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ON THE \textit{m}th ROOTS OF A COMPLEX MATRIX\textasteriskcentered

PANAYIOTIS J. PSARRAKOS

\textbf{Abstract.} If an \(n \times n\) complex matrix \(A\) is nonsingular, then for every integer \(m > 1\), \(A\) has an \(m\)th root \(B\), i.e., \(B^m = A\). In this paper, we present a new simple proof for the Jordan canonical form of the root \(B\). Moreover, a necessary and sufficient condition for the existence of \(m\)th roots of a singular complex matrix \(A\) is obtained. This condition is in terms of the dimensions of the null spaces of the powers \(A^k\) \((k = 0, 1, 2, \ldots)\).

\textbf{Key words.} Ascent sequence, eigenvalue, eigenvector, Jordan matrix, matrix root.

\textbf{AMS subject classifications.} 15A18, 15A21, 15A22, 47A56

1. Introduction and preliminaries. Let \(\mathcal{M}_n\) be the algebra of all \(n \times n\) complex matrices and let \(A \in \mathcal{M}_n\). For an integer \(m > 1\), a matrix \(B \in \mathcal{M}_n\) is called an \(m\)th root of \(A\) if \(B^m = A\). If the matrix \(A\) is nonsingular, then it always has an \(m\)th root \(B\). This root is not unique and its Jordan structure is related to the Jordan structure of \(A\) [2, pp. 231-234]. In particular, \((\lambda - \mu_0)^k\) is an elementary divisor of \(B\) if and only if \((\lambda - \mu_0^m)^k\) is an elementary divisor of \(A\). If \(A\) is a singular complex matrix, then it may have no \(m\)th roots. For example, there is no matrix \(B\) such that \(B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\). As a consequence, the problem of characterizing the singular matrices, which have \(m\)th roots, is of interest [1], [2].

Consider the (associated) matrix polynomial \(P(\lambda) = I_n \lambda^m - A\), where \(I_n\) is the identity matrix of order \(n\) and \(\lambda\) is a complex variable. A matrix \(B \in \mathcal{M}_n\) is an \(m\)th root of \(A\) if and only if \(P(B) = B^m - A = 0\). As a consequence, the problem of computation of \(m\)th roots of \(A\) is strongly connected with the spectral analysis of \(P(\lambda)\). The suggested references for matrix polynomials are [3] and [7].

A set of vectors \(\{x_0, x_1, \ldots, x_k\}\), which satisfies the equations

\[
P(\omega_0)x_0 = 0 \\
P(\omega_0)x_1 + \frac{1}{1!} P^{(1)}(\omega_0)x_0 = 0 \\
\vdots \\
P(\omega_0)x_k + \frac{1}{1!} P^{(1)}(\omega_0)x_{k-1} + \cdots + \frac{1}{k!} P^{(k)}(\omega_0)x_0 = 0,
\]

where the indices on \(P(\lambda)\) denote derivatives with respect to the variable \(\lambda\), is called a \textit{Jordan chain} of length \(k+1\) of \(P(\lambda)\) corresponding to the \textit{eigenvalue} \(\omega_0 \in \mathbb{C}\) and the \textit{eigenvector} \(x_0 \in \mathbb{C}^n\). The vectors in a Jordan chain are not uniquely defined and for \(m > 1\), they need not be linearly independent [3], [6]. If we set \(m = 1\), then the

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The Jordan structure of the linear pencil $I_n \lambda - A$ coincides with the Jordan structure of $A$, and the vectors of each Jordan chain are chosen to be linearly independent [2], [6]. Moreover, there exist a matrix

\[
J_A = \bigoplus_{j=1} \left( I_{k_j} \omega_j + N_{k_j} \right) \quad (k_1 + k_2 + \ldots + k_\xi = n),
\]

where $N_{k_j}$ is the nilpotent matrix of order $k_j$ having ones on the super diagonal and zeros elsewhere, and an $n \times n$ nonsingular matrix

\[
X_A = \begin{bmatrix} x_{1,1} & \ldots & x_{1,k_1} & x_{2,1} & \ldots & x_{2,k_2} & \ldots & x_{\xi,1} & \ldots & x_{\xi,k_\xi} \end{bmatrix},
\]

where for every $j = 1, 2, \ldots, \xi$, $\{x_{j,1}, x_{j,2}, \ldots, x_{j,k_j}\}$ is a Jordan chain of $A$ corresponding to $\omega_j \in \sigma(A)$, such that (see [2], [4], [6])

\[
A = X_A J_A X_A^{-1}.
\]

