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BLOCK GENERALIZED LOCALLY TOEPLITZ SEQUENCES: THE CASE OF MATRIX FUNCTIONS AND AN ENGINEERING APPLICATION*

CARLO GARONI† AND STEFANO SERRA-CAPIZZANO‡

Abstract. The theory of block generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the spectral distribution of block-structured matrices arising from the discretization of differential problems, with a special reference to systems of differential equations (DEs) and to the higher-order finite element or discontinuous Galerkin approximation of both scalar and vectorial DEs. In the present paper, the theory of block GLT sequences is extended by proving that $\{f(A_n)\}_n$ is a block GLT sequence as long as f is continuous and $\{A_n\}_n$ is a block GLT sequence formed by Hermitian matrices. It is also provided a relevant application of this result to the computation of the distribution of the numerical eigenvalues obtained from the higher-order isogeometric Galerkin discretization of second-order variable-coefficient differential eigenvalue problems (a topic of interest not only in numerical analysis but also in engineering).

Key words. Block generalized locally Toeplitz sequences, Matrix functions, Differential eigenvalue problems, Higher-order isogeometric Galerkin method, B-splines.

AMS subject classifications. 15B05, 65F60, 65N25, 65N30, 65D07.

1. Introduction.

The theory of generalized locally Toeplitz (GLT) sequences stems from Tilli's work on locally Toeplitz (LT) sequences [37] and from the spectral theory of Toeplitz matrices [1, 13, 14, 15, 16, 26, 30, 36, 38, 39, 40]; we refer the reader to [12] for a gentle introduction to this subject, to [21, 22, 34, 35] for advanced studies, and to [2, 3, 4, 5, 6, 8] for further recent developments. Starting from the original intuition in [35, Section 3.3], the theory of block GLT sequences has been recently developed in [23] as an extension of the theory of (scalar) GLT sequences. Just as the latter, the theory of block GLT sequences has been devised in order to solve a specific application problem, namely the problem of computing/analyzing the spectral distribution of matrices arising from the numerical discretization of differential problems. In particular, this theory applies to block-structured matrices arising from either the discretization of systems of differential equations (DEs) or the higher-order finite element (FE) or discontinuous Galerkin (DG) approximation of both scalar and vectorial DEs; see [9, 19, 20]. More details on the theory of block GLT sequences and its applications can be found in [20, 23].

This paper was born from the observation that the theory of block GLT sequences covered in [23] is actually incomplete if compared to the theory of GLT sequences [21]. In particular, a crucial result in [21] states that if $\{A_n\}_n$ is a GLT sequence formed by Hermitian matrices and $f : \mathbb{C} \to \mathbb{C}$ is a continuous function, then $\{f(A_n)\}_n$ is again a GLT sequence. The version of this result with "GLT sequence" replaced by "block GLT sequence" is important not only from a numerical analysis point of view but also from an engineering point of view. Consider, for example, the isogeometric Galerkin discretization (based on B-splines with arbitrary degree and smoothness) of second-order variable-coefficient differential eigenvalue

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problems, as in the engineering review [24]. The distribution of the associated numerical eigenvalues has been deeply investigated in [24] as it allows one to provide analytical predictions for the so-called eigenvalue errors (i.e., the errors occurring when approximating the exact eigenvalues with the numerical eigenvalues), thus extending several spectral results from the engineering literature [18, 28, 29, 31]. Despite the careful study conducted in [24], all the main results contained therein have been stated without a proof, because their formal mathematical derivation requires precisely the aforementioned result about functions of block GLT sequences.

In this paper, we extend the theory of block GLT sequences developed in [23] by proving the aforementioned result. As an application, we provide formal mathematical proofs to the main results appeared in [24]. Further applications include the computation of the spectral distribution of matrix functions of the form $f(A_n)$, where n represents the mesh-fineness parameter and A_n is the matrix arising from the discretization of a system of DEs or the higher-order FE/DG approximation of a scalar/vectorial DE. Indeed, the matrices A_n arising from these discretization processes are often block-structured matrices such that $\{A_n\}_n$ is a block GLT sequence.

The paper is organized as follows. In Section 2, we recall the definition and some properties of matrix functions. In Section 3, we report from [23] a summary of the theory of block GLT sequences, including additional topics that we shall need later on. In Section 4, we prove our main result about functions of block GLT sequences. In Section 5, we provide formal mathematical proofs to the main results of [24].

2. Matrix functions. Given a diagonalizable matrix $A \in \mathbb{C}^{m\times m}$, if $\lambda_1,\ldots,\lambda_t$ are the *distinct* eigenvalues of A and $V_1,\ldots,V_t$ are their respective eigenspaces, we have $\mathbb{C}^m = \bigoplus_{i=1}^t V_i$. For each function $f:\Lambda(A)\to\mathbb{C}$, we define $f(A)$ as the matrix such that

$$f(A)\mathbf{v} = f(\lambda_i)\mathbf{v} \quad \text{for every } \mathbf{v}\in V_i \text{ and every } i=1,\ldots,t. \tag{2.1}$$

In practice, $f(A)$ is the matrix that possesses the same eigenspaces $V_1,\ldots,V_t$ as A with corresponding eigenvalues $f(\lambda_1),\ldots,f(\lambda_t)$. Note that such a matrix $f(A)$ exists and is unique. To show the uniqueness, simply note that, if B is another matrix such that $B\mathbf{v} = f(\lambda_i)\mathbf{v} = f(A)\mathbf{v}$ for every $\mathbf{v}\in V_i$ and every $i=1,\ldots,t$, then B coincides with $f(A)$ on each basis of $\mathbb{C}^m$ formed by eigenvectors of A, hence $B=f(A)$. To show the existence, fix a basis of $\mathbb{C}^m$ formed by eigenvectors of A, define $f(A)$ on this basis in the unique possible way to meet (2.1), and extend the definition to the whole $\mathbb{C}^m$ by linearity. It can be checked that the matrix $f(A)$ defined in this way satisfies (2.1).

Now, let $A\in\mathbb{C}^{m\times m}$ be diagonalizable and let $\lambda_1,\ldots,\lambda_m$ denote *all* the eigenvalues of A, i.e., all the roots of the characteristic polynomial of A, each of them counted with its multiplicity. If $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is a basis of $\mathbb{C}^m$ formed by eigenvectors of A,

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i, \quad i=1,\ldots,m,$$

then for each $f:\Lambda(A)\to\mathbb{C}$, we have

$$f(A)\mathbf{v}_i = f(\lambda_i)\mathbf{v}_i, \quad i=1,\ldots,m.$$

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This is a spectral decomposition of $f(A)$, which can be rewritten in matrix form as

$$f(A) = V \begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_m) \end{bmatrix} V^{-1}, \quad V = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array}\right]. \tag{2.2}$$

As a straightforward consequence of (2.2), if $f(\lambda) = \lambda$, then $f(A) = A$. Moreover, if A is invertible and $f(\lambda) = \lambda^{-1}$, then $f(A) = A^{-1}$. Further properties of matrix functions are given in the next lemma.

Lemma 2.1. *Let A be a diagonalizable matrix. Then, the following properties hold.*

1. *If $\alpha, \beta \in \mathbb{C}$ and $f, g : \Lambda(A) \to \mathbb{C}$ then $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$.*
2. *If $f, g : \Lambda(A) \to \mathbb{C}$ then $(fg)(A) = f(A)g(A)$ and $(gf)(A) = g(A)f(A)$. In particular, two functions of the same matrix always commute.*
3. *Suppose $\Lambda(A) \subset (0, \infty)$, so that the functions $\lambda^{1/2}$, $\lambda^{-1/2} : \Lambda(A) \to \mathbb{R}$ are well-defined and hence also the matrices $A^{1/2}$, $A^{-1/2}$ via definition (2.1). Then, we have $A^{1/2}A^{-1/2} = I$, $(A^{1/2})^2 = A$, $(A^{-1/2})^2 = A^{-1}$.*
4. *If $p(\lambda) = \sum_{j=0}^{r} a_j \lambda^j$ is a polynomial, then the matrix $p(A)$ obtained from definition (2.1) coincides with $\sum_{j=0}^{r} a_j A^j$.*

For more on matrix functions, including the definition of $f(A)$ in the case where A is not diagonalizable, we refer the reader to Higham's book [27].

3. The theory of block GLT sequences.

In this section, we summarize the theory of block GLT sequences, which was originally introduced in [35, Section 3.3] and has been recently revised and systematically developed in [23]. We also cover additional topics that we shall need later on.

