Subdirect sums of nonsingular M-matrices and of their inverses

Rafael Bru
Francisco Pedroche
Daniel B. Szyld
szyld@euclid.math.temple.edu

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1159
SUBDIRECT SUMS OF NONSINGULAR $M$-MATRICES AND OF THEIR INVERSES

RAFAEL BRU†, FRANCISCO PEDROCHE†, AND DANIEL B. SZYLD‡

Abstract. The question of when the subdirect sum of two nonsingular $M$-matrices is a nonsingular $M$-matrix is studied. Sufficient conditions are given. The case of inverses of $M$-matrices is also studied. In particular, it is shown that the subdirect sum of overlapping principal submatrices of a nonsingular $M$-matrix is a nonsingular $M$-matrix. Some examples illustrating the conditions presented are also given.

AMS subject classifications. 15A48.

Key words. Subdirect sum, $M$-matrices, Inverse of $M$-matrix, Overlapping blocks.

1. Introduction. Subdirect sum of matrices are generalizations of the usual sum of matrices (a $k$-subdirect sum is formally defined below in Section 2). They were introduced by Fallat and Johnson in [3], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric $M$-matrices, are positive definite or symmetric $M$-matrices, respectively. They also showed that this is not the case for $M$-matrices; the sum of two $M$-matrices may not be an $M$-matrix. One goal of the present paper is to give sufficient conditions so that the subdirect sum of nonsingular $M$-matrices is a nonsingular $M$-matrix. We also treat the case of the subdirect sum of inverses of $M$-matrices.

Subdirect sums of two overlapping principal submatrices of a nonsingular $M$-matrix appear naturally when analyzing additive Schwarz methods for Markov chains or other matrices [2], [4]. In this paper we show that the subdirect sum of two overlapping principal submatrices of a nonsingular $M$-matrix is a nonsingular $M$-matrix.

The paper is structured as follows. In Section 2 we focus on the nonsingularity of the subdirect sum of any pair of nonsingular matrices, giving an explicit expression for the inverse. In Section 2.1 we study the $k$-subdirect sum of two nonsingular $M$-matrices and in particular, the case of subdirect sums of overlapping blocks of nonsingular $M$-matrices. In Section 2.3 we extend some results to the subdirect sum of more than two nonsingular $M$-matrices. In Section 3 we analyze the subdirect sum of two inverses. Finally, in Section 4 we mention some open questions on subdirect sums of $P$-matrices. Throughout the paper we give examples which help illustrate the theoretical results.

†Received by the editors 16 February 2005. Accepted for publication 22 June 2005. Handling Editor: Michael Neumann.
‡Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València. C. de Vera s/n. 46022 València, Spain (rbu@mat.upv.es, pedroche@mat.upv.es). Supported by Spanish DGI and FEDER grant MTM2001-02998 and by the Oficina de Ciència i Tecnologia de la Presidència de la Generalitat Valenciana under project GRUPO03/002.
§Department of Mathematics, Temple University, Philadelphia, PA 19122-6094, U.S.A. (szyl@math.temple.edu). Supported in part by the U.S. National Science Foundation under grant DMS-0207825.
2. **Subdirect sums of nonsingular matrices.** Let \( A \) and \( B \) be two square matrices of order \( n_1 \) and \( n_2 \), respectively, and let \( k \) be an integer such that \( 1 \leq k \leq \min(n_1, n_2) \). Let \( A \) and \( B \) be partitioned into \( 2 \times 2 \) blocks as follows:

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}, \quad (2.1)
\]

where \( A_{22} \) and \( B_{11} \) are square matrices of order \( k \). Following [3], we call the following square matrix of order \( n = n_1 + n_2 - k \),

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} + B_{11} & B_{12} \\
0 & B_{21} & B_{22}
\end{bmatrix} \quad (2.2)
\]

the \( k \)-subdirect sum of \( A \) and \( B \) and denote it by \( C = A \oplus_k B \).

We are interested in the case when \( A \) and \( B \) are nonsingular matrices. We partition the inverses of \( A \) and \( B \) conformably to (2.1) and denote its blocks as follows:

\[
A^{-1} = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
\hat{B}_{11} & \hat{B}_{12} \\
\hat{B}_{21} & \hat{B}_{22}
\end{bmatrix}, \quad (2.3)
\]

where, as before, \( \hat{A}_{22} \) and \( \hat{B}_{11} \) are square of order \( k \).

In the following result we show that nonsingularity of matrix \( \hat{A}_{22} + \hat{B}_{11} \) is a necessary and sufficient condition for the \( k \)-subdirect sum \( C \) to be nonsingular. The proof is based on the use of the relation \( n = n_1 + n_2 - k \) to properly partition the indicated matrices.