The matrix $J_A$ is called the Jordan matrix of $A$, and it is unique up to permutations of the diagonal Jordan blocks $I_{k_j} \omega_j + N_{k_j}$ ($j = 1, 2, \ldots, \xi$) [2], [4].

The set of all eigenvalues of $P(\lambda)$, that is, $\sigma(P) = \{ \mu \in \mathbb{C} : \det P(\mu) = 0 \}$, is called the spectrum of $P(\lambda)$. Denoting by $\sigma(A) = \sigma(I_n \lambda - A)$ the spectrum of the matrix $A$, it is clear that $\sigma(P) = \{ \mu \in \mathbb{C} : \mu^m \in \sigma(A) \}$. If $J_A$ is the Jordan matrix of $A$ in (1.1), then it will be convenient to define the $J$-spectrum of $A$, $\sigma_J(A) = \{\omega_1, \omega_2, \ldots, \omega_\xi\}$, where the eigenvalues of $A$ follow exactly the order of their appearance in $J_A$ (obviously, repetitions are allowed). For example, the $J$-spectrum of the matrix $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is $\sigma_J(M) = \{0, 0, 1\}$.

In this article, we study the Jordan structure of the $m$th roots ($m > 1$) of a complex matrix. In Section 2, we consider a nonsingular matrix and present a new constructive proof for the Jordan canonical form of its $m$th roots. This proof is simple and based on spectral analysis of matrix polynomials [2], [3], [7]. Furthermore, it yields directly the Jordan chains of the $m$th roots. We also generalize a known uniqueness statement [5]. In Section 3, using a methodology of Cross and Lancaster [1], we obtain a necessary and sufficient condition for the existence of $m$th roots of a singular matrix.

2. The nonsingular case. Consider a nonsingular matrix $A \in \mathcal{M}_n$ and an integer $m > 1$. If $A$ is diagonalizable and $S \in \mathcal{M}_n$ is a nonsingular matrix such that

\[
A = S \text{diag}\{r_1 e^{i \phi_1}, r_2 e^{i \phi_2}, \ldots, r_m e^{i \phi_m}\} S^{-1},
\]

where $r_j > 0$, $\phi_j \in [0, 2\pi)$ ($j = 1, 2, \ldots, n$), then for every $m$-tuple $(s_1, s_2, \ldots, s_n)$, $s_j \in \{1, 2, \ldots, m\}$ ($j = 1, 2, \ldots, n$), the matrix

\[
B = S \text{diag}\{\frac{1}{r_1} e^{i \phi_1 + \frac{2 \pi}{m} (s_1 - 1)}, \frac{1}{r_2} e^{i \phi_2 + \frac{2 \pi}{m} (s_2 - 1)}, \ldots, \frac{1}{r_n} e^{i \phi_n + \frac{2 \pi}{m} (s_n - 1)}\} S^{-1}
\]

is an $m$th root of $A$. Hence, the investigation of the $m$th roots of a nonsingular (and not diagonalizable) matrix $A$ via the Jordan canonical form of $A$ arises in a natural way [2]. The following lemma is necessary and of independent interest.
Lemma 2.1. Let \( \{x_0, x_1, \ldots, x_k\} \) be a Jordan chain of \( A \in \mathcal{M}_n \) (with linearly independent terms) corresponding to a nonzero eigenvalue \( \omega_0 = r_0 e^{i \phi_0} \in \sigma(A) \) \((r_0 > 0, \phi_0 \in [0, 2\pi))\), and let \( P(\lambda) = I_n \lambda^m - A \). Then for every eigenvalue \( \mu_1^m 0 e^{i \phi_0 + 2(t-1)\pi} \in \sigma(P) ; \ t = 1, 2, \ldots, m, \) the matrix polynomial \( P(\lambda) \) has a Jordan chain of the form

