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NOTES ON TWO RECENT RESULTS OF AUDENAERT*

YAZHOU HAN† AND JINGJING SHAO‡

Abstract. In this article, some inequalities for τ-measurable operators which are related to two recent results of Audenaert are proved.

Key words. Noncommutative L_p spaces, Von Neumann algebras, τ-Measurable operators.

AMS subject classifications. 47A63, 46L52.

1. Introduction. Let $\mathbb{M}_n$ be the space of $n\times n$ complex matrices. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$.

Let $\mathbb{M}_n^+$ be the positive part of $\mathbb{M}_n$. In [1], Audenaert proved that if $A_i, B_i \in \mathbb{M}_n^+$ $(i = 1, 2, \ldots, n)$, such that $A_iB_i = B_iA_i$, then

$$\left|\left|\left|\sum_{i=1}^{n} A_iB_i\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^{n} A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)\left(\sum_{i=1}^{n} B_i\right)\right|\right|\right|. \tag{1.1}$$

A special case of inequality (1.1) confirms a conjecture of Hayajneh and Kittaneh in [9] and answers a question of Bourin.

In another paper [2], Audenaert proved that for $X, Y \in \mathbb{M}_n$ and $0 \le q \le 1$,

$$|||XY^*|||^2 \le |||qX^*X + (1-q)Y^*Y|||\,|||(1-q)X^*X + qY^*Y|||. \tag{1.2}$$

As is explained in [2], inequality (1.2) interpolates between the Arithmetic-Geometric mean and Cauchy-Schwarz matrix norm inequalities. Very recently Lin [12] gave another proof of inequality (1.1) and (1.2).

Using the notion of the generalized singular numbers studied by Fack and Kosaki [7], we show that the inequality (1.1) and (1.2) hold for the norm on noncommutative L_p spaces. Our idea of proof follows the one given in [12].

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†College of Mathematics and System Sciences, Xinjiang University Urumqi 830046, China (hyz0080@aliyun.com). Supported by the National Natural Science Foundation of China No. 11401507 and the Natural Science Foundation of Xinjiang University (Starting Fund for Doctors, Grant No. BS150202).

‡College of Information Science and Technology, Xiamen University Tan Kah Kee College, Xiamen 363123, China (jingjing.shao86@yahoo.com).

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2. Preliminaries. Unless stated otherwise, $\mathcal{M}$ will always denote a semifinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful finite normalized trace τ. We refer to [14] for noncommutative integration. We denote the identity of $\mathcal{M}$ by 1. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, we define its distribution function by $\lambda_s(x) = \tau(e_s^{\perp}(|x|))$ and x will be called τ-measurable if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. The measure topology in $L_0(\mathcal{M})$ is the vector space topology defined via the neighbourhood base $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, where $N(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \tau(e_{(\varepsilon,\infty)}(|x|)) \leq \delta\}$ and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε, ∞). With respect to the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$-algebra.

Definition 2.1. Let $x \in L_0(\mathcal{M})$ and $t > 0$. The *t-th singular number (or generalized singular number) of* x, $\mu_t(x)$, is defined by

$$\mu_t(x) = \inf \left\{ \|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^{\perp}) \leq t \right\}.$$

If $x, y \in L_0(\mathcal{M})$, then we say that x is submajorized by y and write $x \prec y$ if and only if

$$\int_0^a \mu_t(x)dt \leq \int_0^a \mu_t(y)dt \quad \text{for all } a \geq 0.$$

We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively. For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all densely-defined closed operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} = \left(\int_0^\infty \mu_t(x)^p dt \right)^{\frac{1}{p}} < \infty.$$

As usual, we put $L^\infty(\mathcal{M}; \tau) = \mathcal{M}$ and denote by $\| \cdot \|_\infty$ $(= \| \cdot \|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\| \cdot \|_p$ $(1 \leq p \leq \infty)$. For every $x \in L_0(\mathcal{M})$, there is a unique polar decomposition $x = u|x|$ where $|x| \in L_0(\mathcal{M})^+$ (the positive part of $L_0(\mathcal{M})$) and u is a partial isometry operator. Let $r(x) = u^*u$ and $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x, respectively. Note that $l(x)$ (resp., $r(x)$) is the least projection e of $\mathcal{B}(\mathcal{H})$ such that $ex = x$ (resp., $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. Let $\mathcal{M}^+ = \{x \in \mathcal{M} : x \geq 0\}$(i.e., the

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positive part of $\mathcal{M}$). We write $S(\mathcal{M})^+ = \{x \in \mathcal{M}^+ : \tau(s(x)) < \infty\}$. Let $S(\mathcal{M})$ be the linear span of $S(\mathcal{M})^+$. It is well known that $(S(\mathcal{M}), \|\cdot\|_p)$ is dense on $L^p(\mathcal{M})$. For further results about noncommutative L_p spaces, the reader is referred to [7, 14].