**Theorem 2.1.** Let \( A \) and \( B \) be nonsingular matrices of order \( n_1 \) and \( n_2 \), respectively, and let \( k \) be an integer such that \( 1 \leq k \leq \min(n_1, n_2) \). Let \( A \) and \( B \) be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let \( C = A \oplus_k B \). Then \( C \) is nonsingular if and only if \( \hat{H} = \hat{A}_{22} + \hat{B}_{11} \) is nonsingular.

**Proof.** Let \( I_m \) be the identity matrix of order \( m \). The theorem follows from the following relation:

\[
\begin{bmatrix}
A^{-1} & O \\
O & I_{n-n_2}
\end{bmatrix}
\begin{bmatrix}
I_{n-n_2} & O \\
O & B^{-1}
\end{bmatrix}
= \begin{bmatrix}
I_{n-n_2} & \hat{A}_{12} & O \\
O & \hat{H} & \hat{B}_{12} \\
O & O & I_{n-n_1}
\end{bmatrix}. \quad (2.4)
\]

### 2.1. Nonsingular M-matrices.

Given \( A = \{a_{ij}\} \in \mathbb{R}^{m \times n} \), we write \( A > O \) (\( A \geq O \)) to indicate \( a_{ij} > 0 \) \( (a_{ij} \geq 0) \), for \( i = 1, \ldots, m, j = 1, \ldots, n \), and such matrices are called positive (nonnegative). Similarly, \( A \geq B \) when \( A - B \geq O \). Square matrices which have nonpositive off-diagonal entries are called Z-matrices. We call a Z-matrix \( M \) a nonsingular \( M \)-matrix if \( M^{-1} \geq O \). We recall some properties of these matrices; see [1], [8]:

(i) The diagonal of a nonsingular \( M \)-matrix is positive.

(ii) If \( B \) is a Z-matrix and \( M \) is a nonsingular \( M \)-matrix, and \( M \leq B \), then \( B \) is also a nonsingular \( M \)-matrix. In particular, any matrix obtained from a nonsingular \( M \)-matrix by setting certain off-diagonal entries to zero is also a nonsingular \( M \)-matrix.
(iii) A matrix $M$ is a nonsingular $M$-matrix if and only if each principal submatrix of $M$ is a nonsingular $M$-matrix.

(iv) A $Z$-matrix $M$ is a nonsingular $M$-matrix if and only if there exists a positive vector $x > 0$ such that $Mx > 0$.

We first consider the $k$-subdirect sum of nonsingular $Z$-matrices. From (2.4) we can explicitly write

$$C^{-1} = \begin{bmatrix} I_{n-n_2} & O \\ O & B^{-1} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & -\hat{A}_{12}\hat{H}^{-1} \hat{B}_{12} \\ O & \hat{H}^{-1} \end{bmatrix} \begin{bmatrix} \hat{A}_{12}\hat{H}^{-1} \hat{B}_{12} \\ \hat{H}^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix}$$

from which we obtain

$$C^{-1} = \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{21} & \hat{A}_{12} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{22} & \hat{A}_{12}\hat{H}^{-1}\hat{B}_{12} \\ \hat{B}_{11}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{11}\hat{H}^{-1}\hat{A}_{22} & \hat{B}_{11}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{12} \\ \hat{B}_{21}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{21}\hat{H}^{-1}\hat{A}_{22} & \hat{B}_{21}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{22} \end{bmatrix}$$

(2.5)

and therefore we can state the following immediate result.

**Theorem 2.2.** Let $A$ and $B$ be nonsingular $Z$-matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let $C = A \oplus_k B$. Let $H = A_{22} + B_{11}$ be nonsingular. Then $C$ is a nonsingular $M$-matrix if and only if each of the nine blocks of $C^{-1}$ in (2.5) is nonnegative.

We consider now the case where $A$ and $B$ are nonsingular $M$-matrices. It was shown in [3] that even if $H = A_{22} + B_{11}$ is a nonsingular $M$-matrix, this does not guarantee that $C = A \oplus_k B$ is a nonsingular $M$-matrix. We point out that this matrix $H$ is not the matrix $\hat{H}$ obtained from $A^{-1}$ and $B^{-1}$ and used in Theorem 2.1. The fact that $H$ is a nonsingular $M$-matrix is a necessary but not a sufficient condition for $C$ to be a nonsingular $M$-matrix. Sufficient conditions are presented in the following result.

**Theorem 2.3.** Let $A$ and $B$ be nonsingular $M$-matrices partitioned as in (2.1). Let $x_1 > 0 \in \mathbb{R}^{(n_1-k)\times 1}$, $y_1 > 0 \in \mathbb{R}^{k\times 1}$, $x_2 > 0 \in \mathbb{R}^{k\times 1}$ and $y_2 > 0 \in \mathbb{R}^{(n_2-k)\times 1}$ be such that

$$A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} > 0, \quad B \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} > 0. \quad (2.6)$$

Let $H = A_{22} + B_{11}$ be a nonsingular $M$-matrix and let

$$y = H^{-1}(A_{22}y_1 + B_{11}x_2). \quad (2.7)$$

Then if $y \leq y_1$ and $y \leq x_2$ the $k$-subdirect sum $C = A \oplus_k B$ is a nonsingular $M$-matrix.

