS \begin{bmatrix} x_{jl} \\ 0 \end{bmatrix} = \begin{bmatrix} D_{11}x_{jl} \\ D_{21}x_{jl} \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} \text{ for some } y \in \operatorname{span}\{x_{jl}, \ldots, x_{jk_j}\}.$$

In particular, $D_{21}x_{jl} = 0$. Since β is a basis for $\mathbb{E}^m$ and $D_{21}x_{jl} = 0$ on β, it follows that $D_{21} = 0$. So,

$$S^{-1}AS = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \quad \text{and} \quad S^{-1}A^qS = \begin{bmatrix} (D_{11})^q & F_{12} \\ 0 & (D_{22})^q \end{bmatrix} = \begin{bmatrix} B_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}.$$

Since $C_{22} = (D_{22})^q$ is a block upper triangular matrix with $k-1$ blocks, by the induction hypothesis, D_{22} is also in block upper triangular form with the same partition. Thus, $S^{-1}AS$ and $S^{-1}A^qS$ are both in block upper triangular form with the same partition. ∎

The following corollary of Theorem 5 is a generalization of Theorem 3.4 in [5].

Corollary 6. *Let $A \in M_n(\mathbb{F})$ with $index_0(A) \le 1$. Choose a positive integer q so that for all distinct eigenvalues λ and μ, it follows that $\lambda^q \neq \mu^q$. Then there is a permutation matrix P such that $P^{-1}AP$ and $P^{-1}A^qP$ are in Frobenius normal form with the same partition.*

Proof. Choose a permutation matrix P so that $P^{-1}A^qP$ is in Frobenius normal form. By Theorem 5, $P^{-1}A^qP$ and $P^{-1}AP$ are block upper triangular with the same partition. Since the diagonal blocks of $P^{-1}A^qP$ are irreducible, the diagonal blocks in $P^{-1}AP$ must also be irreducible. Hence, $P^{-1}AP$ is in Frobenius normal form with the same partition. ∎

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Theorem 7. *Let $A \in M_n(\mathbb{F})$ with $\text{index}_0(A) \leq 1$. Choose q to be a positive integer such that for all distinct eigenvalues λ and μ it follows that $\lambda^q \neq \mu^q$ and let α be any partition of $\langle n \rangle$ for which $(S^{-1}A^qS)_\alpha$ is block upper triangular. Then $\overline{\mathcal{R}_\alpha(S^{-1}AS)} = \overline{\mathcal{R}_\alpha(S^{-1}A^qS)}$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a partition of $\langle n \rangle$ such that $(S^{-1}A^qS)_\alpha$ is block upper triangular. By Theorem 5 the matrix $(S^{-1}AS)_\alpha$ is also in block upper triangular form.

Let $j \in \langle n \rangle$ and $l \in \langle n \rangle$. If $j > l$, since both matrices are block upper triangular, there is no path from α_j to α_l in either graph. Assume $j < l$ and that α_j has access to α_l in $\mathcal{R}_\alpha(S^{-1}A^qS)$. Then there must be a path from some vertex α_j to some vertex α_l in $\mathcal{D}(S^{-1}AS)$, and hence, α_j must have access to α_l in $\mathcal{R}_\alpha(S^{-1}AS)$. If α_j does not have access to α_l in $\mathcal{R}_\alpha(S^{-1}A^qS)$, then let

$$T = \{t | \, \alpha_t \text{ has access to } \alpha_l \text{ but not } \alpha_j \text{ in } \mathcal{R}_\alpha(S^{-1}A^qS)\},$$

and

$$P = \{p | \, \alpha_p \text{ is accessed by } \alpha_j \text{ in } \mathcal{R}_\alpha(S^{-1}A^qS), \text{ but not by } \alpha_t \text{ for any } t \in T\}.$$

Notice $\alpha_l \in T$ and $\alpha_j \in P$. Let $p \in P$ and $t \in T$. Suppose there is a path from α_p to α_t in $\mathcal{R}_\alpha(S^{-1}A^qS)$. Then there are paths from α_j to α_p, from α_p to α_t, and from α_t to α_l that together form a path from α_j to α_l, contradicting that α_j does not have access to α_l in $\mathcal{R}_\alpha(S^{-1}A^qS)$. Thus $(S^{-1}A^qS)_{PT} = 0$ and $(S^{-1}A^qS)_{TP} = 0$. It follows then that there is an ordering of the $\{\alpha_1, \ldots, \alpha_k\}$ such that α_l appears in the list before α_j, and for which $S^{-1}A^qS$ is in block upper triangular form. By Theorem 5, $S^{-1}AS$ would also be in block upper triangular form with respect to the reordering of the partition, establishing that α_j does not have access to α_l in $\mathcal{R}_\alpha(S^{-1}AS)$. Since the access relationships in the two graphs are the same, their transitive closures are equal. ∎

Finally, we observe that a well know theorem in the context of matrices in $M_n(\mathbb{C})$ is true over any field.