Sequences of matrices and block matrix-sequences. If $A \in \mathbb{C}^{m\times m}$, its singular values and eigenvalues are denoted by $\sigma_1(A), \ldots, \sigma_m(A)$ and $\lambda_1(A), \ldots, \lambda_m(A)$, respectively. Throughout this paper, a sequence of matrices is any sequence of the form $\{A_n\}_n$, where A_n is a square matrix of size d_n and $d_n \to \infty$ as $n \to \infty$. We say that the sequence of matrices $\{A_n\}_n$, with A_n of size d_n, is sparsely unbounded (s.u.) if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\#\{i \in \{1, \ldots, d_n\} : \ \sigma_i(A_n) > M\}}{d_n} = 0.$$

For the next result, which is fundamental to our purposes, see [22, Section 2.6.4].

S 1. If $\{A_n\}_n$ is a s.u. sequence of Hermitian matrices, with A_n of size d_n, then the following property holds: for every $M > 0$ there exists n_M such that, for $n \ge n_M$,

$$A_n = \hat{A}_{n,M} + \tilde{A}_{n,M}, \quad \operatorname{rank}(\hat{A}_{n,M}) \le r(M) d_n, \quad \|\tilde{A}_{n,M}\| \le M,$$

where $r(M) \to 0$ as $M \to \infty$, the matrices $\hat{A}_{n,M}$ and $\tilde{A}_{n,M}$ are Hermitian, and for all functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$g(\hat{A}_{n,M} + \tilde{A}_{n,M}) = g(\hat{A}_{n,M}) + g(\tilde{A}_{n,M}).$$

Let $s \ge 1$ be a fixed positive integer independent of n; an s-block matrix-sequence (or simply a matrix-sequence if s can be inferred from the context or we do not need/want to specify it) is a special sequence of matrices $\{A_n\}_n$ in which the size of A_n is $d_n = sn$.

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Singular value and eigenvalue distribution of a sequence of matrices. Let μ_k be the Lebesgue measure in $\mathbb{R}^k$. Throughout this paper, all the terminology from measure theory (such as "measurable set", "measurable function", "a.e.", etc.) is referred to the Lebesgue measure. A matrix-valued function $f : D \subseteq \mathbb{R}^k \to \mathbb{C}^{r\times r}$ is said to be measurable (resp., continuous, Riemann-integrable, in $L^p(D)$, etc.) if its components $f_{\alpha\beta} : D \to \mathbb{C},\ \alpha, \beta = 1, \dots, r$, are measurable (resp., continuous, Riemann-integrable, in $L^p(D)$, etc.). We denote by $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) the space of continuous complex-valued functions with bounded support defined on $\mathbb{R}$ (resp., $\mathbb{C}$).

Definition 3.1. Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n, and let $f : D \subset \mathbb{R}^k \to \mathbb{C}^{r\times r}$ be a measurable function defined on a set D with $0 < \mu_k(D) < \infty$.

- We say that $\{A_n\}_n$ has a (asymptotic) singular value distribution described by f, and we write $\{A_n\}_n \sim_\sigma f$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D \frac{\sum_{i=1}^{r} F(\sigma_i(f(\mathbf{x})))}{r} \mathrm{d}\mathbf{x}, \quad \forall\, F \in C_c(\mathbb{R}). \tag{3.3}$$

- We say that $\{A_n\}_n$ has a (asymptotic) spectral (or eigenvalue) distribution described by f, and we write $\{A_n\}_n \sim_\lambda f$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D \frac{\sum_{i=1}^{r} F(\lambda_i(f(\mathbf{x})))}{r} \mathrm{d}\mathbf{x}, \quad \forall\, F \in C_c(\mathbb{C}). \tag{3.4}$$

If $\{A_n\}_n$ has both a singular value and an eigenvalue distribution described by f, we write $\{A_n\}_n \sim_{\sigma,\lambda} f$.

We note that Definition 3.1 is well-posed as the functions

$$\mathbf{x} \mapsto \sum_{i=1}^{r} F(\sigma_i(f(\mathbf{x}))) \quad \text{and} \quad \mathbf{x} \mapsto \sum_{i=1}^{r} F(\lambda_i(f(\mathbf{x})))$$

are measurable [23, Lemma 2.1]. Whenever we write a relation such as $\{A_n\}_n \sim_\sigma f$ or $\{A_n\}_n \sim_\lambda f$, it is understood that f is as in Definition 3.1; that is, f is a measurable function defined on a subset D of some $\mathbb{R}^k$ with $0 < \mu_k(D) < \infty$, and f takes values in $\mathbb{C}^{r\times r}$ for some $r \geq 1$. We refer the reader to [20, Remark 1] for the informal meaning behind the singular value and spectral distributions (3.3) and (3.4).

In what follows, the conjugate transpose of the matrix A is denoted by A^* and the spectrum of A by $\Lambda(A)$. If $A \in \mathbb{C}^{m\times m}$ and $1 \leq p \leq \infty$, we denote by $\|A\|_p$ the Schatten p-norm of A, i.e., the p-norm of the vector $(\sigma_1(A), \dots, \sigma_m(A))$. The Schatten ∞-norm $\|A\|_\infty$ is the largest singular value of A and coincides with the spectral norm $\|A\|$. The Schatten 1-norm $\|A\|_1$ is the sum of the singular values of A and is often referred to as the trace-norm of A. The Schatten 2-norm $\|A\|_2$ coincides with the Frobenius norm of A. For more on Schatten p-norms, see [10].

D 1. If $\{A_n\}_n \sim_\sigma f$, then $\{A_n\}_n$ is s.u.
D 2. If $\{A_n\}_n \sim_\lambda f$ and $\Lambda(A_n) \subseteq S$ for all n, then $\Lambda(f) \subseteq \overline{S}$ a.e.

For the proof of D 1, see [23, Proposition 2.3]. Property D 2 is stated in [25, Theorem 4.2] for Lebesgue-integrable functions f; it can be proved for general measurable functions f by the same line of argument as in the proof of [25, Theorem 2.4]; a formal proof is given in [7, Lemma 2.2 and Corollary 2.13].

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Block Toeplitz matrices. Given a function $f:[-\pi,\pi]\to\mathbb{C}^{s\times s}$ in $L^1([-\pi,\pi])$, its Fourier coefficients are denoted by

$$f_k=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta\in\mathbb{C}^{s\times s},\quad k\in\mathbb{Z},$$

where the integrals are computed componentwise. The nth block Toeplitz matrix generated by f is defined as

$$T_n(f)=[f_{i-j}]_{i,j=1}^n\in\mathbb{C}^{sn\times sn}.$$

It is not difficult to see that all the matrices $T_n(f)$ are Hermitian if f is Hermitian a.e.

Block diagonal sampling matrices. For $n\in\mathbb{N}$ and $a:[0,1]\to\mathbb{C}^{s\times s}$, we define the block diagonal sampling matrix $D_n(a)$ as the block diagonal matrix

$$D_n(a)=\operatorname*{diag}_{i=1,\ldots,n} a\Big(\frac{i}{n}\Big)=\begin{bmatrix} a(\frac{1}{n}) & & & \\ & a(\frac{2}{n}) & & \\ & & \ddots & \\ & & & a(1)\end{bmatrix}\in\mathbb{C}^{sn\times sn}.$$

Zero-distributed sequences. A sequence of matrices $\{Z_n\}_n$ such that $\{Z_n\}_n\sim_\sigma 0$ is referred to as a zero-distributed sequence. Note that, for any $r\geq 1$, $\{Z_n\}_n\sim_\sigma 0$ is equivalent to $\{Z_n\}_n\sim_\sigma O_r$ (throughout this paper, O_m and I_m denote the $m\times m$ zero matrix and the $m\times m$ identity matrix, respectively). In what follows, we use the natural convention $1/\infty=0$. For the next results, see [22, Section 2.6.3].

Z 1. Let $\{Z_n\}_n$ be a sequence of matrices, with Z_n of size d_n. We have $\{Z_n\}_n\sim_\sigma 0$ if and only if $Z_n=R_n+N_n$ with $\operatorname{rank}(R_n)/d_n\to 0$ and $\|N_n\|\to 0$ as $n\to\infty$.

Z 2. Let $\{Z_n\}_n$ be a sequence of matrices, with Z_n of size d_n. If there exists a $p\in[1,\infty]$ such that $\|Z_n\|_p/(d_n)^{1/p}\to 0$ as $n\to\infty$, then $\{Z_n\}_n\sim_\sigma 0$.

Approximating classes of sequences. The notion of approximating classes of sequences (a.c.s.) is the fundamental concept on which the theory of block GLT sequences is based.

Definition 3.2. Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n, and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of sequences of matrices, with $B_{n,m}$ of size d_n. We say that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if the following condition is met: for every m there exists n_m such that, for $n\geq n_m$,

$$A_n=B_{n,m}+R_{n,m}+N_{n,m},\quad \operatorname{rank}(R_{n,m})\leq c(m)d_n,\quad \|N_{n,m}\|\leq\omega(m),$$

where n_m, $c(m)$, $\omega(m)$ depend only on m and $\lim_{m\to\infty}c(m)=\lim_{m\to\infty}\omega(m)=0$.

Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if, for large m, the sequence $\{B_{n,m}\}_n$ approximates $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size d_n) plus a small-norm matrix. It turns out that, for each fixed sequence of positive integers d_n such that $d_n\to\infty$, the notion of a.c.s. is a notion of convergence in the space $\mathscr{E}=\{\{A_n\}_n:\ A_n\in\mathbb{C}^{d_n\times d_n}\}$.

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More precisely, there exists a pseudometric $d_{\text{a.c.s.}}$ in $\mathscr{E}$ such that $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if and only if $d_{\text{a.c.s.}}(\{B_{n,m}\}_n, \{A_n\}_n) \to 0$ as $m \to \infty$; see [22, Section 2.7.1]. We will therefore use the convergence notation $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ to indicate that $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$. A useful criterion to identify an a.c.s. is provided below [22, Section 2.7.4].

ACS 1. Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n, let $\{\{B_{n,m}\}_n\}_m$ be a sequence of sequences of matrices, with $B_{n,m}$ of size d_n, and let $p \in [1, \infty]$. Suppose that for every m there exists n_m such that, for $n \geq n_m$,
$$\|A_n - B_{n,m}\|_p \leq \varepsilon(m, n)(d_n)^{1/p},$$
where $\lim_{m\to\infty} \limsup_{n\to\infty} \varepsilon(m, n) = 0$. Then $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

Block GLT sequences. We intentionally omit the formal definition of block GLT sequences for two reasons. First, the definition is rather cumbersome as it requires to introduce other related (and complicated) concepts such as "block LT operators" and "block LT sequences". Second, the definition is not necessary to our purposes, because everything that can be derived from it can also be derived (and in a much easier way) from the properties GLT 1 – GLT 4 reported below. The reader who is interested in the formal definition of block GLT sequences can find it in [23, Section 5] along with the proofs of properties GLT 1 – GLT 4.

Let $s \geq 1$ be a fixed positive integer. An s-block GLT sequence (or simply a GLT sequence if s can be inferred from the context or we do not need/want to specify it) is a special s-block matrix-sequence $\{A_n\}_n$ equipped with a measurable function $\kappa : [0, 1] \times [-\pi, \pi] \to \mathbb{C}^{s\times s}$, the so-called symbol. We use the notation $\{A_n\}_n \sim_{\text{GLT}} \kappa$ to indicate that $\{A_n\}_n$ is an s-block GLT sequence with symbol κ. The symbol of an s-block GLT sequence is unique in the sense that if $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{A_n\}_n \sim_{\text{GLT}} \varsigma$ then $\kappa = \varsigma$ a.e. in $[0, 1] \times [-\pi, \pi]$. The main properties of s-block GLT sequences proved in [23] are listed below. If A is a matrix, we denote by $A^\dagger$ the Moore–Penrose pseudoinverse of A (recall that $A^\dagger = A^{-1}$ whenever A is invertible). If $f_m, f : D \subseteq \mathbb{R}^k \to \mathbb{C}^{r\times r}$ are measurable matrix-valued functions, we say that f_m converges to f in measure (resp., a.e., in $L^p(D)$, etc.) if $(f_m)_{\alpha\beta}$ converges to $f_{\alpha\beta}$ in measure (resp., a.e., in $L^p(D)$, etc.) for all $\alpha, \beta = 1, \dots, r$.

GLT 1. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ then $\{A_n\}_n \sim_\sigma \kappa$. If moreover each A_n is Hermitian, then κ is Hermitian a.e. in $[0, 1] \times [-\pi, \pi]$ and $\{A_n\}_n \sim_\lambda \kappa$.

GLT 2. We have:
- $\{T_n(f)\}_n \sim_{\text{GLT}} \kappa(x, \theta) = f(\theta)$ if $f : [-\pi, \pi] \to \mathbb{C}^{s\times s}$ is in $L^1([-\pi, \pi])$;
- $\{D_n(a)\}_n \sim_{\text{GLT}} \kappa(x, \theta) = a(x)$ if $a : [0, 1] \to \mathbb{C}^{s\times s}$ is Riemann-integrable;
- $\{Z_n\}_n \sim_{\text{GLT}} \kappa(x, \theta) = O_s$ if and only if $\{Z_n\}_n \sim_\sigma 0$.

GLT 3. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{B_n\}_n \sim_{\text{GLT}} \varsigma$ then:
- $\{A_n^*\}_n \sim_{\text{GLT}} \kappa^*$;
- $\{\alpha A_n + \beta B_n\}_n \sim_{\text{GLT}} \alpha\kappa + \beta\varsigma$ for all $\alpha, \beta \in \mathbb{C}$;
- $\{A_n B_n\}_n \sim_{\text{GLT}} \kappa\varsigma$;
- $\{A_n^\dagger\}_n \sim_{\text{GLT}} \kappa^{-1}$ provided that κ is invertible a.e.

GLT 4. $\{A_n\}_n \sim_{\text{GLT}} \kappa$ if and only if there exist s-block GLT sequences $\{B_{n,m}\}_n \sim_{\text{GLT}} \kappa_m$ such that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ and $\kappa_m \to \kappa$ in measure.

4. Main result. This section is devoted to stating and proving the main theoretical result of the paper. If $f : \mathbb{K} \to \mathbb{K}$, with $\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{C}$, and if $\kappa : [0, 1] \times [-\pi, \pi] \to \mathbb{C}^{s\times s}$ is measurable and Hermitian a.e., we denote by $f(\kappa)$ the function (defined a.e. in $[0, 1] \times [-\pi, \pi]$) that associates with $(x, \theta) \in [0, 1] \times [-\pi, \pi]$

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the matrix $f(\kappa(x,\theta))$. If E is any set, we denote by χ_E and E^c the characteristic (indicator) function of E and the complementary set of E, respectively.

THEOREM 4.1. *Let $\{A_n\}_n$ be an s-block matrix-sequence and let $\kappa : [0,1]\times[-\pi,\pi] \to \mathbb{C}^{s\times s}$ be measurable. If $\{A_n\}_n \sim_{\rm GLT} \kappa$ and each A_n is Hermitian then $\{f(A_n)\}_n \sim_{\rm GLT} f(\kappa)$ for any continuous function $f : \mathbb{C} \to \mathbb{C}$.*

Proof. It suffices to prove the theorem for real continuous functions $f : \mathbb{R} \to \mathbb{R}$, since every A_n is Hermitian by assumption and κ is Hermitian a.e. by GLT 1. Indeed, suppose we have proved the theorem for real continuous functions and let $f : \mathbb{C} \to \mathbb{C}$ be any continuous complex function. Denote by $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ the real and imaginary parts of the restriction of f to $\mathbb{R}$. Then, α, β are continuous functions such that $f(x) = \alpha(x) + {\rm i}\beta(x)$ for all $x \in \mathbb{R}$, and since the eigenvalues of A_n are real we have $f(A_n) = \alpha(A_n) + {\rm i}\beta(A_n)$. In view of the relations $\{\alpha(A_n)\}_n \sim_{\rm GLT} \alpha(\kappa)$ and $\{\beta(A_n)\}_n \sim_{\rm GLT} \beta(\kappa)$, GLT 3 yields $\{f(A_n)\}_n \sim_{\rm GLT} \alpha(\kappa) + {\rm i}\beta(\kappa) = f(\kappa)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real continuous function. For each $M > 0$, let $\{p_{m,M}\}_m$ be a sequence of polynomials that converges uniformly to f over $[-M, M]$:

$$\lim_{m\to\infty} \|f - p_{m,M}\|_{\infty,[-M,M]} = 0.$$

Note that such a sequence exists by the Weierstrass theorem; see, e.g., [32, Theorem 7.26]. Since any block GLT sequence is s.u. (by GLT 1 and D 1), the sequence $\{A_n\}_n$ is s.u. Hence, by S 1, for all $M > 0$, there exists n_M such that, for $n \ge n_M$,

$$A_n = \hat{A}_{n,M} + \tilde{A}_{n,M}, \quad \operatorname{rank}(\hat{A}_{n,M}) \le r(M)sn, \quad \|\tilde{A}_{n,M}\| \le M,$$

where $r(M) \to 0$ as $M \to \infty$, the matrices $\hat{A}_{n,M}$ and $\tilde{A}_{n,M}$ are Hermitian, and for all functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$g(\hat{A}_{n,M} + \tilde{A}_{n,M}) = g(\hat{A}_{n,M}) + g(\tilde{A}_{n,M}).$$

Thus, for every $M > 0$, every m and every $n \ge n_M$, we can write

$$\begin{aligned} f(A_n) &= p_{m,M}(A_n) + f(A_n) - p_{m,M}(A_n) \\ &= p_{m,M}(A_n) + f(\hat{A}_{n,M}) + f(\tilde{A}_{n,M}) - p_{m,M}(\hat{A}_{n,M}) - p_{m,M}(\tilde{A}_{n,M}) \\ &= p_{m,M}(A_n) + (f - p_{m,M})(\hat{A}_{n,M}) + (f - p_{m,M})(\tilde{A}_{n,M}). \end{aligned} \tag{4.5}$$