**Proof.** We will show that there exists $u > 0$ such that $Cu > 0$. We first note that from (2.6) we get

$$A_{11}x_1 + A_{12}y_1 > 0, \quad B_{11}x_2 + B_{12}y_2 > 0, \quad A_{21}x_1 + A_{22}y_1 > 0, \quad B_{21}x_2 + B_{22}y_2 > 0. \quad (2.8)$$
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses

Taking $u = \begin{bmatrix} x_1 \\ y \\ y_2 \end{bmatrix}$ and partitioning $C$ as in (2.2) we obtain

$$Cu = \begin{bmatrix} A_{11}x_1 + A_{12}y \\ A_{21}x_1 + (A_{22} + B_{11})y + B_{12}y_2 \\ B_{21}y + B_{22}y_2 \end{bmatrix}. \quad (2.9)$$

Since $A_{21} \leq O$ and $B_{12} \leq O$, from (2.8) it follows that $A_{22}y_1 > 0$ and $B_{11}x_2 > 0$. Since $H^{-1} \geq O$, from (2.7) we have that $y$ is positive, and consequently, so is $u$, i.e., $u > 0$. We will show that $Cu > 0$ one block of rows in (2.9) at a time. If $y \leq y_1$, as $A_{12} \leq 0$, we have that $A_{12}y \geq A_{12}y_1$ and again using (2.8) we obtain that the first block of rows of $Cu$ is positive. In a similar way, the condition $y \leq x_2$ together with the last equation of (2.8) allows to conclude that the third block of rows of $Cu$ is positive. Finally, substituting $y$ given by (2.7) in the second row of $Cu$ and considering (2.8) we conclude that the second block of rows of $Cu$ is also positive. ~\(\blacksquare\)

Note that $A$ and $B$ are nonsingular $M$-matrices and therefore the positive vectors $(x_1, y_1)$ and $(x_2, y_2)$ of (2.6) always exist. This theorem gives sufficient but not necessary conditions for $C = A \oplus_k B$ to be a nonsingular $M$-matrix, as illustrated in Example 2.5 further below.

**Example 2.4.** The matrices

$$A = \begin{bmatrix} 3 & -2 & -1 \\ -1/2 & 2 & -3 \\ -1 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & -1/3 \\ -3 & 9 & 0 \\ -2 & -1/2 & 0 \end{bmatrix},$$

and the vectors

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}$$

satisfy the inequalities (2.6), and computing the vector $y$ from (2.7) we get $y \approx (1.95, 0.87)^T$, which satisfy $y \leq y_1$ and $y \leq x_2$. Therefore the 2-subdirect sum

$$C = \begin{bmatrix} 3 & -2 & -1 & 0 \\ -1/2 & 3 & -5 & -1/3 \\ -1 & -4 & 13 & 0 \\ 0 & -2 & -1/2 & 6 \end{bmatrix}$$

is a nonsingular $M$-matrix in accordance with Theorem 2.3.

**Example 2.5.** The matrices

$$A = \begin{bmatrix} 5 & -1/2 & -1/3 \\ -1 & 4 & -2 \\ -1 & -6 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & -1/3 \\ -3 & 9 & 0 \\ -2 & -1/2 & 0 \end{bmatrix},$$

and the vectors

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}$$
satisfy the inequalities (2.6), but computing vector \( y \) from (2.7) we obtain

\[
y \approx (1.18, 0.85)^T,
\]

which does not satisfy the conditions of Theorem 2.3. Nevertheless the 2-subdirect sum

\[
C = A \oplus_2 B = \begin{bmatrix}
5 & -1/2 & -1/3 & 0 \\
-1 & 5 & -4 & -1/3 \\
-1 & -9 & 19 & 0 \\
0 & -2 & -1/2 & 6
\end{bmatrix}
\]

is a nonsingular \( M \)-matrix.

In the special case of \( A \) and \( B \) block lower and upper triangular nonsingular \( M \)-matrices, respectively, the results of Theorems 2.2 and 2.3 are easy to establish. Let

\[
A = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix},
\]

with \( A_{22} \) and \( B_{11} \) square matrices of order \( k \).

**Theorem 2.6.** Let \( A \) and \( B \) be nonsingular lower and upper block triangular nonsingular \( M \)-matrices, respectively, partitioned as in (2.10). Then \( C = A \oplus_k B \) is a nonsingular \( M \)-matrix.