Given $x, y \in L_0(\mathcal{M})$ and $0 < p < \infty$, from Theorem 4.2 of [7], we have

$$\int_0^t \mu_s(xy)^p ds \le \int_0^t \mu_s(y)^p \mu_s(x)^p ds, \quad t > 0.$$

Let $0 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $x \in L^q(\mathcal{M})$ and $y \in L^r(\mathcal{M})$, then the usual Hölder inequality implies that

$$\left(\int_0^\infty \mu_s(xy)^p ds\right)^{\frac{1}{p}} \le \left(\int_0^\infty (\mu_s(y)\mu_s(x))^p ds\right)^{\frac{1}{p}} \le \left(\int_0^\infty \mu_s(x)^q ds\right)^{\frac{1}{q}} \left(\int_0^\infty \mu_s(y)^r ds\right)^{\frac{1}{r}}.$$

That is

$$\|xy\|_p \le \|x\|_q \|y\|_r. \tag{2.1}$$

3. Main result.

We start this section by two simple lemmas.

LEMMA 3.1. *Let $x_1, x_2, \ldots, x_n \in L_0(\mathcal{M})^+$ and $1 \le p < \infty$.*

(1) *If f is any nonnegative convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$\left\|\sum_{i=1}^n f(x_i)\right\|_p \le \left\|f\left(\sum_{i=1}^n x_i\right)\right\|_p. \tag{3.1}$$

(2) *If f is any nonnegative concave function on $[0, \infty)$, then*

$$\left\|f\left(\sum_{i=1}^n x_i\right)\right\|_p \le \left\|\sum_{i=1}^n f(x_i)\right\|_p. \tag{3.2}$$

Proof. (1) If $f(\sum_{i=1}^n x_i) \notin L^p(\mathcal{M})$, then $\|f(\sum_{i=1}^n x_i)\|_p = \infty$. Thus, the inequality (3.1) is clear. If $f\left(\sum_{i=1}^n x_i\right) \in L^p(\mathcal{M})$, it follows from Theorem 5.3 (i) of [6] that

$$\int_0^t \mu_s\left(\sum_{i=1}^n f(x_i)\right) ds \le \int_0^t \mu_s\left(f\left(\sum_{i=1}^n x_i\right)\right) ds, \quad t > 0.$$

Then Theorem 2.1 of [4] tells us that

$$\int_0^t \mu_s\left(\sum_{i=1}^n f(x_i)\right)^p ds \le \int_0^t \mu_s\left(f\left(\sum_{i=1}^n x_i\right)\right)^p ds, \quad t > 0.$$

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Hence, $\| \sum_{i=1}^{n} f(x_i)\|_p \leq \|f(\sum_{i=1}^{n} x_i)\|_p$.

(2) The proof can be done similarly to (1) by using Theorem 5.3(ii) of [6]. The details are omitted. □

The matrix version of Lemma 3.1 appears in [11].

Let $x, y, z \in L_0(\mathcal{M})$. The block matrix $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is positive partial transpose (i.e., PPT) if $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ and $\begin{pmatrix} x & z^* \\ z & y \end{pmatrix} \geq 0$.

Lemma 3.2. *Let $x, y \in S(\mathcal{M})^+$ and $z \in S(\mathcal{M})$. If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT and $1 \leq p < \infty$, then $\|z^*z\|_p \leq \|xy\|_p$.*

Proof. From the fact $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT, we deduce $z = x^{\frac{1}{2}} k_1 y^{\frac{1}{2}}$ and $z^* = x^{\frac{1}{2}} k_2 y^{\frac{1}{2}}$, where k_1, k_2 are contraction operators. By Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned}
\int_0^t \mu_s(z^*z)^p ds &= \int_0^t \mu_s \left(x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} k_1 y^{\frac{1}{2}} \right)^p ds \\
&= \int_0^t \mu_s \left(y^{\frac{1}{2}} x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} k_1 \right)^p ds \\
&\leq \int_0^t \mu_s \left(y^{\frac{1}{2}} x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^p ds \\
&\leq \int_0^t \mu_s \left(\left(y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^2 \right)^p ds, \quad t > 0.
\end{aligned}$$

It follows from Theorem 2 of [10] that

$$\begin{aligned}
\int_0^t \mu_s(z^*z)^p ds &\leq \int_0^t \mu_s \left(\left(y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^2 \right)^p ds \\
&\leq \int_0^t \mu_s(yx)^p ds \\
&= \int_0^t \mu_s(xy)^p ds, \quad t > 0.
\end{aligned}$$

This completes the proof. □

The matrix version of Lemma 3.2 appears in [13].

