2007

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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1183

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NOTE ON DELETING A VERTEX AND WEAK INTERLACING OF THE LAPLACIAN SPECTRUM

ZVI LOTKER

Abstract. The question of what happens to the eigenvalues of the Laplacian of a graph when we delete a vertex is addressed. It is shown that

$$\lambda_i - 1 \leq \lambda^v_i \leq \lambda_i + 1,$$

where $$\lambda_i$$ is the $$i$$th smallest eigenvalues of the Laplacian of the original graph and $$\lambda^v_i$$ is the $$i$$th smallest eigenvalues of the Laplacian of the graph $$G[V - v]$$; i.e., the graph obtained after removing the vertex $$v$$. It is shown that the average number of leaves in a random spanning tree $$\mathcal{F}(G) > \frac{2|E|e^{-\lambda_2}}{\lambda_1}$$, if $$\lambda_2 > \alpha n$$.

Key words. Spectrum, Random spanning trees, Cayley formula, Laplacian, Number of leaves.

AMS subject classifications. 05C30, 34L15, 34L40.

1. Introduction. Given a graph $$G = (V, E)$$ with $$n$$ vertices $$V = \{1, ..., n\}$$ and $$E$$ edges, let $$A$$ be the adjacency matrix of $$G$$, i.e. $$a_{i,j} = 1$$ if vertex $$i \in V$$ is adjacent to vertex $$j \in V$$ and $$a_{i,j} = 0$$ otherwise. The Laplacian matrix of graph $$G$$ is $$L = D - A$$, where $$D$$ is a diagonal matrix where $$d_{i,i}$$ is equal to the degree $$d_i$$ of vertex $$i$$ in $$G$$. The Laplacian of a graph is one of the basic matrices associated with a graph. The spectrum of the Laplacian fully characterizes the Laplacian (for more detail see [1]). Since $$L$$ is symmetric and positive semidefinite, its eigenvalues are all nonnegative. We denote them by $$\lambda_1 \leq ... \leq \lambda_n$$. One of the elementary operations on a graph is deleting a vertex $$v \in V$$, we denote the graph obtained from deleting the node $$v$$ by $$G[V - v]$$, and the Laplacian Matrix of $$G[V - v]$$ by $$L^v$$. Finally let $$\lambda^v_i \leq ... \leq \lambda^n_{v-1}$$ be the eigenvalues of $$L^v$$.

A well known theorem in Algebraic Graph theory is the interlacing of Laplacian spectrum under addition/deletion of an edge; see for example [1, Thm. 13.6.2]) quoted next.

Theorem 1.1. Let $$X$$ be a graph with $$n$$ vertices and let $$Y$$ be obtained from $$X$$ by adding an edge joining distinct vertices of $$X$$ then

$$\lambda_{i-1}(L(Y)) \leq \lambda_i(L(X)) \leq \lambda_i(L(Y)),$$

for all $$i = 1, ..., n$$, (we assume that $$\lambda_0 = -\infty$$).

We remark that the eigenvalues of adjacency matrices $$A(G)$$ and $$A(G[V - v])$$ also interlace; see, for example, [1, Thm. 9.1.1]. A natural question is whether we get a similar behavior for the Laplacian when we add/delete a vertex. In this note we study this question.
Related Work. This work uses two theorems from Matrix Analysis. The first is Cauchy’s Interlacing theorem which states that the eigenvalues of a Hermitian matrix \( A \) of order \( n \) interlace the eigenvalues of the principal submatrix of order \( n - 1 \), obtained by removing the \( i \)th row and the \( i \)th column for each \( i \in \{1, \ldots, n\} \).

**Theorem 1.2.** Let \( A \) be a Hermitian matrix of order \( n \) and let \( B \) be a principal submatrix of \( A \) of order \( n - 1 \). Then the eigenvalues of \( A \) and \( B \) are interlacing i.e. \( \lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(B) \leq \lambda_n(A) \).

Proof of this theorem can be found in [2].

The second theorem we use is the Courant-Fischer Theorem. This theorem is an extremely useful characterization of the eigenvalues of symmetric matrices.

**Theorem 1.3.** Let \( L \) be a symmetric matrix. Then

1. the \( i \)th eigenvalue \( \lambda_i \) of \( L \) is given by

\[
\lambda_i = \min_{U} \max_{x \in U} \{x^tLx : x \in U, x \in U, \dim(U) = i, x \in U = \text{span}(U)\},
\]

2. the \((n - i + 1)\)st eigenvalue \( \lambda_{n-i+1} \) of \( L \) is given by

\[
\lambda_{n-i+1} = \max_{U} \min_{x \in U} \{x^tLx : x \in U, x \in U, \dim(U) = i, x \in U = \text{span}(U)\},
\]

where \( U \) ranges over all \( i \) dimensional subspaces.

