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SUBDIRECT SUMS OF S-STRICLY DIAGONALLY DOMINANT MATRICES∗
RAFAEL BRU†, FRANCISCO PEDROCHE†, AND DANIEL B. SZYLD‡

Abstract. Conditions are given which guarantee that the $k$-subdirect sum of $S$-strictly diagonally dominant matrices ($S$-SDD) is also $S$-SDD. The same situation is analyzed for SDD matrices. The converse is also studied: given an SDD matrix $C$ with the structure of a $k$-subdirect sum and positive diagonal entries, it is shown that there are two SDD matrices whose subdirect sum is $C$.

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Key words. Subdirect sum, Diagonally dominant matrices, Overlapping blocks.

1. Introduction. The concept of $k$-subdirect sum of square matrices emerges naturally in several contexts. For example, in matrix completion problems, overlapping subdomains in domain decomposition methods, global stiffness matrix in finite elements, etc.; see, e.g., [1], [2], [5], and references therein.

Subdirect sums of matrices are generalizations of the usual sum of matrices (a $k$-subdirect sum is formally defined below in section 2). They were introduced by Fallat and Johnson in [5], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric $M$-matrices, is positive definite or symmetric $M$-matrices, respectively. They also showed that this is not the case for $M$-matrices: the subdirect sum of two $M$-matrices may not be an $M$-matrix, and therefore the subdirect sum of two $H$-matrices may not be an $H$-matrix.

In this paper we show that for a subclass of $H$-matrices the $k$-subdirect sum of matrices belongs to the same class. We show this for certain strictly diagonally dominant matrices (SDD) and for $S$-strictly diagonally dominant matrices ($S$-SDD), introduced in [4]; see also [3], [9], for further properties and analysis. We also show that the converse holds: given an SDD matrix $C$ with the structure of a $k$-subdirect sum and positive diagonal entries, then there are two SDD matrices whose subdirect sum is $C$.

2. Subdirect sums. Let $A$ and $B$ be two square matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned into $2 \times 2$ blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

(2.1)
where $A_{22}$ and $B_{11}$ are square matrices of order $k$. Following [5], we call the square matrix of order $n = n_1 + n_2 - k$ given by

$$C = \begin{bmatrix}
A_{11} & A_{12} & O \\
A_{21} & A_{22} + B_{11} & B_{12} \\
O & B_{21} & B_{22}
\end{bmatrix}$$

(2.2)

the $k$-subdirect sum of $A$ and $B$ and denote it by $C = A \oplus_k B$.

It is easy to express each element of $C$ in terms of those of $A$ and $B$. To that end, let us define the following set of indices

$$S_1 = \{1, 2, \ldots, n_1 - k\},$$
$$S_2 = \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\},$$
$$S_3 = \{n_1 + 1, n_1 + 2, \ldots, n\}.$$

(3)

Denoting $C = (c_{ij})$ and $t = n_1 - k$, we can write

$$c_{ij} = \begin{cases}
    a_{ij} & i \in S_1, \ j \in S_1 \cup S_2 \\
    0 & i \in S_1, \ j \in S_3 \\
    a_{ij} & i \in S_2, \ j \in S_1 \\
    a_{ij} + b_{i-t,j-t} & i \in S_2, \ j \in S_2 \\
    b_{i-t,j-t} & i \in S_2, \ j \in S_3 \\
    0 & i \in S_3, \ j \in S_1 \\
    b_{i-t,j-t} & i \in S_3, \ j \in S_2 \cup S_3.
\end{cases}$$

(4)

Note that $S_1 \cup S_2 \cup S_3 = \{1, 2, \ldots, n\}$ and that $n = t + n_2$; see Figure 2.1.