\[
y_0 = x_0
y_1 = a_{1,1} x_1
y_2 = a_{2,1} x_1 + a_{2,2} x_2
\vdots 
\vdots 
y_k = a_{k,1} x_1 + a_{k,2} x_2 + \cdots + a_{k,k} x_k,
\]

where the coefficients \( a_{i,j} \) \((1 \leq j \leq i \leq k)\) depend on the integer \( t \) and for every \( i = 1, 2, \ldots, k, \ a_{i,i} = (m r_0^{m+2(t-1)} e^{i(m-1) \phi_0 + 2(t-1)\pi})^i \neq 0. \) Moreover, the vectors \( y_0, y_1, \ldots, y_k \) are linearly independent.

Proof. Since \( \{x_0, x_1, \ldots, x_k\} \) is a Jordan chain of the matrix \( A \) corresponding to the eigenvalue \( \omega_0 \neq 0, \) we have

\[
(A - I_n \omega_0)x_0 = 0
\]

and

\[
(A - I_n \omega_0)x_i = x_{i-1} ;\ i = 1, 2, \ldots, k.
\]

Let \( \mu_0 \) be an eigenvalue of \( P(\lambda) \) such that \( \mu_0^m = \omega_0. \) By the equation

\[
(I_n \omega_0 - A)x_0 = (I_n \mu_0^m - A)x_0 = 0,
\]

it is obvious that \( y_0 = x_0 \) is an eigenvector of \( P(\lambda) \) corresponding to \( \mu_0 \in \sigma(P). \) Assume now that there exists a vector \( y_1 \in \mathbb{C}^n \) such that

\[
P(\mu_0)y_1 + \frac{P^{(1)}(\mu_0)}{1!} y_0 = 0.
\]

Then

\[
(I_n \mu_0^m - A)y_1 = -m \mu_0^{m-1} y_0,
\]

or equivalently,

\[
(I_n \omega_0 - A)y_1 = m \mu_0^{m-1}(I_n \omega_0 - A)x_1.
\]

Hence, we can choose \( y_1 = a_{1,1} x_1, \) where \( a_{1,1} = m \mu_0^{m-1} \neq 0. \) Similarly, if we consider the equation

\[
P(\mu_0)y_2 + \frac{P^{(1)}(\mu_0)}{1!} y_1 + \frac{P^{(2)}(\mu_0)}{2!} y_0 = 0,
\]

and
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then it follows

$$(I_n\mu_0^m - A)y_2 = -m\mu_0^{m-1}y_1 - \frac{m(m-1)}{2}\mu_0^{m-2}y_0,$$

or equivalently,

$$(I_n\omega_0 - A)y_2 = (I_n\omega_0 - A)\left((m\mu_0^{m-1})^2x_2 + \frac{m(m-1)}{2}\mu_0^{m-2}x_1\right).$$

Thus, we can choose $y_2 = a_{2,1}x_1 + a_{2,2}x_2$, where $a_{2,1} = \frac{m(m-1)}{2}\mu_0^{m-2}$ and $a_{2,2} = (m\mu_0^{m-1})^2 \neq 0$. Repeating the same steps implies that the matrix polynomial $P(\lambda)$ has a Jordan chain $\{y_0, y_1, \ldots, y_k\}$ as in (2.1).

Define the $n \times (k+1)$ matrices

$$X_0 = \begin{bmatrix} x_0 & x_1 & \cdots & x_k \end{bmatrix} \quad \text{and} \quad Y_0 = \begin{bmatrix} y_0 & y_1 & \cdots & y_k \end{bmatrix}.$$

Since the vectors $x_0, x_1, \ldots, x_k \in \mathbb{C}^n$ are linearly independent, $\text{rank}(X_0) = k+1$. Moreover, $Y_0 = X_0T_0$, where the $(k+1) \times (k+1)$ upper triangular matrix

$$T_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{k,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{k,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k,k} \end{bmatrix}$$

is nonsingular. As a consequence, $\text{rank}(Y_0) = k+1$, and the proof is complete.