The matrix $(f - p_{m,M})(\hat{A}_{n,M})$ can be written as the sum of two terms, namely

$$(f - p_{m,M})(\hat{A}_{n,M}) = R_{n,m,M} + N'_{n,m,M},$$

where

$$\begin{aligned} R_{n,m,M} &= (f - p_{m,M})(\hat{A}_{n,M}) \cdot \chi_{S^c}((f - p_{m,M})(\hat{A}_{n,M})) = \alpha(\hat{A}_{n,M}), \\ N'_{n,m,M} &= (f - p_{m,M})(\hat{A}_{n,M}) \cdot \chi_S((f - p_{m,M})(\hat{A}_{n,M})) = \beta(\hat{A}_{n,M}), \end{aligned}$$

S is the singleton $S = \{(f - p_{m,M})(0)\} \subset \mathbb{C}$, and

$$\begin{aligned} \alpha(\lambda) &= (f - p_{m,M})(\lambda) \cdot \chi_{S^c}((f - p_{m,M})(\lambda)), \\ \beta(\lambda) &= (f - p_{m,M})(\lambda) \cdot \chi_S((f - p_{m,M})(\lambda)). \end{aligned}$$

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Note that $R_{n,m,M}$ is obtained from the spectral decomposition of $(f - p_{m,M})(\hat{A}_{n,M})$ by setting to 0 all the eigenvalues that are equal to $(f - p_{m,M})(0)$, while $N'_{n,m,M}$ is obtained from the spectral decomposition of $(f - p_{m,M})(\hat{A}_{n,M})$ by setting to 0 all the eigenvalues that are different from $(f - p_{m,M})(0)$. Note that [1]

$$\operatorname{rank}(R_{n,m,M}) \le \operatorname{rank}(\hat{A}_{n,M}) \le r(M)sn,$$
$$\|N'_{n,m,M}\| \le |f(0) - p_{m,M}(0)|.$$

Concerning the matrix $N''_{n,m,M} = (f - p_{m,M})(\tilde{A}_{n,M})$, the inequality $\|\tilde{A}_{n,M}\| \le M$ yields

$$\|N''_{n,m,M}\| \le \|f - p_{m,M}\|_{\infty,[-M,M]}.$$

Let

$$N_{n,m,M} = N'_{n,m,M} + N''_{n,m,M}.$$

By (4.5), for every $M > 0$, every m and every $n \ge n_M$, we have

$$f(A_n) = p_{m,M}(A_n) + R_{n,m,M} + N_{n,m,M},$$

where

$$\operatorname{rank}(R_{n,m,M}) \le r(M)sn,$$
$$\|N_{n,m,M}\| \le \|N'_{n,m,M}\| + \|N''_{n,m,M}\| \le 2\|f - p_{m,M}\|_{\infty,[-M,M]}.$$

Choose a sequence $\{M_m\}_m$ such that

$$M_m \to \infty, \quad \|f - p_{m,M_m}\|_{\infty,[-M_m,M_m]} \to 0. \tag{4.6}$$

Then, for every m and every $n \ge n_{M_m}$,

$$f(A_n) = p_{m,M_m}(A_n) + R_{n,m,M_m} + N_{n,m,M_m},$$
$$\operatorname{rank}(R_{n,m,M_m}) \le r(M_m)sn,$$
$$\|N_{n,m,M_m}\| \le 2\|f - p_{m,M_m}\|_{\infty,[-M_m,M_m]},$$

which implies that

$$\{p_{m,M_m}(A_n)\}_n \xrightarrow{\text{a.c.s.}} \{f(A_n)\}_n.$$

Moreover, by GLT 3,

$$\{p_{m,M_m}(A_n)\}_n \sim_{\text{GLT}} p_{m,M_m}(\kappa).$$

Finally, by (4.6),

$$\|f(\kappa) - p_{m,M_m}(\kappa)\| = \max_{i=1,\ldots,s} |(f - p_{m,M_m})(\lambda_i(\kappa))|$$
$$\le \|f - p_{m,M_m}\|_{\infty,[-\|\kappa\|,\|\kappa\|]} \to 0 \text{ a.e.},$$

which implies that

$$p_{m,M_m}(\kappa) \to f(\kappa) \text{ a.e.}$$

All the hypotheses of GLT 4 are then satisfied and $\{f(A_n)\}_n \sim_{\text{GLT}} f(\kappa)$. □

[1] The inequality $\operatorname{rank}(R_{n,m,M}) \le \operatorname{rank}(\hat{A}_{n,M})$ follows from the observation that if $\lambda_i(\hat{A}_{n,M}) = 0$, then we have $(f - p_{m,M})(\lambda_i(\hat{A}_{n,M})) = (f - p_{m,M})(0) \in S$ and $\lambda_i(R_{n,m,M}) = \alpha(\lambda_i(\hat{A}_{n,M})) = 0$.

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5. Higher-order isogeometric Galerkin discretization of second-order variable-coefficient differential eigenvalue problems. Let $\mathbb{R}^+$ be the set of positive real numbers. Consider the following eigenvalue problem: find eigenvalues $\lambda_j \in \mathbb{R}^+$ and eigenfunctions u_j, for $j = 1, 2, \ldots, \infty$, such that

$$\begin{cases} -(a(x)u_j'(x))' = \lambda_j b(x)u_j(x), & x \in \Omega, \\ u_j(x) = 0, & x \in \partial\Omega, \end{cases} \tag{5.7}$$

where Ω is a bounded open interval in $\mathbb{R}$ and we assume that $a, b \in L^1(\Omega)$ and $a, b > 0$ a.e. in Ω. The corresponding weak formulation reads as follows: find eigenvalues $\lambda_j \in \mathbb{R}^+$ and eigenfunctions $u_j \in H_0^1(\Omega)$, for $j = 1, 2, \ldots, \infty$, such that

$$\mathrm{a}(u_j, w) = \lambda_j (b\, u_j, w), \quad \forall\, w \in H_0^1(\Omega),$$

where

$$\mathrm{a}(u_j, w) = \int_\Omega a(x)u_j'(x)w'(x)\mathrm{d}x, \quad (b\, u_j, w) = \int_\Omega b(x)u_j(x)w(x)\mathrm{d}x.$$

Isogeometric Galerkin discretization. In the standard Galerkin method, we fix a set of basis functions $\{\varphi_1, \ldots, \varphi_N\} \subset H_0^1(\Omega)$, we define the so-called approximation space $\mathscr{W} = \mathrm{span}(\varphi_1, \ldots, \varphi_N)$, and we find approximations of the exact eigenpairs (λ_j, u_j), $j = 1, 2, \ldots, \infty$, by solving the following (Galerkin) problem: find $\lambda_{j,\mathscr{W}} \in \mathbb{R}^+$ and $u_{j,\mathscr{W}} \in \mathscr{W}$, for $j = 1, \ldots, N$, such that

$$\mathrm{a}(u_{j,\mathscr{W}}, w) = \lambda_{j,\mathscr{W}} (b\, u_{j,\mathscr{W}}, w), \quad \forall\, w \in \mathscr{W}. \tag{5.8}$$

Assuming the exact and numerical eigenvalues are arranged in non-decreasing order, the pair $(\lambda_{j,\mathscr{W}}, u_{j,\mathscr{W}})$ is taken as an approximation of the pair (λ_j, u_j) for all $j = 1, \ldots, N$. The numbers $\lambda_{j,\mathscr{W}}/\lambda_j - 1,\ j = 1, \ldots, N$, are referred to as the (relative) eigenvalue errors. In view of the canonical identification of each function $w \in \mathscr{W}$ with its coefficient vector with respect to the basis $\{\varphi_1, \ldots, \varphi_N\}$, solving the Galerkin problem (5.8) is equivalent to solving the generalized eigenvalue problem

$$K\mathbf{u}_{j,\mathscr{W}} = \lambda_{j,\mathscr{W}} M\mathbf{u}_{j,\mathscr{W}}, \tag{5.9}$$

where $\mathbf{u}_{j,\mathscr{W}}$ is the coefficient vector of $u_{j,\mathscr{W}}$ with respect to $\{\varphi_1, \ldots, \varphi_N\}$ and

$$K = [\mathrm{a}(\varphi_j, \varphi_i)]_{i,j=1}^N = \left[\int_\Omega a(x)\varphi_j'(x)\varphi_i'(x)\mathrm{d}x\right]_{i,j=1}^N, \tag{5.10}$$

$$M = [(b\,\varphi_j, \varphi_i)]_{i,j=1}^N = \left[\int_\Omega b(x)\varphi_j(x)\varphi_i(x)\mathrm{d}x\right]_{i,j=1}^N. \tag{5.11}$$