**Proof.** We can repeat the same argument as in the proof of Theorem 2.3 with the advantage of having \( A_{12} = O \) and \( B_{21} = O \). Note that conditions \( y \leq y_1 \) and \( y \leq x_2 \) are not necessary here because the first and last block of rows of \( C \) in (2.9) are automatically positive in this case. \( \square \)

**Remark 2.7.** The expression of \( C^{-1} \) is given by (2.5). In this particular case of block triangular matrices we have \( A_{12} = O, \ B_{21} = O, \ A_{22} = A_{22}^{-1}, \ B_{11} = B_{11}^{-1} \), from which \( H = A_{22}^{-1} + B_{11}^{-1} \). If, in addition, \( A_{22} = B_{11} \), then we obtain

\[
C^{-1} = \begin{bmatrix}
-rac{1}{2}A_{11}^{-1}A_{22}^{-1} & O & 0 & O \\
0 & -rac{1}{2}A_{22}^{-1}B_{11}^{-1} & -rac{1}{2}A_{22}^{-1}B_{22}^{-1} & 0
\end{bmatrix} \geq O.
\]

**Example 2.8.** The matrices

\[
A = \begin{bmatrix}
3 & 0 & 0 \\
-1 & 5 & -1 \\
-1 & -9 & 5
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
6 & -2 & -1 \\
-4 & 3 & -3 \\
0 & 0 & 2
\end{bmatrix}
\]

satisfy the hypotheses of Theorem 2.6. The matrices \( C = A \oplus_2 B \) and \( C^{-1} \) are

\[
C = \begin{bmatrix}
3 & 0 & 0 & 0 \\
-1 & 11 & -3 & -1 \\
-1 & -13 & 8 & -3 \\
0 & 0 & 0 & 2
\end{bmatrix}, \quad C^{-1} = \begin{bmatrix}
1/3 & 0 & 0 & 0 \\
11/147 & 8/49 & 3/49 & 17/98 \\
8/49 & 13/49 & 11/49 & 23/49 \\
0 & 0 & 0 & 1/2
\end{bmatrix}
\]

and therefore \( C \) is a nonsingular \( M \)-matrix as expected.
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses

In some applications, such as in domain decomposition \[6], \[7\], matrices $A$ and $B$ partitioned as in (2.1) arise with a common block, i.e., $A_{22} = B_{11}$. In the next example we show that even if $A$ and $B$ are nonsingular $M$-matrices, and so is the common block, we cannot ensure that $C = A \oplus_k B$ is a nonsingular $M$-matrix.

**Example 2.9.** The matrices

$$A = \begin{bmatrix} 370 & -342 & -318 \\ -448 & 737 & -107 \\ -46 & -190 & 444 \end{bmatrix}, \quad B = \begin{bmatrix} 737 & -107 & -134 \\ -190 & 444 & -440 \\ -885 & -182 & 603 \end{bmatrix}$$

are nonsingular $M$-matrices with $A_{22} = B_{11}$ an $M$-matrix, but $C = A \oplus_2 B$ is not an $M$-matrix, since we have

$$C = \begin{bmatrix} 370 & -342 & -318 & 0 \\ -448 & 1474 & -214 & -134 \\ -46 & -380 & 888 & -440 \\ 0 & -885 & -182 & 603 \end{bmatrix}$$

and $C^{-1} \approx \begin{bmatrix} -0.0291 & -0.0242 & -0.0204 & -0.0203 \\ -0.0145 & -0.0109 & -0.0098 & -0.0096 \\ -0.0214 & -0.0163 & -0.0132 & -0.0133 \\ -0.0277 & -0.0210 & -0.0183 & -0.0164 \end{bmatrix}$.

In the next section we shall see that when $A$ and $B$ share a block and they are submatrices of a given nonsingular $M$-matrix, the resulting $k$-subdirect sum is in fact a nonsingular $M$-matrix.

**2.2. Overlapping $M$-matrices.** In this section we restrict $A$ and $B$ to be principal submatrices of a given nonsingular $M$-matrix and such that they have a common block. Let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$  \hspace{1cm} (2.11)$$

be a nonsingular $M$-matrix with $M_{22}$ square matrix of order $k \geq 1$ and let

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$ \hspace{1cm} (2.12)$$

be of order $n_1$ and $n_2$, respectively. The $k$-subdirect sum of $A$ and $B$ is thus given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ 2M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}.$$  \hspace{1cm} (2.13)$$

In the following theorem we show that $C$ is a nonsingular $M$-matrix.
Theorem 2.10. Let $M$ be a nonsingular $M$-matrix partitioned as in (2.11), and let $A$ and $B$ be two overlapping principal submatrices given by (2.12). Then the $k$-subdirect sum $C = A \oplus_k B$ is a nonsingular $M$-matrix.