The Jordan chain $\{y_0, y_1, \ldots, y_k\}$ of $P(\lambda)$ in the above lemma, is said to be associated to the Jordan chain $\{x_0, x_1, \ldots, x_k\}$ of $A$, and clearly depends on the choice of $t \in \{1, 2, \ldots, m\}$. Consider now the nonsingular matrix $X_A \in \mathcal{M}_n$ in (1.2), and for any $(s_1, s_2, \ldots, s_\xi)$, $s_j \in \{1, 2, \ldots, m\}$ ($j = 1, 2, \ldots, \xi$), define the matrix

$$Y_A(s_1, s_2, \ldots, s_\xi) = \begin{bmatrix} y_{1,1} & \cdots & y_{1,k_1} & y_{2,1} & \cdots & y_{\xi,1} & \cdots & y_{\xi,k_\xi} \end{bmatrix},$$

where for every $j = 1, 2, \ldots, \xi$, the set $\{y_{j,1}, y_{j,2}, \ldots, y_{j,k_j}\}$ is the associated Jordan chain of $P(\lambda)$ corresponding to the Jordan chain $\{x_{j,1}, x_{j,2}, \ldots, x_{j,k_j}\}$ of $A$ and the integer $s_j$.

**Corollary 2.2.** For every $(s_1, s_2, \ldots, s_\xi)$, $s_j \in \{1, 2, \ldots, m\}$ ($j = 1, 2, \ldots, \xi$), the associated matrix $Y_A(s_1, s_2, \ldots, s_\xi)$ is nonsingular.

**Proof.** By Lemma 2.1, there exist upper triangular matrices $T_1, T_2, \ldots, T_\xi$, which depend on the choice of the $\xi$-tuple $(s_1, s_2, \ldots, s_\xi)$ and have nonzero diagonal elements, such that

$$Y_A(s_1, s_2, \ldots, s_\xi) = X_A \left( \bigoplus_{j=1}^{\xi} T_j \right).$$

Since $X_A$ is also nonsingular the proof is complete. □
THEOREM 2.3. Let \( A \in \mathcal{M}_n \) be a nonsingular complex matrix with Jordan matrix \( J_A = \bigoplus_{j=1}^{\xi} (I_{k_j} \omega_j + N_{k_j}) \) as in (1.1) and \( J \)-spectrum
\[
\sigma_J(A) = \{ \omega_1 = r_1 e^{i \theta_1}, \omega_2 = r_2 e^{i \theta_2}, \ldots, \omega_{\xi} = r_{\xi} e^{i \theta_{\xi}} \}.
\]
Consider an integer \( m > 1 \), the nonsingular matrix \( X_A \in \mathcal{M}_n \) in (1.2) such that
\[ A = X_A J_A X_A^{-1}, \]
a \( \xi \)-tuple \( (s_1, s_2, \ldots, s_{\xi}) \), \( s_j \in \{1, 2, \ldots, m\} \) \( (j = 1, 2, \ldots, \xi) \) and the associated matrix \( Y_A(s_1, s_2, \ldots, s_{\xi}) \). Then the matrix
\[
B = Y_A(s_1, s_2, \ldots, s_{\xi}) \left( \bigoplus_{j=1}^{\xi} \left( I_{k_j} r_j^{\frac{m}{e} e^{i \frac{2(\theta_j j - 1)\pi}{m}} + N_{k_j}} \right) \right) Y_A(s_1, s_2, \ldots, s_{\xi})^{-1}
\]
(2.2)
is an \( m \)th root of \( A \).

Proof. Since the associated matrix \( Y_A(s_1, s_2, \ldots, s_{\xi}) \) is nonsingular, by Corollary 7.11 in [3], the linear pencil \( I_n \lambda - B \) is a right divisor of \( P(\lambda) = I_n \lambda^m - A \), i.e., there exists an \( n \times n \) matrix polynomial \( Q(\lambda) \) of degree \( m - 1 \) such that
\[
P(\lambda) = Q(\lambda) (I_n \lambda - B).
\]
Consequently, by [2, pp. 81-82] (see also Lemma 22.9 in [7]), \( P(B) = B^m - A = 0 \), and hence \( B \) is an \( m \)th root of the matrix \( A \). \( \square \)

At this point, we remark that the associated matrix \( Y_A(s_1, s_2, \ldots, s_{\xi}) \) can be computed directly by the method described in the proof of Lemma 2.1. Moreover, it is clear that a nonsingular matrix may have \( m \)th roots with common eigenvalues. Motivated by [5], we obtain a spectral condition that implies the uniqueness of an \( m \)th root.