The matrices K and M are referred to as the stiffness and mass matrices, respectively. Due to our assumption that $a, b > 0$ a.e. on Ω, both K and M are symmetric positive definite, regardless of the chosen basis functions $\varphi_1, \ldots, \varphi_N$. Moreover, it is clear from (5.9) that the numerical eigenvalues $\lambda_{j,\mathscr{W}}$, $j = 1, \ldots, N$, are just the eigenvalues of the matrix

$$L = M^{-1}K. \tag{5.12}$$

In the isogeometric Galerkin method [17], we assume that the physical domain Ω is described by a global geometry function $G : [0, 1] \to \overline{\Omega}$, which is invertible and satisfies $G(\partial([0, 1])) = \partial\overline{\Omega}$. We fix a set of basis

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functions $\{\hat\varphi_1,\ldots,\hat\varphi_N\}$ defined on the reference (parametric) domain $[0,1]$ and vanishing on the boundary $\partial([0,1])$, and we find approximations to the exact eigenpairs (λ_j,u_j), $j=1,2,\ldots,\infty$, by using the standard Galerkin method described above, in which the approximation space is chosen as $\mathscr{W}=\mathrm{span}(\varphi_1,\ldots,\varphi_N)$, where

$$\varphi_i(x)=\hat\varphi_i(G^{-1}(x))=\hat\varphi_i(\hat x),\quad x=G(\hat x),\quad i=1,\ldots,N. \tag{5.13}$$

The resulting stiffness and mass matrices K and M are given by (5.10) and (5.11), with the basis functions φ_i defined as in (5.13). If we assume that G and $\hat\varphi_i$, $i=1,\ldots,N$, are sufficiently regular, we can apply standard differential calculus to obtain for K and M the following expressions:

$$K=\left[\int_0^1\frac{a(G(\hat x))}{|G'(\hat x)|}\hat\varphi_j'(\hat x)\hat\varphi_i'(\hat x)\mathrm{d}\hat x\right]_{i,j=1}^N, \tag{5.14}$$

$$M=\left[\int_0^1 b(G(\hat x))|G'(\hat x)|\hat\varphi_j(\hat x)\hat\varphi_i(\hat x)\mathrm{d}\hat x\right]_{i,j=1}^N. \tag{5.15}$$

p-Degree C^k B-spline basis functions. Following the higher-order isogeometric Galerkin approach as in [24], the basis functions $\hat\varphi_1,\ldots,\hat\varphi_N$ will be chosen as piecewise polynomial functions of degree $p\ge 1$. More precisely, for $p,n\ge 1$ and $0\le k\le p-1$, let $B_{1,[p,k]},\ldots,B_{n(p-k)+k+1,[p,k]}:\mathbb{R}\to\mathbb{R}$ be the B-splines of degree p and smoothness C^k defined on the knot sequence

$$\begin{aligned}&\{\tau_1,\ldots,\tau_{n(p-k)+p+k+2}\}\\ &=\Big\{\underbrace{0,\ldots,0}_{p+1},\ \underbrace{\frac1n,\ldots,\frac1n}_{p-k},\ \underbrace{\frac2n,\ldots,\frac2n}_{p-k},\ \ldots,\ \underbrace{\frac{n-1}{n},\ldots,\frac{n-1}{n}}_{p-k},\ \underbrace{1,\ldots,1}_{p+1}\Big\}.\end{aligned} \tag{5.16}$$

The basis functions $\hat\varphi_1,\ldots,\hat\varphi_N$ are defined as follows:

$$\hat\varphi_i=B_{i+1,[p,k]},\quad i=1,\ldots,n(p-k)+k-1. \tag{5.17}$$

In particular, we have $N=n(p-k)+k-1$.

We collect below a few properties of $B_{1,[p,k]},\ldots,B_{n(p-k)+k+1,[p,k]}$ that we shall use in this paper. For the formal definition of B-splines, as well as for the proof of the properties listed below, see [11, 33].

- The support of the ith B-spline is given by

$$\mathrm{supp}(B_{i,[p,k]})=[\tau_i,\tau_{i+p+1}],\quad i=1,\ldots,n(p-k)+k+1. \tag{5.18}$$

- Except for the first and the last one, all the other B-splines vanish on the boundary of $[0,1]$, i.e.,

$$B_{i,[p,k]}(0)=B_{i,[p,k]}(1)=0,\quad i=2,\ldots,n(p-k)+k. \tag{5.19}$$

- $\{B_{1,[p,k]},\ldots,B_{n(p-k)+k+1,[p,k]}\}$ is a basis for the space of piecewise polynomial functions of degree p and smoothness C^k, that is,

$$\mathscr{V}_{n,[p,k]}=\left\{v\in C^k([0,1]):\ v|_{\left[\frac in,\frac{i+1}{n}\right]}\in\mathbb{P}_p\ \text{ for all }\ i=0,\ldots,n-1\right\},$$

where $\mathbb{P}_p$ is the space of polynomials of degree $\le p$. Moreover, $\{B_{2,[p,k]},\ldots,B_{n(p-k)+k,[p,k]}\}$ is a basis for the space

$$\mathscr{W}_{n,[p,k]}=\left\{w\in\mathscr{V}_{n,[p,k]}:\ w(0)=w(1)=0\right\}.$$

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- The B-splines form a non-negative partition of unity over $[0,1]$:

$$B_{i,[p,k]} \geq 0 \quad \text{over} \quad \mathbb{R}, \quad i = 1, \ldots, n(p-k)+k+1, \tag{5.20}$$

$$\sum_{i=1}^{n(p-k)+k+1} B_{i,[p,k]} = 1 \quad \text{over} \quad [0,1]. \tag{5.21}$$

- The derivatives of the B-splines satisfy

$$\sum_{i=1}^{n(p-k)+k+1} |B'_{i,[p,k]}| \leq C_p n \quad \text{over} \quad [0,1], \tag{5.22}$$

where C_p is a constant depending only on p. Note that the derivatives $B'_{i,[p,k]}$ may not be defined at some of the grid points $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$ in the case of C^0 smoothness ($k=0$). In (5.22), it is assumed that the undefined values are excluded from the summation.

- For every $\mathbf{y} = (y_1, \ldots, y_{n(p-k)+k+1}) \in \mathbb{R}^{n(p-k)+k+1}$, we have

$$\left\| \sum_{i=1}^{n(p-k)+k+1} y_i B_{i,[p,k]} \right\|^2_{L^2([0,1])} = \int_0^1 \left(\sum_{i=1}^{n(p-k)+k+1} y_i B_{i,[p,k]} \right)^2 \geq \frac{c_p}{n} \|\mathbf{y}\|^2, \tag{5.23}$$

where c_p is a constant depending only on p and $\|\mathbf{y}\|^2 = \sum_{i=1}^{n(p-k)+k+1} y_i^2$.

- All the B-splines, except for the first $k+1$ and the last $k+1$, are uniformly shifted-scaled versions of $p-k$ fixed reference functions $\beta_{1,[p,k]}, \ldots, \beta_{p-k,[p,k]}$, namely the first $p-k$ B-splines defined on the reference knot sequence

$$\underbrace{0, \ldots, 0}_{p-k}, \underbrace{1, \ldots, 1}_{p-k}, \ldots, \underbrace{\eta, \ldots, \eta}_{p-k}, \quad \eta = \left\lceil \frac{p+1}{p-k} \right\rceil.$$

In formulas, setting

$$\nu = \left\lceil \frac{k+1}{p-k} \right\rceil, \tag{5.24}$$

for the B-splines $B_{k+2,[p,k]}, \ldots, B_{k+1+(n-\nu)(p-k),[p,k]}$ we have

$$\begin{aligned} &B_{k+1+(p-k)(r-1)+q,[p,k]}(t) = \beta_{q,[p,k]}(nt-r+1), \\ &r = 1, \ldots, n-\nu, \quad q = 1, \ldots, p-k. \end{aligned} \tag{5.25}$$

We point out that the supports of the reference B-splines $\beta_{q,[p,k]}$ satisfy

$$\mathrm{supp}(\beta_{1,[p,k]}) \subseteq \mathrm{supp}(\beta_{2,[p,k]}) \subseteq \cdots \subseteq \mathrm{supp}(\beta_{p-k,[p,k]}) = [0,\eta].$$

Figures 1 and 2 show the graphs of the B-splines $B_{1,[p,k]}, \ldots, B_{n(p-k)+k+1,[p,k]}$ for the degree $p=3$ and the smoothness $k=1$, and the graphs of the associated reference B-splines $\beta_{1,[p,k]}, \beta_{2,[p,k]}$.

