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A SIMPLE PROOF OF THE CLASSIFICATION OF NORMAL TOEPLITZ MATRICES*

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Abstract. An easy proof to show that every complex normal Toeplitz matrix is classified as either of type I or of type II is given. Instead of difference equations on elements in the matrix used in past studies, polynomial equations with coefficients of elements are used. In a similar fashion, it is shown that a real normal Toeplitz matrix must be one of four types: symmetric, skew-symmetric, circulant, or skew-circulant. Here trigonometric polynomials in the complex case and algebraic polynomials in the real case are used.

Key words. Normal matrices, Toeplitz matrices.

AMS subject classifications. 15A57, 47B15, 47B35

1. Introduction: Normal Toeplitz Matrices. Ikramov [3] first showed that a normal Toeplitz matrix over real field must be one of four types: symmetric, skew-symmetric (up to the principal diagonal), circulant, or skew-circulant. Later on, the complex version of the problem was solved by Ikramov et al. [6] and Gel'fgat [2]. The history of the theory was described in [5]. The complex case was also proved by Farenick, Krupnik, Krupnik and Lee [1], and Ito [2], independently of [6]. We give here another proof which, we believe, is the simplest one. The complex case and the real case will be proved in a similar fashion, although trigonometric polynomials will be used in the complex case and algebraic polynomials will be used in the real case. Now denote Toeplitz matrix of order $N + 1$ by T_N :

$$T_N = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-N} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_{-1} \\ a_N & a_{N-1} & a_{N-2} & \cdots & a_0 \end{bmatrix}.$$

The matrix T_N is called normal Toeplitz if $T_N T_N^* - T_N^* T_N = 0$, where T_N^* is the transposed conjugate of T_N . We can simply state classifications of normal Toeplitz matrices in the following way. A normal Toeplitz matrix T_N is of type I if and only if for some α_0 , with $|\alpha_0| = 1$ and ,

$$(1) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = \alpha_0 [\bar{a}_1 \ \bar{a}_2 \ \cdots \ \bar{a}_N],$$

whereas T_N is of type II if and only if for some β_0 , with $|\beta_0| = 1$,

$$(2) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = \beta_0 [a_N \ a_{N-1} \ \cdots \ a_1].$$

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Surely the complex case contains the real case, but when all the a'_k 's are real valued, it will be found that α_0 and β_0 must be either $+1$ or -1 , hence we may have the further concrete classification of four types: A normal Toeplitz matrix T_N is symmetric if and only if

$$(3) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = [a_1 \ a_2 \ \cdots \ a_N],$$

T_N is skew-symmetric if and only if

$$(4) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = - [a_1 \ a_2 \ \cdots \ a_N],$$

T_N is circulant if and only if

$$(5) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = [a_N \ a_{N-1} \ \cdots \ a_1],$$

T_N is skew-circulant if and only if

$$(6) \quad [a_{-1} \ a_{-2} \ \cdots \ a_{-N}] = - [a_N \ a_{N-1} \ \cdots \ a_1].$$

Throughout this paper, we assume a_0 to be zero without loss of generality since a_0 does not play any role in the above classifications.

2. Proof of Theorem in the Complex Case. Now we define trigonometric polynomials $s(x)$ and $t(x)$ with coefficients a_1, a_2, \dots, a_N and $a_{-1}, a_{-2}, \dots, a_{-N}$, respectively:

$$s(x) = a_1 e^{ix} + a_2 e^{2ix} + \cdots + a_N e^{Nix}$$

$$t(x) = a_{-1} e^{-ix} + a_{-2} e^{-2ix} + \cdots + a_{-N} e^{-Nix}.$$

Using these trigonometric polynomials, we can restate conditions (1) and (2) in the previous section as

(1') type I: $t(x) = \overline{\alpha_0 s(x)}$, with $|\alpha_0| = 1$,

(2') type II: $t(x) = \beta_0 s(x) e^{-i(N+1)x}$, with $|\beta_0| = 1$.

THEOREM 2.1. *Every normal Toeplitz matrix is either of type I or of type II.*

Proof. Our proof is based on the expression in [7, p. 998] or [1, p. 1038]. Necessary and sufficient conditions for T_N to be normal in terms of a_n are that

$$(7) \quad a_m \bar{a}_n - \bar{a}_{-m} a_{-n} + \bar{a}_{N+1-m} a_{N+1-n} - a_{-(N+1-m)} \bar{a}_{-(N+1-n)} = 0,$$

$(1 \leq n, m \leq N)$.

We now rewrite (7) by using polynomials $s(x)$ and $t(x)$. Multiply both sides of (7) by $e^{imx} e^{-iny}$ and sum these new expressions over all m and n to obtain

$$(8) \quad s(x) \overline{s(y)} - \overline{t(x)} t(y) + \overline{s(x)} s(y) e^{i(N+1)(x-y)} - t(x) \overline{t(y)} e^{i(N+1)(x-y)} = 0.$$

It easily seen that (8) is equivalent to (7), and that (1') or (2') implies (8). Hence both (1') and (2') are sufficient conditions for T_N to be normal. Thus it only remains to show the necessity of the condition. When we take $x = y$ in (8), we have

$$(9) \quad s(x) \overline{s(x)} - t(x) \overline{t(x)} = 0, \text{ or } |s(x)| = |t(x)|$$

for each x . Hence except for trivial case $t(x) \equiv 0$, we can find an x_0 such that $t(x_0) \neq 0$ and there exist α and β such that $s(x_0) = \alpha t(x_0)$ and $\overline{t(x_0)} = \beta \overline{t(x_0)}$, with $|\alpha| = |\beta| = 1$. By setting $y = x_0$ and then by dividing through by $t(x_0)$, equation (8) becomes

$$s(x) \overline{\alpha} - \overline{t(x)} \beta + \overline{s(x)} \alpha \beta e^{i(N+1)(x-x_0)} - t(x) e^{i(N+1)(x-x_0)} = 0,$$

from which we have

$$(10) \quad \overline{t(x)} = s(x) \overline{\alpha} \beta + \overline{s(x)} \alpha e^{i(N+1)(x-x_0)} - t(x) \overline{\beta} e^{i(N+1)(x-x_0)}.$$