Proof. Let us construct an $n \times n$ $Z$-matrix $T$ as follows:

$$T = \begin{bmatrix} M_{11} & 2M_{12} & M_{13} \\ M_{21} & 2M_{22} & M_{23} \\ M_{31} & 2M_{32} & M_{33} \end{bmatrix}.$$  \hspace{1cm} (2.14)

Then $T = M \text{diag}(I, 2I, I)$ and we get $T^{-1} = \text{diag}(I, (1/2)I, I)M^{-1} \geq O$. Then $T$ is a nonsingular $M$-matrix. Finally since $C$ is a $Z$-matrix and $C \geq T$ we conclude that $C$ is a nonsingular $M$-matrix. \]

Example 2.11. The following nonsingular $M$-matrix is partitioned as in (2.11):


Taking overlapping submatrices $A$ and $B$ as in (2.12) the 3-subdirect sum $C = A \oplus_3 B$ is given by


and it is a nonsingular $M$-matrix according to Theorem 2.10. In fact, we have that

$$C^{-1} \approx \begin{bmatrix} 1.3500 & 0.3977 & 0.2624 & 0.1609 & 0.2103 & 0.1232 \\ 0.7628 & 1.4108 & 0.3383 & 0.2085 & 0.2185 & 0.1478 \\ 0.3007 & 0.2845 & 0.7422 & 0.2006 & 0.1824 & 0.1763 \\ 1.1024 & 1.1571 & 0.8927 & 1.6092 & 1.3118 & 0.8940 \\ 0.4854 & 0.5256 & 0.5116 & 0.4379 & 0.9664 & 0.4013 \\ 0.4543 & 0.4743 & 0.4564 & 0.5941 & 0.5634 & 1.4679 \end{bmatrix}.$$  

2.3. $k$-subdirect sum of $p$ $M$-matrices. In this section we extend Theorems 2.3 and 2.10 to the subdirect sum of several nonsingular $M$-matrices. Example 2.14 later in the section illustrates the notation used in the proofs.

Theorem 2.12. Let $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, p$, be nonsingular $M$-matrices partitioned as

$$A_i = \begin{bmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{bmatrix}.$$  \hspace{1cm} (2.16)
with \( A_{i,11} \) a square matrix of order \( k_{i-1} \geq 1 \) and \( A_{i,22} \) a square matrix of order \( k_i \geq 1 \), i.e., \( n_i = k_{i-1} + k_i \). Since \( A_i \) are nonsingular \( M \)-matrices we have that there exist \( x_i > 0 \in \mathbb{R}^{(n_i-k_i) \times 1} \) and \( y_i > 0 \in \mathbb{R}^{k_i \times 1} \) such that

\[
A_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} > 0, \quad i = 1, \ldots, p.
\] (2.17)

Let \( C_0 = A_1 \) and define the following \( p-1 \) \( k_i \)-subdirect sums:

\[
C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1,
\] (2.18)
i.e.,

\[
\begin{align*}
C_1 &= A_1 \oplus_{k_1} A_2, \\
C_2 &= (A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 = C_1 \oplus_{k_2} A_3, \\
&\vdots \\
C_{p-1} &= (A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_{p-2}} A_{p-1}) \oplus_{k_{p-1}} A_p = C_{p-2} \oplus_{k_{p-1}} A_p.
\end{align*}
\]

Each subdirect sum \( C_i \) is of order \( m_i \), such that \( m_0 = n_1 \) and

\[
m_i = m_{i-1} + n_{i+1} - k_i = m_{i-1} + k_{i+1}, \quad i = 1, \ldots, p-1.
\]

Let us partition \( C_i \) in the form

\[
C_i = \begin{bmatrix} C_{i,11} & C_{i,12} \\ C_{i,21} & C_{i,22} \end{bmatrix}, \quad i = 1, \ldots, p-1,
\] (2.19)

with \( C_{i,22} \) a square matrix of order \( k_{i+1} \). Let

\[
H_i = C_{i-1,22} + A_{i+1,11}, \quad i = 1, \ldots, p-1,
\]
be nonsingular \( M \)-matrices and let

\[
z_i = H_i^{-1}(C_{i-1,22}y_i + A_{i+1,11}x_{i+1}), \quad i = 1, \ldots, p-1.
\]

Then, if \( z_i \leq y_i \) and \( z_i \leq x_{i+1} \), the subdirect sums \( C_i \) given by (2.18) are nonsingular \( M \)-matrices for \( i = 1, \ldots, p-1 \).