![Fig. 2.1. Sets for the subdirect sum $C = A \oplus_k B$, with $t = n_1 - k$ and $p = t + 1$; cf. (2.4).](image)

3. Subdirect sums of S-SDD matrices. We begin with some definitions which can be found, e.g., in [4], [9].

**Definition 3.1.** Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, let us define the $i$th deleted absolute row sum as

$$r_i(A) = \sum_{j \neq i, j=1}^n |a_{ij}|, \ \forall i = 1, 2, \ldots, n,$$
and the $i$th deleted absolute row-sum with columns in the set of indices $S = \{i_1, i_2, \ldots\} \subseteq N := \{1, 2, \ldots, n\}$ as

$$r^S_i(A) = \sum_{j \neq i, j \in S} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n.$$  

Given any nonempty set of indices $S \subseteq N$ we denote its complement in $N$ by $\bar{S} := N \setminus S$. Note that for any $A = (a_{ij}) \in C^{n \times n}$ we have that $r_i(A) = r^S_i(A) + r_{\bar{S}}^i(A)$.

**Definition 3.2.** Given a matrix $A = (a_{ij}) \in C^{n \times n}$, $n \geq 2$ and given a nonempty subset $S$ of $\{1, 2, \ldots, n\}$, then $A$ is an $S$-strictly diagonally dominant matrix if the following two conditions hold:

\[
\begin{align*}
\text{i)} & \quad |a_{ii}| > r^S_i(A), \\
\text{ii)} & \quad \left(|a_{ii}| - r^S_i(A)\right)\left(|a_{jj}| - r^S_j(A)\right) > r^S_i(A) r^S_j(A) \quad \forall i \in S, \forall j \in \bar{S}.
\end{align*}
\]

(3.1)

It was shown in [4] that an $S$-strictly diagonally dominant matrix ($S$-SDD) is a nonsingular $H$-matrix. In particular, when $S = \{1, 2, \ldots, n\}$, then $A = (a_{ij}) \in C^{n \times n}$ is a strictly diagonally dominant matrix (SDD). It is easy to show that an SDD matrix is an $S$-SDD matrix for any proper subset $S$, but the converse is not always true as we show in the following example.

**Example 3.3.** Consider the following matrix

\[
A = \begin{bmatrix}
2.6 & -0.4 & -0.7 & -0.2 \\
-0.4 & 2.6 & -0.5 & -0.7 \\
-0.6 & -0.7 & 2.2 & -1.0 \\
-0.8 & -0.7 & -0.5 & 2.2 \\
\end{bmatrix},
\]

which is a $\{1, 2\}$-SDD matrix but is not an SDD matrix. A natural question is to ask if the subdirect sum of S-SDD matrices is in the class, but in general this is not true. For example, the 2-subdirect sum $C = A \oplus_2 A$ gives

\[
C = \begin{bmatrix}
2.6 & -0.4 & -0.7 & -0.2 & 0 & 0 \\
-0.4 & 2.6 & -0.5 & -0.7 & 0 & 0 \\
-0.6 & -0.7 & 4.8 & -1.4 & -0.7 & -0.2 \\
-0.8 & -0.7 & -0.9 & 4.8 & -0.5 & -0.7 \\
0 & 0 & -0.6 & -0.7 & 2.2 & -1.0 \\
0 & 0 & -0.8 & -0.7 & -0.5 & 2.2 \\
\end{bmatrix}
\]

which is not a $\{1, 2\}$-SDD matrix; condition ii) of (3.1) fails for the matrix $C$ for the cases $i = 1, j = 5$ and $i = 2, j = 5$. It can also be observed that $C$ is not an SDD matrix.

This example motivates the search of conditions such that the subdirect sum of S-SDD matrices is in the class of S-SDD matrices (for a fixed set $S$).

We now proceed to show our first result. Let $A$ and $B$ be matrices of order $n_1$ and $n_2$, respectively, partitioned as in (2.1) and consider the sets $S_i$ defined in (2.3). Then we have the following relations

\[
\begin{align*}
&r^{S_1}_i(C) = r^{S_1}_i(A), \\
&r^{S_2,S_3}_i(C) = r^{S_2}_i(A) \quad \forall i \in S_1,
\end{align*}
\]

(3.2)
which are easily derived from (2.4).