**THEOREM 2.4.** Suppose \( A \in \mathcal{M}_n \) is a nonsingular complex matrix and its spectrum \( \sigma(A) \) lies in a cone
\[
K_0 = \{ z \in \mathbb{C} : \theta_1 \leq \text{Arg} \, z \leq \theta_2, \ 0 < \theta_2 - \theta_1 \leq \vartheta_0 < 2 \pi \}.
\]
Then for every \( k = 1, 2, \ldots, m \), \( A \) has a unique \( m \)th root \( B_k \) such that
\[
\sigma(B_k) \subset \left\{ z \in \mathbb{C} : \frac{\theta_1 + 2(k-1)\pi}{m} \leq \text{Arg} \, z \leq \frac{\theta_2 + 2(k-1)\pi}{m} \right\}.
\]
In particular, for every \( k = 2, 3, \ldots, m \), \( B_k = e^{i \frac{2(k-1)\pi}{m}} B_1 \).

Proof. Observe that the spectrum of \( P(\lambda) \), \( \sigma(P) = \{ \mu \in \mathbb{C} : \mu^m \in \sigma(A) \} \), lies in the union
\[
\bigcup_{k=1}^{m} \left\{ z \in \mathbb{C} : \frac{\theta_1 + 2(k-1)\pi}{m} \leq \text{Arg} \, z \leq \frac{\theta_2 + 2(k-1)\pi}{m} \right\},
\]
and for every \( k = 1, 2, \ldots, m \), denote
\[
\Sigma_k = \sigma(P) \cap \left\{ z \in \mathbb{C} : \frac{\theta_1 + 2(k-1)\pi}{m} \leq \text{Arg} \, z \leq \frac{\theta_2 + 2(k-1)\pi}{m} \right\}
\]
\[
= \left\{ r \frac{m}{\text{Arg} \, z} e^{i \frac{2(k-1)\pi}{m}} : r e^{i \varphi} \in \sigma(A), \ r > 0, \ \varphi \in [\theta_1, \theta_2] \right\}.
\]
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Then by Theorem 2.3, for every $k = 1, 2, \ldots, m$, the matrix $A$ has an $m$th root $B_k$ such that $\sigma(B_k) = \Sigma_k$. Since the sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ are mutually disjoint, Lemma 22.8 in [7] completes the proof.

It is worth noting that if $J_A = \bigoplus_{j=1}^{\ell} (I_{k_j}, \omega_j + N_{k_j})$ is the Jordan matrix of $A$ in (1.1), and $\sigma_J(A) = \{ \omega_1 = r_1 e^{i\omega_1}, \omega_2 = r_2 e^{i\omega_2}, \ldots, \omega_{\ell} = r_{\ell} e^{i\omega_{\ell}} \}$ is the $J$-spectrum of $A$, then for every $k = 1, 2, \ldots, m$, the $m$th root $B_k$ in the above theorem is given by (2.2) for $s_1 = s_2 = \cdots = s_\ell = k$. Furthermore, if we allow $\theta_1 \longrightarrow -\pi^+$ and $\theta_2 \longrightarrow -\pi^-$, then for $m = 2$, we have the following corollary.

**Corollary 2.5.** (Theorem 5 in [5]) Let $A \in \mathcal{M}_n$ be a complex matrix with $\sigma(A) \cap (-\infty, 0) = \emptyset$. Then $A$ has a unique square root $B$ such that $\sigma(B) \subset \{ z \in \mathbb{C} : \Re z > 0 \}$.