Proof of this theorem can be found in [3, p. 186]. Let \( v \in V \) be a vertex. Let \( P \) be the principal submatrix after we delete the row and column that correspond to the vertex \( v \) of the Laplacian. Denote the eigenvalues of \( P \) by \( \rho_1 \leq \cdots \leq \rho_{n-1} \).

**2. Weak Interlace for the \( L, L^v \).** In this section we show a weak interlacing connection between the \( L \) and \( L^v \). Since \( L \) is a symmetric matrix we can use Cauchy’s interlacing theorem. The next corollary simply applies this theorem for \( L \) and \( P \).

**Corollary 2.1.** \( \lambda_1 \leq \rho_1 \leq \cdots \leq \rho_{n-1} \leq \lambda_n \).

The next lemma uses the Courant-Fischer Theorem in order to prove weak interlacing for \( L, P \).

**Lemma 2.2.** For all \( i = 1, \ldots, n - 1 \), \( \rho_i \leq \lambda_i^v + 1 \)

Proof. Let \( I_v = P - L^v \). Note that \( I_v \) is a \((0,1)\) diagonal matrix whose \( j \)th diagonal entry is 1 if and only if \( j \) is connected to \( v \) in \( G \). Fix \( i \in \{1, \ldots, n - 1\} \). Using the Courant-Fischer Theorem it follows that

\[
\rho_{n-i+1} = \max_{U} \min_{x \in U} \{x^tL^v x : x \in U, \dim(U) = i, x \in U = \text{span}(U)\},
\]

where \( x^t \) is the transpose of \( x \). Substituting \( L^v + I_v \) in \( P \) it follows that

\[
\rho_{n-i+1} = \max_{U} \min_{x \in U} \{x^t(L^v + I_v) x : x \in U, \dim(U) = i, x \in U = \text{span}(U)\}.
\]

Using standard calculus we get

\[
\rho_{n-i+1} \leq \max_{U} \min_{x \in U} \{x^tL^v x : x \in U, \dim(U) = i, x \in U = \text{span}(U)\}.
\]
\[ + \max_{x \in U} \min_{y \in \mathbb{R}^n, \text{dim}(U) = i, x \in U = \text{span}(U)} \{ a^t I_x x : x^t \}
\leq \lambda_{n-i+1}^v + 1. \]

We now use the previous lemma to get a lower bound on \( \lambda_i^v \).

**Lemma 2.3.** For all \( v = 1, \ldots, n \) and all \( i = 1, \ldots, n - 1 \),
\[ \lambda_i - 1 \leq \lambda_i^v. \]

**Proof.** Fix \( i \in \{1, \ldots, n-1\} \). From Lemma 2.2 it follows that \( \rho_i \leq \lambda_i^v + 1 \). Now this lemma follows from substituting the conclusion of Corollary 2.1 into the previous inequality \( \lambda_i \leq \rho_i \leq \lambda_i^v + 1 \).

The next lemma provides an upper bound on \( \lambda_i^v \).

**Lemma 2.4.** For all \( v = 1, \ldots, n \) and all \( i = 1, \ldots, n - 1 \),
\[ \lambda_i^v \leq \lambda_{i+1}. \]

**Proof.** We prove this lemma by induction on \( d_v \), the degree of the node \( v \). If the degree is \( d_v = 0 \), then by removing the node \( v \) we reduce the multiplicity of the small eigenvalues, which is 0. Formally \( \lambda_i^v = \lambda_{i+1} \) for \( i = 1, \ldots, n - 1 \). Therefore the lemma holds in this case. For the induction step, suppose that the statement holds for \( d_v = k \) and consider the case \( d_v = k + 1 \). Since \( d_v > 0 \) it follows that there exists an edge \( e \) connecting the vertex \( v \) to some other node \( u \). Denote the graph obtained by removing the edge \( e \) from the graph \( G \) by \( X \). Let \( \sigma_1 \leq \ldots \leq \sigma_{n-1} \) be the eigenvalues of the Laplacian of the graph \( X \). From Theorem 1.1 it follows that \( \sigma_i \leq \lambda_i \) for all \( i = 1, \ldots, n \). Using induction we obtain that \( \lambda_{i-1}^v \leq \sigma_i \leq \lambda_i \), for all \( i = 2, \ldots, n \) \( \square \)

Now we present our main theorem.

**Theorem 2.5.** For all \( v = 1, \ldots, n \) and all \( i = 1, \ldots, n - 1 \),
\[ \lambda_i - 1 \leq \lambda_i^v \leq \lambda_{i+1}. \]

**Proof.** The proof is a direct consequence of Lemmas 2.3 and 2.4. \( \square \)

We remark that both inequalities above are tight. To see that, we show there exist graphs such that \( \lambda_i - 1 = \lambda_i^v \). Consider the graph \( K_n \). It is well known that the eigenvalues of \( K_n \) are \( 0, n, \ldots, n \), where the multiplicity of the eigenvalue \( n \) is \( n - 1 \) and 0 is a simple eigenvalue. Now removing a vertex from \( K_n \) produces the graph \( K_{n-1} \). Again the eigenvalues of \( K_{n-1} \) are \( 0, n-1, \ldots, n-1 \), where the multiplicity of the eigenvalue \( n - 1 \) is \( n - 2 \) and 0 is a simple eigenvalue. To see that there are graphs that satisfy \( \lambda_i^v = \lambda_{i+1} \), consider the graph without any edges.