Now, using Lemma 3.1 and Lemma 3.2, we get our first main result of this note.

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THEOREM 3.3. *Let* $1 \leq p,q,r < \infty$ *with* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *For* $i = 1,2,\ldots,n$, *let* $x_i \in L^q(\mathcal{M})^+$, $y_i \in L^r(\mathcal{M})^+$ *such that* $x_iy_i = y_ix_i$. *Then*

$$\left\|\sum_{i=1}^n x_iy_i\right\|_p \leq \left\|\left(\sum_{i=1}^n x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\right)^2\right\|_p \leq \left\|\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)\right\|_p.$$

Proof. By inequality (2.1), we obtain

$$\sum_{i=1}^n x_iy_i \in L^p(\mathcal{M}), \ \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) \in L^p(\mathcal{M}).$$

Since $x_iy_i = y_ix_i$, we have $x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} = y_i^{\frac{1}{2}}x_i^{\frac{1}{2}}$. Thus, $x_iy_i \geq 0$ and $x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \geq 0$. From Lemma 3.1, we deduce

$$\left\|\sum_{i=1}^n x_iy_i\right\|_p = \left\|\sum_{i=1}^n \left(x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\right)^2\right\|_p \leq \left\|\left(\sum_{i=1}^n x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\right)^2\right\|_p.$$

Let $0 < s \leq \infty$. It is well known that $S(\mathcal{M})$ is dense in $L^s(\mathcal{M})$. Hence, there exist

$$\{x_{i,k}\}_{k=1}^\infty, \{y_{i,k}\}_{k=1}^\infty, \{z_{i,k}\}_{k=1}^\infty \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x_{i,k}\|_q \to 0, \ \ \|y_i - y_{i,k}\|_r \to 0, \ \ \|z_i - z_{i,k}\|_{2p} \to 0, \ \ k \to \infty.$$

Note that

$$\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}}x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix} = \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \geq 0.$$

It is easy to see that $\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}}x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix}$ is PPT. Hence,

$$\begin{pmatrix} \sum_{i=1}^n x_{i,k} & \sum_{i=1}^n x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}} \\ \sum_{i=1}^n y_{i,k}^{\frac{1}{2}}x_{i,k}^{\frac{1}{2}} & \sum_{i=1}^n y_{i,k} \end{pmatrix}$$

is PPT. It follows from Lemma 3.2 that

$$\left\|\left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}}y_{i,k}^{\frac{1}{2}}\right)^2\right\|_p \leq \left\|\left(\sum_{i=1}^n x_{i,k}\right)\left(\sum_{i=1}^n y_{i,k}\right)\right\|_p. \tag{3.3}$$

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On the other hand, inequality (2.1) implies that

$$\begin{aligned}\left\| x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} &\le \left\| x_i^{\frac{1}{2}} \left(y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right) \right\|_{2p} + \left\| \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right) y_i^{\frac{1}{2}} \right\|_{2p} \\ &\le \left\| x_i^{\frac{1}{2}} \right\|_{2q} \left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} + \left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \left\| y_i^{\frac{1}{2}} \right\|_{2r} \\ &= \| x_i \|_q^{\frac{1}{2}} \left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} + \left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \| y_i \|_r^{\frac{1}{2}}.\end{aligned}$$

Put $g(t) = t^{\frac{1}{2}}$, then g is nonnegative and operator monotone. According to Theorem 1.1 in [5], we obtain

$$\int_0^t \mu_s \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right) ds \le \int_0^t \mu_s \left(|x_i - x_{i,k}|^{\frac{1}{2}} \right) ds, \quad t > 0.$$

From the fact $2q > 1$ and Theorem 2.1 of [4] and Lemma 2.5(iv) of [7], we get

$$\int_0^t \mu_s \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right)^{2q} ds \le \int_0^t \mu_s (x_i - x_{i,k})^q ds, \quad t > 0.$$

This implies that

$$\left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \le \| x_i - x_{i,k} \|_q^{\frac{1}{2}} \to 0, \;\; k \to \infty.$$

Similarly,

$$\left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} \le \| y_i - y_{i,k} \|_r^{\frac{1}{2}} \to 0, \;\; k \to \infty.$$

Thus, $\left\| x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \to 0$ as $k \to \infty$, and so

$$\left\| \sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \to 0, \;\; k \to \infty.$$

Therefore, $\left\| \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \to \left\| \sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right\|_{2p}$, $k \to \infty$. Hence,

$$\left\| \left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right)^2 \right\|_p \to \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p, \;\; k \to \infty. \tag{3.4}$$