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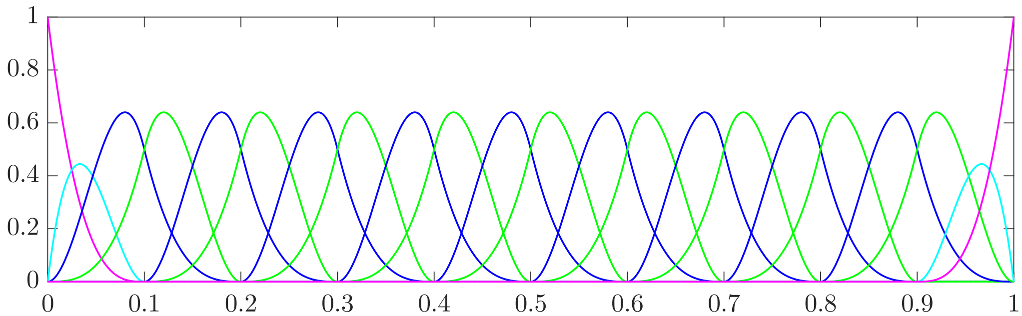

FIGURE 1. *B-splines* $B_{1,[p,k]},\dots,B_{n(p-k)+k+1,[p,k]}$ *for* $p=3$ *and* $k=1$*, with* $n=10$.

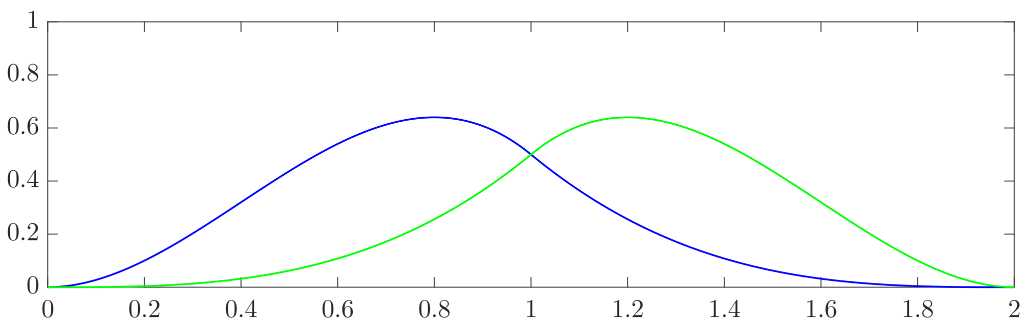

FIGURE 2. *Reference B-splines* $\beta_{1,[p,k]},\beta_{2,[p,k]}$ *for* $p=3$ *and* $k=1$.

GLT analysis of the higher-order isogeometric Galerkin matrices. For any functions $\alpha,\beta\in L^1([0,1])$, let

$$K_{n,[p,k]}(\alpha)=\left[\int_0^1\alpha(\hat{x})B'_{j+1,[p,k]}(\hat{x})B'_{i+1,[p,k]}(\hat{x})\mathrm{d}\hat{x}\right]_{i,j=1}^{n(p-k)+k-1}, \tag{5.26}$$

$$M_{n,[p,k]}(\beta)=\left[\int_0^1\beta(\hat{x})B_{j+1,[p,k]}(\hat{x})B_{i+1,[p,k]}(\hat{x})\mathrm{d}\hat{x}\right]_{i,j=1}^{n(p-k)+k-1}. \tag{5.27}$$

The stiffness and mass matrices (5.14) and (5.15) resulting from the choice of the basis functions as in (5.17) are nothing else than $K_{n,[p,k]}(a_G)$ and $M_{n,[p,k]}(b_G)$, where

$$a_G(\hat{x})=\frac{a(G(\hat{x}))}{|G'(\hat{x})|},\quad b_G(\hat{x})=b(G(\hat{x}))|G'(\hat{x})|.$$

The main result of this section is Theorem 5.4. It provides formal mathematical proofs to the main results appeared in the engineering review [24] by giving the spectral distribution of the normalized sequences $\{n^{-1}K_{n,[p,k]}(a_G)\}_n$, $\{nM_{n,[p,k]}(a_G)\}_n$, $\{n^{-2}L_{n,[p,k]}(a_G,b_G)\}_n$, where

$$L_{n,[p,k]}(a_G,b_G)=(M_{n,[p,k]}(b_G))^{-1}K_{n,[p,k]}(a_G)$$

is the matrix whose eigenvalues are just the numerical eigenvalues produced by the considered higher-order isogeometric Galerkin method. The proof of Theorem 5.4 is entirely based on the theory of block GLT sequences and it is therefore referred to as (block) GLT analysis. In view of what follows, define the

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$(p-k)\times(p-k)$ blocks

$$K_{[p,k]}^{[\ell]}=\left[\int_{\mathbb{R}}\beta'_{j,[p,k]}(t)\beta'_{i,[p,k]}(t-\ell)\mathrm{d}t\right]_{i,j=1}^{p-k},\quad \ell\in\mathbb{Z}, \tag{5.28}$$

$$M_{[p,k]}^{[\ell]}=\left[\int_{\mathbb{R}}\beta_{j,[p,k]}(t)\beta_{i,[p,k]}(t-\ell)\mathrm{d}t\right]_{i,j=1}^{p-k},\quad \ell\in\mathbb{Z}, \tag{5.29}$$

and the matrix-valued functions $\kappa_{[p,k]},\mu_{[p,k]}:[-\pi,\pi]\to\mathbb{C}^{(p-k)\times(p-k)}$,

$$\kappa_{[p,k]}(\theta)=\sum_{\ell\in\mathbb{Z}}K_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta}=K_{[p,k]}^{[0]}+\sum_{\ell>0}\Big(K_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta}+(K_{[p,k]}^{[\ell]})^T\mathrm{e}^{-\mathrm{i}\ell\theta}\Big), \tag{5.30}$$

$$\mu_{[p,k]}(\theta)=\sum_{\ell\in\mathbb{Z}}M_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta}=M_{[p,k]}^{[0]}+\sum_{\ell>0}\Big(M_{[p,k]}^{[\ell]}\mathrm{e}^{\mathrm{i}\ell\theta}+(M_{[p,k]}^{[\ell]})^T\mathrm{e}^{-\mathrm{i}\ell\theta}\Big). \tag{5.31}$$

Due to the compact support of the reference functions $\beta_{1,[p,k]},\ldots,\beta_{p-k,[p,k]}$, there is only a finite number of nonzero blocks $K_{[p,k]}^{[\ell]}$ and $M_{[p,k]}^{[\ell]}$; consequently, the series in (5.30) and (5.31) are actually finite sums.

We are now ready to state and prove a few lemmas that we will use in the proof of Theorem 5.4. In what follows, we use the following notation.

- If $p,n\ge 1$, $0\le k\le p-1$ and A is a matrix of size $n(p-k)+k-1$, we denote by $\tilde{A}$ the principal submatrix of A corresponding to the row and column indices $i,j=k+1,\ldots,k+(n-\nu)(p-k)$, where $\nu=\lceil(k+1)/(p-k)\rceil$ as in (5.24).
- If $p,n\ge 1$, $0\le k\le p-1$ and A is a matrix of size $n(p-k)+k-1$, we denote by $\hat{A}$ the block diagonal matrix

$$\hat{A}=\begin{bmatrix}I_{k(p-k)-k} & & \\ & A & \\ & & 1\end{bmatrix},$$

  where it is understood that the block $I_{k(p-k)-k}$ is not present if $k(p-k)-k=0$, i.e., if $k=p-1$. Note that $\hat{A}$ has the following key properties:
  - its size $(n+k)(p-k)$ is a multiple of $p-k$ and it is such that the difference $(n+k)(p-k)-(n(p-k)+k-1)=k(p-k)-k+1>0$ is independent of n;
  - it contains A as a principal submatrix in such a way that $\tilde{A}$ is the principal submatrix of $\hat{A}$ corresponding to the row and column indices $i,j=k(p-k)+1,\ldots,k(p-k)+(n-\nu)(p-k)$;
  - its eigenvalues (resp., singular values) are given by the eigenvalues (resp., singular values) of A plus further $k(p-k)-k+1$ eigenvalues (resp., singular values) that are equal to 1.

Lemma 5.1. *Let $p,n\ge 1$ and $0\le k\le p-1$. Then,*

$$\tilde{K}_{n,[p,k]}(1)=nT_{n-\nu}(\kappa_{[p,k]}), \tag{5.32}$$

$$\tilde{M}_{n,[p,k]}(1)=n^{-1}T_{n-\nu}(\mu_{[p,k]}). \tag{5.33}$$

Proof. The proof of (5.32) is given in [20, Lemma 2], where the matrices $\tilde{K}_{n,[p,k]}(1)$ and $K_{n,[p,k]}(1)$ are denoted by $\widetilde{A}_{n,[p,k]}(1)$ and $A_{n,[p,k]}(1)$, respectively. The proof of (5.33) is essentially the same as the proof of (5.32). ∎

Lemma 5.2. *Let $p\ge 1$ and $0\le k\le p-1$. Then, for all functions $g\in L^1([0,1])$,*

$$\{n^{-1}\hat{K}_{n,[p,k]}(g)\}_n\sim_{\mathrm{GLT}} g(\hat{x})\kappa_{[p,k]}(\theta), \tag{5.34}$$

$$\{n\hat{M}_{n,[p,k]}(g)\}_n\sim_{\mathrm{GLT}} g(\hat{x})\mu_{[p,k]}(\theta). \tag{5.35}$$

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Proof. We only prove (5.34) as the proof of (5.35) is completely analogous. The proof consists of the following three steps.