**Proof.** It is easy to see that applying Theorem 2.3 to each consecutive pair of matrices \( C_i \) we have that \( C_1, C_2, \ldots, C_{p-1} \) are nonsingular \( M \)-matrices. This can be shown by induction. \( \square \)

We now extend Theorem 2.10 to the sub-direct sum of \( p \) submatrices of a given nonsingular \( M \)-matrix \( M \). To that end, we first define \( M(S) \) a principal submatrix of \( M \) with rows and columns with indices in the set of indices \( S = \{i, i+1, i+2, \ldots, j\} \). In [2] we call these consecutive principal submatrices. For example, matrices \( A \) and \( B \) given by (2.12) can be expressed as a submatrices of \( M \) given by (2.11) as \( A = M(S_1), B = M(S_2) \) with \( S_1 = \{1, 2\} \) and \( S_2 = \{2, 3\} \).
Theorem 2.13. Let $M$ be a nonsingular $M$-matrix. Let $A_i = M(S_i)$, $i = 1, \ldots, p$, be principal consecutive submatrices of $M$ and consider the $p - 1$ $k_i$-subdirect sums given by

$$C_i = C_{i-1} \oplus k_i A_{i+1}, \quad i = 1, \ldots, p - 1,$$

(2.20)

in which $C_0 = A_1$. Then each of the $k_i$-subdirect sums $C_i$ is a nonsingular $M$-matrix.

Proof. It is easy to relate the structure of each $C_i$ to that of the submatrices $A_i$ involved. We consider that $A_i$ are overlapping principal submatrices of the form (2.12) but allowing that each $A_i$ has different number of blocks. Let $M$ be partitioned as

$$M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\
M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\
M_{31} & M_{32} & M_{33} & \cdots & M_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{n1} & M_{n2} & M_{n3} & \cdots & M_{nn}
\end{bmatrix} \quad (2.21)$$

according with the size of the principal submatrices $A_i$. Each block $M_{ij}$ may be a submatrix of more than one $A_m$, $m = 1, \ldots, p$. Let $b_{ij}^{(l)} \geq 0$ be the number of matrices $A_m$ such that $M_{ij}$ is a submatrix of $A_m$, for $m = 1, \ldots, l + 1$. Of course we can have $b_{ij}^{(l)} = 0$. Let us consider the $l$th subdirect sum $C_l$, $1 \leq l \leq p - 1$, which is of the form

$$C_l = \begin{bmatrix}
b_{11}^{(l)} M_{11} & b_{12}^{(l)} M_{12} & b_{13}^{(l)} M_{13} & \cdots & b_{1n}^{(l)} M_{1n} \\
b_{21}^{(l)} M_{21} & b_{22}^{(l)} M_{22} & b_{23}^{(l)} M_{23} & \cdots & b_{2n}^{(l)} M_{2n} \\
b_{31}^{(l)} M_{31} & b_{32}^{(l)} M_{32} & b_{33}^{(l)} M_{33} & \cdots & b_{3n}^{(l)} M_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{n1}^{(l)} M_{n1} & b_{n2}^{(l)} M_{n2} & b_{n3}^{(l)} M_{n3} & \cdots & b_{nn}^{(l)} M_{nn}
\end{bmatrix}. \quad (2.22)$$

Observe that $C_l$ is a $Z$-matrix and that $b_{ij}^{(l)} > 0$. Furthermore, for each column it holds that $b_{ij}^{(l)} \geq b_{ji}^{(l)}$, $j = 1, \ldots, l$.

The proof proceeds in a manner similar to that of Theorem 2.10. Consider the $Z$-matrix (partitioned in the same manner as $M$)

$$T_l = M_l \text{diag}(b_{11}^{(l)} I, b_{22}^{(l)} I, b_{33}^{(l)} I, \ldots, b_{ll}^{(l)} I),$$

where $M_l$ is the principal submatrix of (2.21) with row and column blocks from 1 to $l$. It follows that $T_l^{-1} \geq O$ and therefore $T_l$ is a nonsingular $M$-matrix. Finally, since $C_l \geq T_l$, we conclude that $C_l$ is a nonsingular $M$-matrix, $l = 1, \ldots, p$. \[\square\]

Example 2.14. Given the nonsingular $M$-matrix $M$ of Example 2.11, let us consider the following overlapping blocks

$$A_1 = M(\{1, 2, 3\}) = \begin{bmatrix} 13/14 & -4/23 & -3/20 \\
-3/7 & 21/23 & -1/5 \\
-1/7 & -7/46 & 17/20 \end{bmatrix}.$$
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses


Then we have the 2-subdirect sum


which is a nonsingular $M$-matrix, and the 3-subdirect sum


which is also a nonsingular $M$-matrix in accordance with Theorem 2.13. Note that in this example we have $k_1 = 2$ and $k_2 = 3$. Now also that, for example, we have $b_{22}^{(1)} = 2, b_{33}^{(1)} = 2, b_{14}^{(1)} = 0, b_{22}^{(2)} = 2, b_{33}^{(2)} = 2, b_{14}^{(2)} = 3, b_{14}^{(3)} = 0$.