By an argument similar to the one presented above, we obtain

$$\| x_i y_j - x_{i,k} y_{j,k} \|_p \to 0, \;\; k \to \infty.$$

It follows that

$$\left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) - \left(\sum_{i=1}^n x_{i,k} \right) \left(\sum_{i=1}^n y_{i,k} \right) \right\|_p \to 0, \;\; k \to \infty,$$

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which tells us that

$$\left\| \left(\sum_{i=1}^{n} x_{i,k} \right) \left(\sum_{i=1}^{n} y_{i,k} \right) \right\|_p \to \left\| \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right) \right\|_p, \quad k \to \infty. \tag{3.5}$$

Combing (3.3) and (3.4) with (3.5), we have

$$\left\| \left(\sum_{i=1}^{n} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \leq \left\| \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right) \right\|_p. \quad \square$$

Theorem 3.3 includes a special case as follows.

COROLLARY 3.4. *Let* $1 \leq p, q, r < \infty$ *with* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ *and* $s, t > 0$*. For* $t, s > 0$*, let* $x \in L^{tq}(\mathcal{M})^+$*,* $y \in L^{sr}(\mathcal{M})^+$*. Then*

$$\|x^{t+s} + y^{t+s}\|_p \leq \|(x^t + y^t)(x^s + y^s)\|_p.$$

Proof. If we replace n, x_1, y_1, x_2, y_2 by $2, x^t, x^s, y^t, y^s$, respectively, in Theorem 3.3, we deduce that

$$\|x^{t+s} + y^{t+s}\|_p \leq \|(x^t + y^t)(x^s + y^s)\|_p. \quad \square$$

LEMMA 3.5. *Let* $x, y \in S(\mathcal{M})^+$ *and* $0 \leq q \leq 1$*. If* $1 \leq p < \infty$*, then*

$$\|x^q y^{1-q}\|_p \leq \|qx + (1-q)y\|_p.$$

Proof. The result follows immediately from Theorem 3.3 of [8]. $\square$

Now, using Lemma 3.5, we get our another main result of this note.

THEOREM 3.6. *Let* $1 \leq p < \infty$ *and* $0 \leq q \leq 1$*. For all* $x, y \in L^{2p}(\mathcal{M})$*, we have*

$$\|xy^*\|_p^2 \leq \|qx^*x + (1-q)y^*y\|_p \|(1-q)x^*x + qy^*y\|_p.$$

Proof. Since $S(\mathcal{M})$ is dense in $L^s(\mathcal{M}) (0 < s \leq \infty)$, then there exist

$$\{x_i\}_{k=1}^{\infty}, \{y_i\}_{k=1}^{\infty} \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x\|_{2p} \to 0, \quad \|y_i - y\|_{2p} \to 0, \quad k \to \infty.$$

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By inequality (2.1), Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned}\|x_iy_i^*\|_p^2 &= \left(\int_0^\infty \mu_t(y_ix_i^*x_iy_i^*)^{\frac{p}{2}}dt\right)^{\frac{2}{p}} = \left(\int_0^\infty \mu_t(x_i^*x_iy_i^*y_i)^{\frac{p}{2}}dt\right)^{\frac{2}{p}}\\ &= \left(\int_0^\infty \mu_t((y_i^*y_i)^q(x_i^*x_i)^{1-q}(x_i^*x_i)^q(y_i^*y_i)^{1-q})^{\frac{p}{2}}dt\right)^{\frac{2}{p}}\\ &= \|(y_i^*y_i)^q(x_i^*x_i)^{1-q}(x_i^*x_i)^q(y_i^*y_i)^{1-q}\|_{\frac{p}{2}}\\ &\le \left\|(y_i^*y_i)^q(x_i^*x_i)^{1-q}\right\|_p \left\|(x_i^*x_i)^q(y_i^*y_i)^{1-q}\right\|_p\end{aligned}$$

for all $0 \le q \le 1$. It follows from Lemma 3.5 that

$$\|(y_i^*y_i)^q(x_i^*x_i)^{1-q}\|_p \le \|q(y_i^*y_i) + (1-q)(x_i^*x_i)\|_p$$

and

$$\|(x_i^*x_i)^q(y_i^*y_i)^{1-q}\|_p \le \|q(x_i^*x_i) + (1-q)(y_i^*y_i)\|_p.$$

Hence,

$$\|x_iy_i^*\|_p^2 \le \|q(y_i^*y_i) + (1-q)(x_i^*x_i)\|_p\|q(x_i^*x_i) + (1-q)(y_i^*y_i)\|_p.$$

A similar argument to the proof of Theorem 3.3 shows that

$$\|xy^*\|_p^2 \le \|q(y^*y) + (1-q)(x^*x)\|_p\|q(x^*x) + (1-q)(y^*y)\|_p. \quad \square$$

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