**Theorem 3.4.** Let $A$ and $B$ be matrices of order $n_1$ and $n_2$, respectively. Let $n_1 \geq 2$, and let $k$ be an integer such that $1 \leq k \leq \min(n_1,n_2)$, which defines the sets $S_1$, $S_2$, $S_3$ as in (2.3). Let $A$ and $B$ be partitioned as in (2.1). Let $S$ be a set of indices of the form $S = \{1, 2, \ldots\}$. Let $A$ be $S$-strictly diagonally dominant, with $\text{card}(S) \leq \text{card}(S_1)$, and let $B$ be strictly diagonally dominant. If all diagonal entries of $A_{22}$ and $B_{11}$ are positive (or all negative), then the $k$-subdirect sum $C = A \oplus_k B$ is $S$-strictly diagonally dominant, and therefore nonsingular.

**Proof.** We first prove the case when $S = S_1$. Since $A$ is $S_1$-strictly diagonally dominant, we have that

\[
\begin{align*}
\text{i)} & \quad |a_{ii}| > r_{i}^{S_1}(A) \quad \forall i \in S_1, \\
\text{ii)} & \quad (|a_{ii}| - r_{i}^{S_1}(A))(|a_{jj}| - r_{j}^{S_2}(A)) > r_{i}^{S_2}(A)r_{j}^{S_1}(A) \quad \forall i \in S_1, \forall j \in S_2.
\end{align*}
\]

Note that $A$ is of order $n_1$ and then the complement of $S_1$ in $\{1, 2, \ldots, n_1\}$ is $S_2$.

We want to show that $C$ is also an $S_1$-strictly diagonally dominant matrix, i.e., we have to show that

1) $|c_{ii}| > r_{i}^{S_1}(C) \quad \forall i \in S_1$, and
2) $(|c_{ii}| - r_{i}^{S_1}(C))(|c_{jj}| - r_{j}^{S_2\cup S_3}(C)) > r_{i}^{S_2\cup S_3}(C)r_{j}^{S_1}(C) \quad \forall i \in S_1, \forall j \in S_2 \cup S_3$.

Note that since $C$ is of order $n$, the complement of $S_1$ in $\{1, 2, \ldots, n\}$ is $S_2 \cup S_3$.

To see that 1) holds we use equations (2.4), (3.2) and part i) of (3.3) (see also Figure 2.1) to obtain

$$|c_{ii}| = |a_{ii}| > r_{i}^{S_1}(A) = r_{i}^{S_1}(C), \quad \forall i \in S_1.$$ 

To see that 2) holds we distinguish two cases: $j \in S_2$ and $j \in S_3$. If $j \in S_2$, from (2.4) we have the following relations (recall that $t = n_1 - k$):

\[
\begin{align*}
r_{j}^{S_2}(C) &= \sum_{j \neq k, k \in S_1} |c_{jk}| = \sum_{j \neq k, k \in S_1} |a_{jk}| = r_{j}^{S_1}(A), \\
r_{j}^{S_2\cup S_3}(C) &= \sum_{j \neq k, k \in S_2 \cup S_3} |c_{jk}| = \sum_{j \neq k, k \in S_2} |c_{jk}| + \sum_{j \neq k, k \in S_3} |c_{jk}|
\end{align*}
\]

\[
= r_{j}^{S_2}(C) + r_{j}^{S_3}(C),
\]

\[
r_{j}^{S_2}(C) = \sum_{j \neq k, k \in S_2} |a_{jk} + b_{j-t,k-t}|,
\]

\[
r_{j}^{S_3}(C) = \sum_{j \neq k, k \in S_3} |b_{j-t,k-t}| = r_{j}^{S_1}(B),
\]

\[
c_{jj} = a_{jj} + b_{j-t,j-t}.
\]