Observation 8. *Let $A \in M_n(\mathbb{F})$. If the index of cyclicity of A is k, then $\lambda \in \sigma(A)$ implies $\lambda\omega \in \sigma(A)$ for all $\omega \in \mathbb{E}$ such that $\omega^k = 1$.*

Proof. Since A has index of cyclicity k we can assume without loss of generality that

$$A = \begin{bmatrix} 0 & A_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{13} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_{k-1k} \\ A_{k1} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Let $\lambda \in \sigma(A)$ with eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$. Choose $\omega \in \mathbb{E}$ such that ω satisfies $\omega^k = 1$. If $y = \begin{bmatrix} x_1 \\ \omega x_2 \\ \vdots \\ \omega^{k-2}x_{k-1} \\ \omega^{k-1}x_k \end{bmatrix}$, then

$$Ay = \begin{bmatrix} \omega A_{12}x_2 \\ \omega^2 A_{13}x_3 \\ \vdots \\ \omega^{k-1}A_{k-1k}x_k \\ \omega^k A_{k1}x_1 \end{bmatrix} = \begin{bmatrix} \omega\lambda x_1 \\ \omega^2\lambda x_2 \\ \vdots \\ \omega^{k-1}\lambda x_{k-1} \\ \omega^k \lambda x_k \end{bmatrix} = \lambda\omega \begin{bmatrix} x_1 \\ \omega x_2 \\ \vdots \\ \omega^{k-2}x_{k-1} \\ \omega^{k-1}\lambda x_k \end{bmatrix}.$$ Hence, $\lambda\omega \in \sigma(A)$. ∎

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Corollary 9. *Let $A \in M_n(\mathbb{F}_p)$ for some prime p. If the index of cyclicity of A is p^k, then $\lambda \in \sigma(A)$ implies $\lambda\omega \in \sigma(A)$ for all nonzero $\omega \in \mathbb{F}_{p^k}$.*

4. Frobenius normal form for cones. In this section, we will restrict our fields to $\mathbb{R}$ and $\mathbb{C}$. Since we will be considering matrices that leave a cone invariant, we need the base field to be ordered, have an absolute value, and have a topology. We begin this section with two interesting examples. The first example shows that a matrix $A \in M_n(\mathbb{R})$ may be $\mathcal{K}$-irreducible for a proper cone $\mathcal{K}$, even though it is reducible with respect to the traditional definition of matrix reducibility. The second example shows that even though a $\mathcal{K}$-nonnegative matrix A may not be $\mathcal{K}$-irreducible, there is no invertible matrix S which puts A in cone Frobenius normal form with respect to the cone $S^{-1}\mathcal{K}$, presenting a counterexample to Theorem 7 in [2].

Example 10. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{K} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^G .$$

It is easy to verify that A is $\mathcal{K}$-irreducible for the proper cone $\mathcal{K}$, but A is clearly a reducible matrix in the traditional sense.

Example 11. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}, \quad \mathcal{L} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 0 \le x,\ 0 \le y, \text{ and } z^2 \le xy \right\} \quad \text{and} \quad \mathcal{F} = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \mid a > 0 \right\}.$$

Notice that even though A is $\mathcal{L}$-nonnegative for the Lorenz cone $\mathcal{L}$, it is not $\mathcal{L}$-irreducible since $\mathcal{F}$ is an invariant face. Moreover, for any invertible matrix S, notice $S^{-1}AS$ leaves $S^{-1}\mathcal{F}$ invariant. The eigenvalue of A associated with $\mathcal{F}$ is 1, however the remaining eigenvalues of A are $\frac{1}{2}$ and $-\frac{2}{3}$, which cannot be the eigenvalues of a cone nonnegative matrix by Theorem 3.2 in [4] since the spectral radius of the lower 2×2 principal submatrix A_{22} is not an eigenvalue of A_{22}.

Theorem 12. *Let $\mathcal{K}$ be a proper cone and $A \in M_n(\mathbb{R})$ a matrix that is eventually $\mathcal{K}$-nonnegative with $\mathrm{index}_0(A) \le 1$. Let q be an odd positive integer such that A^q is $\mathcal{K}$-nonnegative and for all distinct eigenvalues λ and μ in $\sigma(A)$, it follows that $\lambda^q \ne \mu^q$. If there exists an invertible matrix S such that $S^{-1}A^qS$ is in cone Frobenius normal form with respect to $\mathcal{L} = S^{-1}\mathcal{K}$, then*