3. **Application to average leafy trees.** In this section we use the weak interlacing Theorem 2.5 to obtain a bound on the average number of leaves in a random spanning tree \( F(G) \). Our bound is useful when \( \lambda_2 > \alpha n \), for fixed \( \alpha > 0 \) and \( |E| = O(n^2) \). We call such a graph a dense expander; in this case we show that the bound is linear in the number of vertices.
It is well known that the smallest eigenvalue of $L$ is 0 and that its corresponding eigenvector is $(1, 1, \ldots, 1)$. If $G$ is connected, all other eigenvalues are greater than 0. Let $P^v$ denote the submatrix of $L$ obtained by deleting the $v$th row and $v$th column. Then, by the Matrix Tree Theorem, for each vertex $v \in V$ we have $t(G) = |\det(P^v)|$, where $t(G)$ is the number of spanning trees of $G$. One can rephrase the Matrix Tree Theorem in terms of the spectrum of the Laplacian matrix. The next theorem appears in [1, p. 284]; it connects the eigenvalues of the Laplacian of $G$ and $t(G)$.

**Theorem 3.1.** Let $G$ be a graph on $n$ vertices and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the Laplacian of $G$. Then the number of labeled spanning trees in $G$ is \( \frac{1}{n} \prod_{i=2}^{n} \lambda_i \).

Let $G$ be a graph. Using the previous theorem it is possible to define the following probability space: \( \Omega(G) = \{T : T \text{ is a spanning tree in } G\} \). On this set we take a spanning tree in a uniform probability. We are interested in finding the average number of leaves in a random spanning tree. Let $T$ be a random spanning tree taken from $\Omega(G)$ with the uniform distribution. Denote by $F(G)$ the expected number of leaves in $T$. Using the matrix theorem we can get a formula to compute the average number of leaves in a random spanning tree.

**Lemma 3.2.**

\[
F(G) = \sum_{v \in V} nd_v \frac{\prod_{i=2}^{n-1} \lambda_i^v}{(n-1) \prod_{i=2}^{n} \lambda_i}.
\]

**Proof.** The number of trees that have vertex $v$ as a leaf is $\frac{d_v \prod_{i=2}^{n-1} \lambda_i^v}{(n-1) \prod_{i=2}^{n} \lambda_i}$. The lemma follows by summing over all vertices and dividing by the total number of trees. \( \square \)

The weak interlacing theorem enables us to bound the average number of leaves in a dense expander graph. More precisely, we show that $F(G) = O(n)$.

**Theorem 3.3.** Let $G$ be a graph. If $\lambda_2 > \alpha n$, then the average number of leaves in $T$ is bigger than $2|E|e^{\frac{1}{\lambda_2}}$.

**Proof.**

\[
F(G) = \sum_{v \in V} nd_v \frac{\prod_{i=2}^{n-1} \lambda_i^v}{(n-1) \prod_{i=2}^{n} \lambda_i} \\
\geq \sum_{v \in V} nd_v \frac{\prod_{i=2}^{n-1} (\lambda_i - 1)}{(n-1) \prod_{i=2}^{n} \lambda_i} \\
= \sum_{v \in V} nd_v \frac{\prod_{i=2}^{n-1} \frac{\lambda_i - 1}{\lambda_i}}{(n-1) \lambda_n} \\
= \sum_{v \in V} nd_v \frac{\prod_{i=2}^{n-1} (1 - \frac{1}{\lambda_i})}{(n-1) \lambda_n} \\
\geq \sum_{k \in V} nd_k (1 - \frac{1}{\lambda_2})^n \\
\geq \sum_{k \in V} nd_k (1 - \frac{1}{\lambda_2})^n
\]
\[ \frac{2|E|e^{-\alpha n}}{\lambda_n} \geq \frac{2|E|e^{-\frac{1}{2}\alpha}}{\lambda_n}. \]

**Corollary 3.4.** For any constant \( \alpha > 0 \), if \( \lambda_2 > \alpha n \), and \( |E| = O(n^2) \), then the average number of leaves in \( T \) is \( O(n) \).

**Conclusion.** In this paper we proved a weak interlacing theorem for the Laplacian. Using this theorem we showed that in a dense expander the average number of leaves is \( O(n) \). A natural open question is to show that the average number of leaves in a random tree is an approximation to the maximal spanning leafy tree.

**Acknowledgment.** I would like to thank Ronald de Wolf for helpful discussions.

**References**