3. **Subdirect sums of inverses.** Let $A$ and $B$ be nonsingular matrices partitioned as in (2.1). In this section we consider the $k$-subdirect sum of their inverses. We will establish counterparts to some of results in the previous sections. Let us denote by $G = A^{-1} \oplus_k B^{-1}$, with $A^{-1}$ and $B^{-1}$ partitioned as in (2.3), i.e.,

\[ G = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 \\ \hat{A}_{21} & \hat{A}_{22} + \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}. \] (3.1)

As a corollary to, and in analogy to Theorem 2.1, the next statement indicates that the nonsingularity of $A_{22} + B_{11}$ is a necessary condition to obtain $G$ nonsingular.

**Theorem 3.1.** Let $A$ and $B$ be nonsingular matrices partitioned as in (2.1) and let their inverses be partitioned as in (2.3). Let $G = A^{-1} \oplus_k B^{-1}$ partitioned as in (3.1) with $k \geq 1$. Then $G$ is nonsingular if and only if $H = A_{22} + B_{11}$ is nonsingular.
We remark that in analogy to the expression (2.5) of $C^{-1}$, the explicit form of $G^{-1}$ is

$$G^{-1} = \begin{bmatrix}
  A_{11} - A_{12}H^{-1}A_{21} & A_{12} - A_{12}H^{-1}A_{22} & A_{12}H^{-1}B_{12} \\
  B_{11}H^{-1}A_{21} & B_{11}H^{-1}A_{22} - B_{11}H^{-1}B_{12} + B_{12} \\
  B_{21}H^{-1}A_{21} & B_{21}H^{-1}A_{22} - B_{21}H^{-1}B_{12} + B_{22}
\end{bmatrix}.$$  \hspace{1cm} (3.2)

**Corollary 3.2.** When $A$ and $B$ are nonsingular $M$-matrices with the common block $A_{22} = B_{11}$ a square matrix of order $k$, i.e., of the form

$$A = \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
  A_{22} & B_{12} \\
  B_{21} & B_{22}
\end{bmatrix},$$  \hspace{1cm} (3.3)

then $H = 2A_{22}$ is nonsingular and therefore $G = A^{-1} \oplus_k B^{-1}$ is nonsingular.

We note that this is the case when $A$ and $B$ are overlapping submatrices of an $M$-matrix, i.e., of the form (2.12) and (2.11) considered in Section 2.2, where we were interested in the subdirect sum of $A$ and $B$. Here we conclude that the subdirect sum of their inverses is always nonsingular.

**Example 3.3.** Let $A$ and $B$ be the matrices of Example 2.11, then according to Corollary 3.2, the 3-subdirect sum of the inverses

$$G = A^{-1} \oplus_3 B^{-1} \approx \begin{bmatrix}
  1.5033 & 0.5513 & 0.5547 & 0.2757 & 0.3912 & 0 \\
  0.9540 & 1.5996 & 0.7158 & 0.3635 & 0.4038 & 0 \\
  0.6004 & 0.5636 & 2.9750 & 0.8144 & 0.7407 & 0.3708 \\
  2.0383 & 2.1242 & 3.5729 & 6.5498 & 5.3372 & 2.0139 \\
  0.8953 & 0.9650 & 2.0470 & 1.8025 & 3.9062 & 0.9048 \\
  0 & 0 & 0.8551 & 1.3803 & 1.2652 & 1.9143
\end{bmatrix}$$

is a nonsingular matrix.

In the above example a direct computation shows that $G^{-1}$ is not an $M$-matrix:

$$G^{-1} \approx \begin{bmatrix}
  -0.8900 & -0.2337 & -0.0750 & -0.0119 & -0.0511 & 0.0512 \\
  -0.4682 & 0.8566 & -0.1000 & -0.0238 & -0.0054 & 0.0470 \\
  -0.0714 & -0.0761 & 0.4250 & -0.0357 & -0.0027 & -0.0435 \\
  -0.0952 & -0.1467 & -0.0333 & 0.2857 & -0.3118 & -0.1467 \\
  -0.0357 & 0.0978 & -0.1500 & -0.0714 & 0.4274 & -0.0978 \\
  0.1242 & 0.2045 & -0.0667 & -0.1429 & -0.0565 & 0.7123
\end{bmatrix}$$

which is not a $Z$-matrix. Note that when $A$ and $B$ are $M$-matrices we have from (3.1) that $G = A^{-1} \oplus B^{-1}$ is nonnegative. Therefore assuming that $G^{-1}$ exists we have $(G^{-1})^{-1} \geq 0$. Then it is a natural question to seek conditions so that $G^{-1}$ is a nonsingular $M$-matrix. We study this question next.

The expressions (3.1) of $G$ and (3.2) of $G^{-1}$, Theorem 3.1, and the observation that for nonsingular $M$-matrices we have $(G^{-1})^{-1} \geq 0$, imply the following result.