(i) *The matrix $S^{-1}AS$ is block upper triangular when partitioned conformally with the cone Frobenius normal form of $S^{-1}A^qS$.*
(ii) *For any positive integer r, the matrix $S^{-1}A^rS$ is block upper triangular when partitioned conformally with the cone Frobenius normal form of $S^{-1}A^qS$.*
(iii) *For any diagonal block B_{jj} of $B = S^{-1}AS$, partitioned conformably with the Frobenius normal form of C=$S^{-1}A^qS$,*
  - **(a)** *The matrix B_{jj} does not leave any nontrivial face of $\tilde{I}_j\mathcal{L}$ invariant.*
  - **(b)** *If $\rho(B_{jj}) > 0$, then $\rho(B_{jj}) \in \sigma(B_{jj})$ is a simple eigenvalue and for each $\lambda \in \pi(B_{jj})$, the $\mathrm{index}_\lambda(B_{jj}) \le 1$.*
  - **(c)** *If $S^{-1}A^{q+1}S$ is also $\mathcal{K}$-nonnegative and in cone Frobenius normal form with respect to $\mathcal{L}$ with the same partition, then each B_{jj} is eventually $\tilde{I}_j\mathcal{L}$-nonnegative.*

Proof. Let $B = S^{-1}AS$ and $C = S^{-1}A^qS = (S^{-1}AS)^q$.

(i) That B is block upper triangular when partitioned conformably with the cone Frobenius normal form of C follows directly from Theorem 5.

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(ii) Result (ii) follows easily from result (i).

(iii) Using (i) write

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ 0 & B_{22} & \cdots & B_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{kk} \end{bmatrix}$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ 0 & C_{22} & \cdots & C_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C_{kk} \end{bmatrix},$$

where with this partition, $C = S^{-1}A^qS$ is in Frobenius normal form with respect to $\mathcal{L}$.

(a) Suppose for some j there is a nontrivial face $\mathcal{F}$ of $\tilde{I}_j\mathcal{L}$ left invariant by B_{jj}. Then $C_{jj}\mathcal{F} = (B_{jj})^q\mathcal{F} \subseteq \mathcal{F}$ contradicting that C_{jj} is $\tilde{I}_j\mathcal{L}$-irreducible.

(b) Recall that if a J is an $r \times r$ Jordan block with nonzero eigenvalue λ, then the Jordan form associated with J^q is a single $r\times r$ Jordan block with eigenvalue λ^q. Since C_{jj} is $\tilde{I}_j\mathcal{L}$-irreducible, for each $\mu \in \pi(C_{jj})$, the $\text{index}_\mu(C_{jj}) \leq 1$ by Theorem 3.23 in [4]. Note that $\pi(C_{jj})$ consists of the elements of $\pi(B_{jj})$ each raised to the q-th power, and thus, for any $\lambda \in \pi(B_{jj})$, the Jordan blocks associated with λ in B_{jj} are the same size as the Jordan blocks associated with λ^q in C_{jj}, thus the $\text{index}_\lambda(B_{jj}) \leq 1$.

Choose $\eta \in \sigma(B_{jj})$ such that $\eta^q = \rho(C_{jj}) > 0$. We claim that η is a positive real number, and hence, $\eta = \rho(B_{jj})$. Suppose not. If η is not real, then $\overline{\eta} \in \sigma(B_{jj})$ and $\overline{\eta}^q = \overline{\eta^q} = \overline{\rho(C_{jj})} = \rho(C_{jj}) = \eta^q$, contradicting our choice of q. Since q is odd, if η was a negative real number, then η^q would be a negative real number, which again would be a contradiction. Thus, $\eta > 0$, and hence, $\eta = \rho(B_{jj})$ is a simple eigenvalue of B_{jj}.

(c) For $g > q^2$, there are nonnegative integers s and r such that $g = sq + r$ where $s \geq q$ and $0 \leq r < q$. But then $g = sq + r(q + 1 - q) = (s - r)q + r(q + 1)$. If $(B_{jj})^q$ and $(B_{jj})^{q+1}$ are both $\tilde{I}_j\mathcal{L}$ invariant then $(B_{jj})^g\tilde{I}_j\mathcal{L} = ((B_{jj})^q)^{s-r}((B_{jj})^{q+1})^r\tilde{I}_j\mathcal{L} \subseteq \tilde{I}_j\mathcal{L}$

Note that in Theorem 12 we have not claimed that the diagonal blocks of $S^{-1}AS$ are $\tilde{I}_jS^{-1}\mathcal{K}$-nonnegative without adding some additional condition. The following example shows that the situation may be more complicated than it first appears.

Example 13. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},\ x_1 = \begin{bmatrix} 1 \\ -1 \\ -\sqrt{3} \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ -1 \\ \sqrt{3} \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Set $\mathcal{K} = \{x_1, x_2, x_3\}^G$. Notice that A^2 is $\mathcal{K}$-irreducible, but A and A^3 are not even $\mathcal{K}$-nonnegative.

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