Step 1. We first prove (5.34) in the constant-coefficient case where $g(\hat{x}) = 1$ identically. In this case, by Lemma 5.1, $n^{-1}\tilde{K}_{n,[p,k]}(1) = T_{n-\nu}(\kappa_{[p,k]})$. Considering that $n^{-1}\tilde{K}_{n,[p,k]}(1)$ is the principal submatrix of $n^{-1}\hat{K}_{n,[p,k]}(1)$ corresponding to the row and column indices $i,j = k(p-k)+1,\ldots,k(p-k)+(n-\nu)(p-k)$, we infer that

$$n^{-1}\hat{K}_{n,[p,k]}(1) = T_{n+k}(\kappa_{[p,k]}) + R_{n,[p,k]}, \quad \mathrm{rank}(R_{n,[p,k]}) \le 2(p-k)(k+\nu).$$

Hence, the desired relation $\{n^{-1}\hat{K}_{n,[p,k]}(1)\}_n \sim_{\mathrm{GLT}} \kappa_{[p,k]}(\theta)$ follows from Z 1, GLT 2 and GLT 3.

Step 2. Now we prove (5.34) in the case where $g \in C([0,1])$. Let

$$Z_{n,[p,k]}(g) = n^{-1}\hat{K}_{n,[p,k]}(g) - n^{-1}D_{n+k}(gI_{p-k})\hat{K}_{n,[p,k]}(1).$$

By (5.16), (5.18) and (5.22), for all $r,R = 1,\ldots,n-\nu$ and $q,Q = 1,\ldots,p-k$ we have

$$\begin{aligned}
&|(n\tilde{Z}_{n,[p,k]}(g))_{(p-k)(r-1)+q,(p-k)(R-1)+Q}| \\
&= \Bigg|(\tilde{K}_{n,[p,k]}(g))_{(p-k)(r-1)+q,(p-k)(R-1)+Q} \\
&\qquad - \bigg(\Big(\operatorname*{diag}_{i=k+1,\ldots,k+n-\nu} g\Big(\frac{i}{n+k}\Big)I_{p-k}\Big)\tilde{K}_{n,[p,k]}(1)\bigg)_{(p-k)(r-1)+q,(p-k)(R-1)+Q}\Bigg| \\
&= \left|\int_0^1 \left[g(x) - g\Big(\frac{k+r}{n+k}\Big)\right] B'_{k+1+(p-k)(R-1)+Q,[p,k]}(\hat{x})B'_{k+1+(p-k)(r-1)+q,[p,k]}(\hat{x})\mathrm{d}\hat{x}\right| \\
&= \Bigg|\int_{\tau_{k+1+(p-k)(r-1)+q}}^{\tau_{k+1+(p-k)(r-1)+q+p+1}} \left[g(\hat{x}) - g\Big(\frac{k+r}{n+k}\Big)\right] \\
&\qquad\qquad \cdot\ B'_{k+1+(p-k)(R-1)+Q,[p,k]}(\hat{x})B'_{k+1+(p-k)(r-1)+q,[p,k]}(\hat{x})\mathrm{d}\hat{x}\Bigg| \\
&\le C_p^2 n^2 \int_{(r-1)/n}^{(r+p)/n} \left|g(\hat{x}) - g\Big(\frac{k+r}{n+k}\Big)\right| \mathrm{d}\hat{x} \le C_p^2(p+1)n\omega_g\Big(\frac{p}{n}\Big),
\end{aligned}$$

where $\omega_g(\cdot)$ is the modulus of continuity of g and the last inequality is justified by the fact that the distance of the point $(k+r)/(n+k)$ from the interval $[(r-1)/n,(r+p)/n]$ is not larger than p/n. It follows that each entry of $\tilde{Z}_{n,[p,k]}(g)$ is bounded in modulus by $D_p\omega_g(1/n)$, where D_p is a constant depending only on p. Moreover, by (5.18), the matrix $\tilde{Z}_{n,[p,k]}(g)$ is banded with bandwidth bounded by a constant w_p depending only on p. Thus, by the inequality

$$\|X\| \le \sqrt{\Big(\max_{i=1,\ldots,N}\sum_{j=1}^N |X_{ij}|\Big)\Big(\max_{j=1,\ldots,N}\sum_{i=1}^N |X_{ij}|\Big)}, \quad X \in \mathbb{C}^{N\times N}, \tag{5.36}$$

which is proved, e.g., in [21, Section 2.4.1], we get $\|\tilde{Z}_{n,[p,k]}(g)\| \le w_pD_p\omega_g(1/n) \to 0$ as $n \to \infty$. Considering that $\tilde{Z}_{n,[p,k]}(g)$ is the principal submatrix of $Z_{n,[p,k]}(g)$ corresponding to the row and column indices $i,j = k(p-k)+1,\ldots,k(p-k)+(n-\nu)(p-k)$, we arrive at

$$Z_{n,[p,k]}(g) = N_{n,[p,k]} + R_{n,[p,k]},$$

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where $\|N_{n,[p,k]}\| = \|\tilde{Z}_{n,[p,k]}(g)\| \to 0$ as $n \to \infty$ and $\mathrm{rank}(R_{n,[p,k]}) \le 2(p-k)(k+\nu)$. It follows from Z 1 that $\{Z_{n,[p,k]}(g)\}_n$ is zero-distributed. Since

$$n^{-1}\hat{K}_{n,[p,k]}(g) = n^{-1}D_{n+k}(gI_{p-k})\hat{K}_{n,[p,k]}(1) + Z_{n,[p,k]}(g),$$

we conclude that $\{n^{-1}\hat{K}_{n,[p,k]}(g)\}_n \sim_{\mathrm{GLT}} g(\hat{x})\kappa_{[p,k]}(\theta)$ by Step 1, GLT 2 and GLT 3.

Step 3. Finally, we prove (5.34) in the general case where $g \in L^1([0,1])$. By the density of $C([0,1])$ in $L^1([0,1])$, there exist functions $g_m \in C([0,1])$ such that $g_m \to g$ in $L^1([0,1])$. By Step 2,

$$\{n^{-1}\hat{K}_{n,[p,k]}(g_m)\}_n \sim_{\mathrm{GLT}} g_m(\hat{x})\kappa_{[p,k]}(\theta). \tag{5.37}$$

Moreover,

$$g_m(\hat{x})\kappa_{[p,k]}(\theta) \to g(\hat{x})\kappa_{[p,k]}(\theta) \text{ in measure.} \tag{5.38}$$

We show that

$$\{n^{-1}\hat{K}_{n,[p,k]}(g_m)\}_n \xrightarrow{\text{a.c.s.}} \{n^{-1}\hat{K}_{n,[p,k]}(g)\}_n. \tag{5.39}$$

Once this is done, the thesis (5.34) follows immediately from GLT 4. To prove (5.39), we recall that

$$\|X\|_1 \le \sum_{i,j=1}^{N} |x_{ij}|, \quad X \in \mathbb{C}^{N\times N}; \tag{5.40}$$

see, e.g., [21, Section 2.4.3]. By (5.22),

$$\begin{aligned}
\|\hat{K}_{n,[p,k]}(g) - \hat{K}_{n,[p,k]}(g_m)\|_1 &= \|K_{n,[p,k]}(g) - K_{n,[p,k]}(g_m)\|_1 \\
&\le \sum_{i,j=1}^{n(p-k)+k-1} \left| \int_0^1 \big[g(\hat{x}) - g_m(\hat{x})\big] B'_{j+1,[p,k]}(\hat{x}) B'_{i+1,[p,k]}(\hat{x}) \mathrm{d}\hat{x} \right| \\
&\le \int_0^1 \big|g(\hat{x}) - g_m(\hat{x})\big| \sum_{i,j=1}^{n(p-k)+k-1} |B'_{j+1,[p,k]}(\hat{x})|\,|B'_{i+1,[p,k]}(\hat{x})| \mathrm{d}\hat{x} \\
&\le C_p^2 n^2 \|g - g_m\|_{L^1}.
\end{aligned}$$

Thus, the a.c.s. convergence (5.39) follows from ACS 1. ∎

LEMMA 5.3. *Let $p \ge 1$ and $0 \le k \le p-1$. Then, $\mu_{[p,k]}(\theta)$ is Hermitian positive definite for all $\theta \in [-\pi,\pi]$.*