**Theorem 3.4.** Let $A$ and $B$ be nonsingular $M$-matrices partitioned as in (2.1) and their inverses partitioned as in (2.3). Let $G = A^{-1} \oplus_k B^{-1}$ with $k \geq 1$, and let $H = A_{22} + B_{11}$ be nonsingular. Then $G^{-1}$ is a nonsingular $M$-matrix if and only if $G^{-1}$ is a $Z$-matrix.
Corollary 3.5. Let $A$ and $B$ be lower and upper block triangular nonsingular $M$-matrices, respectively, partitioned as in (2.10) with $A_{22}$ and $B_{11}$ square matrices of order $k$ and $H = A_{22} + B_{11}$ nonsingular. Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular $M$-matrix if and only if the following conditions hold:

i) $B_{11}H^{-1}A_{21} \leq O$.

ii) $B_{11}H^{-1}A_{22}$ is a Z-matrix.

iii) $-B_{11}H^{-1}B_{12} + B_{12} \leq O$.

Proof. From (3.2) and (2.10) we have that

$$G^{-1} = \begin{bmatrix} A_{11} & 0 & 0 \\ B_{11}H^{-1}A_{21} & 0 & -B_{11}H^{-1}B_{12} + B_{12} \\ 0 & 0 & B_{22} \end{bmatrix} \quad (3.4)$$

and therefore $G^{-1}$ is a Z-matrix if and only if the conditions i), ii) and iii) hold. □

Conditions i), ii) and iii) in the corollary are not as stringent as they may appear. For example, let $A$ and $B$ be block triangular nonsingular $M$-matrices partitioned as in (2.10) with a common block $A_{22} = B_{11}$, a square matrix of order $k$, i.e.,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{22} & B_{12} \\ 0 & B_{22} \end{bmatrix}. \quad (3.5)$$

Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular $M$-matrix, since we have from (3.4) that

$$G^{-1} = \begin{bmatrix} A_{11} & O & O \\ \frac{1}{2}A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}B_{12} \\ O & O & B_{22} \end{bmatrix},$$

and therefore $G^{-1}$ is a Z-matrix. In fact, in this case, we have

$$G = \begin{bmatrix} A_{11}^{-1} & O & O \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & O & -A_{22}^{-1}B_{12}B_{22}^{-1} \\ O & O & B_{22}^{-1} \end{bmatrix} \geq O.$$

The next example illustrates this situation.

Example 3.6. Let $A$ and $B$ be the matrices of Example 2.8, then

$$G = A^{-1} \oplus_2 B^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 1/8 & 49/80 & 21/80 & 9/20 \\ 7/24 & 77/80 & 73/80 & 11/10 \\ 0 & 0 & 0 & 1/2 \end{bmatrix},$$

and

$$G^{-1} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -18/49 & 146/49 & -6/7 & -39/49 \\ -4/7 & -22/7 & 2 & -11/7 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$
is a nonsingular $M$-matrix in accordance with Corollary 3.5.

Note that if the hypotheses of Corollary 3.5 are satisfied, and recalling Theorem 2.6, we have that each of the matrices $C = A \oplus_k B$ and $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ are both nonsingular $M$-matrices.

4. $P$-matrices. A square matrix is a $P$-matrix if all its principal minors are positive. As a consequence we have that all the diagonal entries of a $P$-matrix are positive. It is also follows that a nonsingular $M$-matrix is a $P$-matrix. It can also be shown that if $A$ is a nonsingular $M$-matrix, then $A^{-1}$ is a $P$-matrix; see, e.g., [5].

In [3] it is shown that the $k$-subdirect sum (with $k > 1$) of two $P$-matrices is not necessarily a $P$-matrix. Our results in Sections 2.1 and 3 hold for nonsingular $M$-matrices and inverses of $M$-matrices, respectively. As these two classes of matrices are subsets of $P$-matrices, it is natural to ask if similar sufficient conditions can be found so that the $k$-subdirect sum of $P$-matrices is a $P$-matrix. The following example indicates that the answer may not be easy to obtain, since even in the simplest case of diagonal submatrices the $k$-subdirect sum may not be a $P$-matrix.

Example 4.1. Given the $P$-matrices

$$A = \begin{bmatrix} 543 & 388 & 322 \\ 69 & 160 & 0 \\ 368 & 0 & 375 \end{bmatrix}, \quad B = \begin{bmatrix} 136 & 0 & 219 \\ 0 & 225 & 159 \\ 61 & 177 & 230 \end{bmatrix}$$

we have that the 2-subdirect sum

$$C = A \oplus_2 B = \begin{bmatrix} 543 & 388 & 322 & 0 \\ 69 & 296 & 0 & 219 \\ 368 & 0 & 600 & 159 \\ 0 & 61 & 177 & 230 \end{bmatrix}$$

is not a $P$-matrix, since $\det(C) < 0$.

Acknowledgment. We thank the referee for a very careful reading of the manuscript and for his comments.

REFERENCES