**Example 2.6.** Consider a $5 \times 5$ complex matrix $A = X_A J_A X_A^{-1}$, where $X_A \in \mathcal{M}_5$ is nonsingular and

$$J_A = \begin{bmatrix}
i & 1 & 0 & 0 & 0 \\
0 & i & 1 & 0 & 0 \\
0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}. $$

Suppose $m = 3$ and for a pair $(s_1, s_2)$, $s_j \in \{1, 2, 3\}$ $(j = 1, 2)$, denote

$$\alpha = e^{i\frac{2\pi^2(2s_1 - 1)i}{3}}, \quad \beta = e^{i\frac{2\pi^2(s_2 - 1)i}{3}}. $$

Then the associated matrix of $X_A$, corresponding to $(s_1, s_2)$, is

$$Y_{A}(s_1, s_2) = X_A \begin{bmatrix}1 & 0 & 0 \\
0 & 3\alpha^2 & 3\alpha \\
0 & 0 & 9\alpha^4 \end{bmatrix} \oplus \begin{bmatrix}1 & 0 \\
0 & 3\beta^2 \end{bmatrix}. $$

One can verify that the matrix

$$B = Y_{A}(s_1, s_2) \begin{bmatrix}\alpha & 1 & 0 & 0 & 0 \\
0 & \alpha & 1 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & \beta & 1 \\
0 & 0 & 0 & 0 & \beta \end{bmatrix} Y_{A}(s_1, s_2)^{-1} = X_A \begin{bmatrix}\alpha & \frac{1}{4}\alpha^{-2} & \frac{1}{9}\alpha^{-5} & 0 & 0 \\
0 & \alpha & \frac{1}{9}\alpha^{-2} & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & \beta & \frac{1}{4}\beta^{-2} \\
0 & 0 & 0 & 0 & \beta \end{bmatrix} X_A^{-1}$$

is a 3rd root of $A$ (see also the equation (58) in [2, p. 232]). Moreover, if we choose $s_1 = s_2 = 1$, then

$$Y_{A}(1, 1) = X_A \begin{bmatrix}1 & 0 & 0 \\
0 & 3e^{i\frac{2\pi}{3}} & 3e^{i\frac{2\pi}{3}} \\
0 & 0 & 9e^{i\frac{4\pi}{3}} \end{bmatrix} \oplus \begin{bmatrix}1 & 0 \\
0 & 3 \end{bmatrix},$$
and the matrix

\[ B_1 = Y_A(1,1) \begin{bmatrix} e^{i\varphi} & 1 & 0 & 0 \\ 0 & e^{i\varphi} & 1 & 0 \\ 0 & 0 & e^{i\varphi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Y_A(1,1)^{-1} = X_A \begin{bmatrix} e^{i\varphi} & \frac{1}{4}e^{-i\varphi} & \frac{1}{4}e^{-i\varphi} & 0 & 0 \\ 0 & e^{i\varphi} & \frac{1}{4}e^{-i\varphi} & 0 & 0 \\ 0 & 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} X_A^{-1} \]

is the unique 3rd root of \( A \) with \( \sigma(B_1) \subset \{ z \in \mathbb{C} : 0 \leq \text{Arg} z \leq \pi/6 \} \).

3. The singular case. Let \( A \in M_n \) be a singular matrix, and let \( J_A = \bigoplus_{j=1}^{\xi} (I_{k_j}, \omega_j + N_{k_j}) \) be its Jordan matrix in (1.1). For the remainder and without loss of generality, we assume that \( \omega_j = 0 \) for \( j = 1, 2, \ldots, \psi \) \((1 \leq \psi \leq \xi)\), with \( \omega_j \neq 0 \) otherwise, and \( k_1 \geq k_2 \geq \cdots \geq k_{\psi} \) [1], [2]. We also denote by

\[ J_0 = \bigoplus_{j=1}^{\psi} (I_{k_j}, \omega_j + N_{k_j}) = \bigoplus_{j=1}^{\psi} N_{k_j} \]

the diagonal block of \( J_A \) corresponding to the zero eigenvalue. Then by [2, pp. 234-239], we have the following lemma.