Substituting the right hand side of (10) into $\overline{t(x)}$ of (9), we have

$$(11) \quad \left\{ s(x) - t(x) \alpha e^{i(N+1)(x-x_0)} \right\} \left\{ \overline{s(x)} - t(x) \overline{\alpha} \beta \right\} = 0,$$

which implies our required results if we take $\alpha_0 = \alpha\beta$ and $\beta_0 = \overline{\alpha} e^{i(N+1)x_0}$. We should notice here that (11) is the product of polynomials. If one of the polynomials in (11) is nonzero at $x = x_0$, then by continuity the polynomial is nonzero in an open neighborhood \mathcal{U} of x_0 , which implies that the other polynomial in the product (11) is identically zero on \mathcal{U} . Because the only polynomial with a continuum of roots is the zero polynomial, equation (11) implies equation (1') or (2'). Hence type I or type II must occur. \square

3. Proof of Theorem in the Real Case.

THEOREM 3.1. *Every real normal Toeplitz matrix is either symmetric, skew-symmetric, circulant or skew-circulant.*

Proof. We define algebraic polynomials $p(x)$ and $q(x)$ such that

$$\begin{aligned} p(x) &= a_1x + a_2x^2 + \cdots + a_Nx^N \\ q(x) &= a_{-1}x + a_{-2}x^2 + \cdots + a_{-N}x^N. \end{aligned}$$

Also in this real case we can apply (7) in the previous section as necessary and sufficient conditions for T_N to be normal in terms of real valued a_n :

$$(12) \quad a_m a_n - a_{-m} a_{-n} + a_{N+1-m} a_{N+1-n} - a_{-(N+1-m)} a_{-(N+1-n)} = 0 \quad (1 \leq n, m \leq N)$$

Now we introduce the reciprocal polynomials to $p(x)$ and $q(x)$

$$\tilde{p}(x) = a_Nx + a_{N-1}x^2 + \cdots + a_1x^N$$

and

$$\tilde{q}(x) = a_{-N}x + a_{-N+1}x^2 + \cdots + a_{-1}x^N,$$

respectively. Multiply both sides of equation (12) by $x^m y^n$ and then sum the resulting expressions over all m and n to obtain the following analog of (8):

$$(13) \quad p(x) p(y) - q(x) q(y) + \tilde{p}(x) \tilde{p}(y) - \tilde{q}(x) \tilde{q}(y) = 0.$$

Noticing the relation $\tilde{q}(x) = x^{N+1}q(1/x)$ and $\tilde{p}(x) = x^{N+1}p(1/x)$, and letting $y = 1/x$ in (13), we have $p(x)\tilde{p}(x) = q(x)\tilde{q}(x)$. By setting $x = y$ and then multiplying both sides of (13) by $p^2(x)$, we have from the relation $p^2(x)\tilde{p}^2(x) = q^2(x)\tilde{q}^2(x)$,

$$p^4(x) - p^2(x)q^2(x) + q^2(x)\tilde{q}^2(x) - p^2(x)\tilde{q}^2(x) = 0,$$

which can be factored as the following way:

$$(p^2(x) - q^2(x))(p^2(x) - \tilde{q}^2(x)) = 0.$$

From this formula, we obtain four types, symmetric $p(x) = q(x)$, skew-symmetric $p(x) = -q(x)$, circulant $p(x) = \tilde{q}(x)$ and skew-circulant $p(x) = -\tilde{q}(x)$, which are equivalent to (3), (4), (5), and (6), respectively.

REFERENCES

- [1] D.R. Farenick, M. Krupnik, N. Krupnik, and W.Y. Lee. Normal Toeplitz matrices. *SIAM J. Matrix Anal. Appl.*, 17:1037–1043, 1996.
- [2] V.I. Gel'fgat. A normality criterion for Toeplitz matrices. *Zh. Vychisl. Mat. Math. Fiz.*, 35:1428–1432, 1995 (in Russian). *Comput. Math. Math. Phys.*, 35:1147–1150, 1995.
- [3] Kh.D. Ikramov. Describing normal Toeplitz matrices. *Zh. Vychisl. Mat. Math. Fiz.*, 34:473–479, 1994 (in Russian). *Comput. Math. Math. Phys.*, 34:399–404, 1994.
- [4] Kh.D. Ikramov. Classification of normal Toeplitz matrices with real entries. *Mat. Zametki*, 57:1428–1432, 1995 (in Russian). *Math. Notes*, 57:463–469, 1995.
- [5] Kh.D. Ikramov. Comments on Normal Toeplitz Matrices by Farenick et al. *SIAM J. Matrix Anal. Appl.*, 18:218, 1997.
- [6] Kh.D. Ikramov and V.N. Chugunov. Normality condition for a complex Toeplitz matrix. *Zh. Vychisl. Mat. Math. Fiz.*, 36:3–10, 1996 (in Russian). *Comput. Math. Math. Phys.*, 36:131–137, 1996.
- [7] T. Ito. Every normal Toeplitz matrix is either of type I or of type II. *SIAM J. Matrix Anal. Appl.*, 17:998–1106, 1996.