Proof. By Lemma 5.2,

$$\{n\hat{M}_{n,[p,k]}(1)\}_n \sim_{\mathrm{GLT}} \mu_{[p,k]}(\theta),$$

and since $n\hat{M}_{n,[p,k]}(1)$ is symmetric, we infer from GLT 1 that

$$\{n\hat{M}_{n,[p,k]}(1)\}_n \sim_{\lambda} \mu_{[p,k]}(\theta). \tag{5.41}$$

By (5.23), for every $\mathbf{y} \in \mathbb{R}^{n(p-k)+k-1}$ we have

$$\begin{aligned}
\mathbf{y}^T(nM_{n,[p,k]}(1))\mathbf{y} &= n\int_0^1 \left(\sum_{i=1}^{n(p-k)+k-1} y_i B_{i+1,[p,k]}(\hat{x}) \right)^2 \mathrm{d}\hat{x} \\
&= n \left\| \sum_{i=1}^{n(p-k)+k-1} y_i B_{i+1,[p,k]} \right\|_{L^2([0,1])}^2 \ge c_p\|\mathbf{y}\|^2.
\end{aligned}$$

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Hence, by the minimax principle for eigenvalues [10, Corollary III.1.2],

$$\lambda_{\min}(nM_{n,[p,k]}(1)) = \min_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^T(nM_{n,[p,k]}(1))\mathbf{y}}{\|\mathbf{y}\|^2} \geq c_p$$

for all n, which implies that

$$\lambda_{\min}(n\hat{M}_{n,[p,k]}(1)) \geq \min(c_p, 1) \tag{5.42}$$

for all n. Taking into account that $\lambda_{\min}(\mu_{[p,k]}(\theta))$ is a continuous function of θ just as $\mu_{[p,k]}(\theta)$, by (5.41), (5.42) and D 2, we have

$$\lambda_{\min}(\mu_{[p,k]}(\theta)) \geq \min(c_p, 1)$$

for almost every $\theta \in [-\pi, \pi]$, that is, for all $\theta \in [-\pi, \pi]$, thanks to the continuity of $\lambda_{\min}(\mu_{[p,k]}(\theta))$. We then conclude that $\mu_{[p,k]}(\theta)$ is Hermitian positive definite for all $\theta \in [-\pi, \pi]$. ∎

THEOREM 5.4. *Let Ω be a bounded open interval in $\mathbb{R}$ and let $a, b \in L^1(\Omega)$ with $a, b > 0$ a.e. Let $p \geq 1$ and $0 \leq k \leq p-1$. Let $G : [0,1] \to \overline{\Omega}$ be such that $G' \neq 0$ a.e. in $[0,1]$ and*

$$a_G(\hat{x}) = \frac{a(G(\hat{x}))}{|G'(\hat{x})|} \in L^1([0,1]),$$
$$b_G(\hat{x}) = b(G(\hat{x}))|G'(\hat{x})| \in L^1([0,1]).$$

Then,

$$\{n^{-1}K_{n,[p,k]}(a_G)\}_n \sim_{\sigma,\lambda} a_G(\hat{x})\kappa_{[p,k]}(\theta) = \frac{a(G(\hat{x}))}{|G'(\hat{x})|}\kappa_{[p,k]}(\theta), \tag{5.43}$$

$$\{nM_{n,[p,k]}(b_G)\}_n \sim_{\sigma,\lambda} b_G(\hat{x})\mu_{[p,k]}(\theta) = b(G(\hat{x}))|G'(\hat{x})|\,\mu_{[p,k]}(\theta), \tag{5.44}$$

$$\begin{aligned}\{n^{-2}L_{n,[p,k]}(a_G, b_G)\}_n &\sim_{\sigma,\lambda} (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1}(a_G(\hat{x})\kappa_{[p,k]}(\theta)) \\ &= \frac{a(G(\hat{x}))}{b(G(\hat{x}))(G'(\hat{x}))^2}(\mu_{[p,k]}(\theta))^{-1}\kappa_{[p,k]}(\theta).\end{aligned} \tag{5.45}$$

Proof. We first note that it is enough to prove (5.43)–(5.45) with $K_{n,[p,k]}$, $M_{n,[p,k]}$, $L_{n,[p,k]}$ replaced by, respectively, $\hat{K}_{n,[p,k]}$, $\hat{M}_{n,[p,k]}$, $\hat{L}_{n,[p,k]}$, that is,

$$\{n^{-1}\hat{K}_{n,[p,k]}(a_G)\}_n \sim_{\sigma,\lambda} a_G(\hat{x})\kappa_{[p,k]}(\theta) = \frac{a(G(\hat{x}))}{|G'(\hat{x})|}\kappa_{[p,k]}(\theta), \tag{5.46}$$

$$\{n\hat{M}_{n,[p,k]}(b_G)\}_n \sim_{\sigma,\lambda} b_G(\hat{x})\mu_{[p,k]}(\theta) = b(G(\hat{x}))|G'(\hat{x})|\,\mu_{[p,k]}(\theta), \tag{5.47}$$

$$\begin{aligned}\{n^{-2}\hat{L}_{n,[p,k]}(a_G, b_G)\}_n &\sim_{\sigma,\lambda} (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1}(a_G(\hat{x})\kappa_{[p,k]}(\theta)) \\ &= \frac{a(G(\hat{x}))}{b(G(\hat{x}))(G'(\hat{x}))^2}(\mu_{[p,k]}(\theta))^{-1}\kappa_{[p,k]}(\theta).\end{aligned} \tag{5.48}$$

Moreover, (5.46) and (5.47) follow immediately from Lemma 5.2 and the symmetry of $\hat{K}_{n,[p,k]}(a_G)$ and $\hat{M}_{n,[p,k]}(b_G)$. It only remains to prove (5.48); this is precisely the proof that requires our main result (Theorem 4.1). The first observation is that

$$n^{-2}\hat{L}_{n,[p,k]}(a_G, b_G) = (n\hat{M}_{n,[p,k]}(b_G))^{-1}(n^{-1}\hat{K}_{n,[p,k]}(a_G)) \tag{5.49}$$

$$\sim (n\hat{M}_{n,[p,k]}(b_G))^{-1/2}(n^{-1}\hat{K}_{n,[p,k]}(a_G))(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}, \tag{5.50}$$

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where $X \sim Y$ means that the matrix X is similar to Y; note that $\hat{M}_{n,[p,k]}(b_G)$ is positive definite because $b_G > 0$ a.e. in $[0,1]$ by the assumptions on b and G, hence $(\hat{M}_{n,[p,k]}(b_G))^{-1/2}$ is well-defined. By combining (5.49) with Lemmas 5.2, 5.3 and GLT 3, we immediately obtain

$$\{n^{-2}\hat{L}_{n,[p,k]}(a_G, b_G)\}_n \sim_{\rm GLT} (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1}(a_G(\hat{x})\kappa_{[p,k]}(\theta)). \tag{5.51}$$

The singular value distribution in (5.48) follows from (5.51) and GLT 1. Moreover, by Lemmas 5.2, 5.3, GLT 3 and Theorem 4.1 (applied with $f(z) = |z|^{1/2}$), we have

$$\begin{aligned}&\{(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}(n^{-1}\hat{K}_{n,[p,k]}(a_G))(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}\}_n\\ &\sim_{\rm GLT} (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2}(a_G(\hat{x})\kappa_{[p,k]}(\theta))(b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2}.\end{aligned}$$

Considering that $(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}(n^{-1}\hat{K}_{n,[p,k]}(a_G))(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}$ is symmetric, from GLT 1, we get

$$\begin{aligned}&\{(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}(n^{-1}\hat{K}_{n,[p,k]}(a_G))(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}\}_n\\ &\sim_\lambda (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2}(a_G(\hat{x})\kappa_{[p,k]}(\theta))(b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\{(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}(n^{-1}\hat{K}_{n,[p,k]}(a_G))(n\hat{M}_{n,[p,k]}(b_G))^{-1/2}\}_n\\ &\sim_\lambda (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1}(a_G(\hat{x})\kappa_{[p,k]}(\theta))\end{aligned}$$

by Definition 3.1, since

$$\begin{aligned}&(b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1}(a_G(\hat{x})\kappa_{[p,k]}(\theta))\\ &\sim (b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2}(a_G(\hat{x})\kappa_{[p,k]}(\theta))(b_G(\hat{x})\mu_{[p,k]}(\theta))^{-1/2}\end{aligned}$$

for all $(\hat{x},\theta) \in [0,1] \times [-\pi,\pi]$. In view of the similarity (5.50), we conclude that the eigenvalue distribution in (5.48) is satisfied. ∎

The reader who is interested in the engineering implications of Theorem 5.4 is referred to the recent review [24].

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