**Lemma 3.1.** The matrix \( A \in M_n \) has an \( m \)th root if and only if \( J_0 \) has an \( m \)th root.

The ascent sequence of \( A \) is said to be the sequence

\[ d_i = \dim \text{Null } A^i - \dim \text{Null } A^{i-1} ; \quad i = 1, 2, \ldots \]

By [1], we have the following properties:

**P1** The ascent sequences of \( A \) and \( J_0 \) are equal.

**P2** For every \( i = 1, 2, \ldots, d_i \) is the number of the diagonal blocks of \( J_0 \) of order at least \( i \). Thus, if \( d_0 = \sum_{j=1}^{\psi} k_j \) is the order of \( J_0 \), then \( d_0 \geq d_1 \geq d_2 \geq \cdots \geq d_{k_1} \geq 0 \) and \( d_{k_1+1} = d_{k_1+2} = \cdots = 0 \).

**Theorem 3.2.** The complex matrix \( A \in M_n \) has an \( m \)th root if and only if for every integer \( \nu \geq 0 \), the ascent sequence of \( A \) has no more than one element between \( m\nu \) and \( m(\nu+1) \).

(\( \text{Note that the result is obvious when the matrix } A \text{ is nonsingular.} \))

**Proof.** By Lemma 3.1 and Property (P1), it is enough to prove the result for \( J_0 \). First assume that \( J_0 \) has an \( m \)th root \( Z \), and that there exist a nonnegative integer \( \nu \) and two terms of the ascent sequence of \( J_0 \), say \( d_t \) and \( d_{t+1} \), such that

\[ m\nu < d_{t+1} \leq d_t < m(\nu+1) \].
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For every $i = 1, 2, \ldots, k_1$, $\text{Null} Z^{m_i} = \text{Null} J_0^{m_i}$, and consequently, if $c_1, c_2, \ldots$ is the ascent sequence of $Z$, then

$$\sum_{j=1}^{m_i} c_j = \sum_{j=1}^{i} d_j.$$  

Thus, we have

$$d_t = c_{mt} + c_{mt-1} + \cdots + c_{m(t-1)}$$

and

$$d_{t+1} = c_{m(t+1)} + c_{m(t+1)-1} + \cdots + c_{mt+1},$$

where

$$c_{mt-(m-1)} \geq \cdots \geq c_{mt} \geq c_{mt+1} \geq \cdots \geq c_{m(t+1)}.$$  

If $c_{mt} \geq \nu + 1$, then $d_t \geq m c_{mt} \geq m (\nu + 1)$, a contradiction. On the other hand, if $c_{mt} \leq \nu$, then $d_{t+1} \leq m c_{mt} \leq m \nu$, which is also a contradiction. Hence, we conclude that if the matrix $A$ has an $m$th root, then for every integer $\nu \geq 0$, the ascent sequence of $A$ has no more than one element between $m \nu$ and $m (\nu + 1)$.

Conversely, we indicate a constructive proof for the existence of an $m$th root of $J_0$ given that between two successive nonnegative multiplies of $m$ there is at most one term of the ascent sequence of $J_0$. Denote $n_i = \sum_{j=1}^{i} k_j$ for $i = 1, 2, \ldots, \psi$, and let $\{e_1, e_2, \ldots, e_{n_\psi}\}$ be the standard basis of $\mathbb{C}^{n_\psi}$. By [1], for the vectors $x_i = e_{n_i}$ ($i = 1, 2, \ldots, \psi$), we can write the standard basis of $\mathbb{C}^{n_\psi}$ in the following scheme:

$$
\begin{align*}
J_0^{k_1-1} x_1 & \quad J_0^{k_2-2} x_1 & \quad \cdots & \quad J_0^{k_1-3} x_1 & \quad \cdots & \quad J_0 x_1 & \quad x_1 \\
J_0^{k_2-1} x_2 & \quad J_0^{k_3-2} x_2 & \quad \cdots & \quad J_0^{k_2-3} x_2 & \quad \cdots & \quad x_2 \\
J_0^{k_3-1} x_3 & \quad J_0^{k_4-2} x_3 & \quad \cdots & \quad \cdots & \quad x_3 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
J_0^{k_\psi-1} x_\psi & \quad J_0^{k_\psi-2} x_\psi & \quad \cdots & \quad x_\psi 
\end{align*}
$$

(3.1)

In this scheme, there are $\psi$ rows of vectors, such that the $j$th row contains $k_j$ vectors ($j = 1, 2, \ldots, \psi$). Recall that $k_1 \geq k_2 \geq \cdots \geq k_\psi$, and hence the rows are of non-increasing length. Moreover, the above scheme has $k_1$ columns and by Property (P2), for $t = 1, 2, \ldots, k_1$, the length of the $t$th column is equal to the $t$th term of the ascent sequence of $J_0$.

With respect to the above scheme (and the order of its elements), we define a linear transformation $\mathcal{F}$ on $\mathbb{C}^{n_\psi}$ by

$$
(j-1, t)\text{th element} \quad \quad \quad \text{if } j \neq 1 \text{ (mod } m),
$$

$$
(j, t)\text{th element} \quad \rightarrow \quad (j + m - 1, t - 1)\text{th element} \quad \text{if } j = 1 \text{ (mod } m) \text{ and } t \neq 1,
$$

$$
0 \quad \quad \quad \text{if } j = 1 \text{ (mod } m) \text{ and } t = 1.
$$
Separating the rows of the scheme in \( m \)-tuples, one can see that our assumption that for every nonnegative integer \( \nu \), the ascent sequence has no more than one element between \( m\nu \) and \( m(\nu + 1) \) ensures the existence of \( F \). If \( B \) is the \( n_\psi \times n_\psi \) matrix whose \( j \)th column is \( F(e_j) \), then \( B^m = J_0 \) and the proof is complete.

Since between two successive even integers there is exactly one odd integer, for \( m = 2 \), Theorem 3.2 yields the main result of [1].

**Corollary 3.3.** (Theorem 2 in [1]) The matrix \( A \in \mathcal{M}_n \) has a square root if and only if no two terms of its ascent sequence are the same odd integer.

The ascent sequence of the \( k \times k \) nilpotent matrix \( N_k \) is \( 1, 1, \ldots, 1, 0, \ldots \) with its first \( k \) terms equal to 1. Thus, for every integer \( m > 1 \), it has \( k \) terms (i.e., the \( k \) ones) between 0 and \( m \). Hence, it is verified that there is no matrix \( M \in \mathcal{M}_n \) such that \( M^m = N_k \) (see also [2]).

**Corollary 3.4.** Let \( d_1, d_2, d_3, \ldots \) be the ascent sequence of a singular complex matrix \( A \in \mathcal{M}_n \).

(i) If \( d_2 = 0 \) (i.e., \( J_0 = 0 \)), then for every integer \( m > 1 \), \( A \) has an \( m \)th root.

(ii) If \( d_2 > 0 \), then for every integer \( m > d_1 \), \( A \) has no \( m \)th roots.

Our methodology is illustrated in the following example.

**Example 3.5.** Consider the Jordan matrix

\[
J_0 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \oplus \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix} \oplus \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

and let \( m = 3 \). The ascent sequence of \( J_0 \) is \( 3, 3, 1, 0, \ldots \), and if \( \{e_1, e_2, \ldots, e_7\} \) is the standard basis of \( \mathbb{C}^7 \), then the scheme in (3.1) is

\[
\begin{align*}
J_0^2 x_1 &= e_1 & J_0 x_1 &= e_2 & x_1 &= e_3 \\
J_0 x_2 &= e_4 & x_2 &= e_5 \\
J_0^2 x_3 &= e_6 & x_3 &= e_7.
\end{align*}
\]

As in the proof of Theorem 3.2, we define the linear transformation \( F \) on \( \mathbb{C}^7 \) by

\[
F(e_1) = 0, \quad F(e_2) = e_6, \quad F(e_3) = e_7, \quad F(e_4) = e_1, \quad F(e_5) = e_2, \quad F(e_6) = e_4 \quad \text{and} \quad F(e_7) = e_5.
\]

One can see that the \( 7 \times 7 \) matrix

\[
B = \begin{bmatrix}
0 & e_6 & e_7 & e_1 & e_2 & e_4 & e_5
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

is a 3rd root of \( J_0 \). Finally, observe that for every integer \( m > 1 \) different than 3, the matrix \( J_0 \) has no \( m \)th roots.
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REFERENCES