Therefore we can write

\[
(|c_{ii}| - r_{i}^{S_1}(C))(|c_{jj}| - r_{j}^{S_2\cup S_3}(C)) =
\]

\[
(|a_{ii}| - r_{i}^{S_1}(A))(|a_{jj} + b_{j-t,j-t}| - r_{j}^{S_2}(C) - r_{j}^{S_3}(C)), \quad \forall i \in S_1, \forall j \in S_2,
\]

\[
(3.10)
\]
where we have used that $c_{ii} = a_{ii}$, for $i \in S_1$ and equations (3.2), (3.6) and (3.9). Using now that $A_{22}$ and $B_{11}$ have positive diagonal (or both negative diagonal) we have that $|a_{jj} + b_{j-t,j-t}| = |a_{jj}| + |b_{j-t,j-t}|$ and therefore we can rewrite (3.10) as

\[
(|c_{ii}| - r_i^{S_1}(C)) (|c_{jj}| - r_j^{S_2 \cup S_3}(C)) = \]

\[
(|a_{ii}| - r_i^{S_1}(A)) (|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C)), \forall i \in S_1, \forall j \in S_2.
\]

(3.11)

Let us now focus on the second term of the right hand side of (3.11). Observe that from (3.7) and the triangle inequality we have that

\[
r_j^{S_2}(C) = \sum_{j \neq k, k \in S_2} |a_{jk} + b_{j-t,k-t}| \leq \sum_{j \neq k, k \in S_2} |a_{jk}| + \sum_{j \neq k, k \in S_2} |b_{j-t,k-t}|
\]

\[
= r_j^{S_2}(A) + r_j^{S_2}(B)
\]

(3.12)

and using (3.8), from (3.12) we can write the inequality

\[
|a_{jj} + b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(A) - r_j^{S_2}(B) - r_j^{S_3}(B).
\]

Since we have $r_j^{S_2}(B) + r_j^{S_3}(B) = r_j^{S_2 \cup S_3}(B)$, we obtain

\[
|a_{jj} + b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_j^{S_2 \cup S_3}(B),
\]

which allows us to transform (3.11) into the following inequality

\[
(|c_{ii}| - r_i^{S_1}(C)) (|c_{jj}| - r_j^{S_2 \cup S_3}(C)) \geq \]

\[
(|a_{ii}| - r_i^{S_1}(A)) (|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_j^{S_2 \cup S_3}(B)), \forall i \in S_1, \forall j \in S_2,
\]

(3.13)

where we have used that $(|a_{ii}| - r_i^{S_1}(A))$ is positive since $A$ is $S_1$-strictly diagonally dominant. Observe now that $|b_{j-t,j-t}| - r_j^{S_2 \cup S_3}(B)$ is also positive since $B$ is strictly diagonally dominant, and thus we can write

\[
|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_j^{S_2 \cup S_3}(B) > |a_{jj}| - r_j^{S_2}(A)
\]

which jointly with (3.13) leads to the strict inequality

\[
(|c_{ii}| - r_i^{S_1}(C)) (|c_{jj}| - r_j^{S_2 \cup S_3}(C)) \geq (|a_{ii}| - r_i^{S_1}(A)) (|a_{jj}| - r_j^{S_2}(A)), \quad (3.14)
\]

for all $i \in S_1$ and for all $j \in S_2$. Finally, using (ii) of (3.3) (i.e., the fact that $A$ is $S_1$-strictly diagonally dominant) and equations (3.2) and (3.5) we can write the inequality

\[
(|a_{ii}| - r_i^{S_1}(A)) (|a_{jj}| - r_j^{S_2}(A)) > r_i^{S_1}(A) r_j^{S_1}(A) r_j^{S_2}(C) r_j^{S_3}(C)
\]

for all $i \in S_1$ and for all $j \in S_2$, which allows to transform equation (3.14) into the inequality

\[
(|c_{ii}| - r_i^{S_1}(C)) (|c_{jj}| - r_j^{S_2 \cup S_3}(C)) > r_i^{S_2 \cup S_3}(C) r_j^{S_1}(C), \forall i \in S_1, \forall j \in S_2.
\]
Therefore we have proved condition 2) for the case $j \in S_2$. In the case $j \in S_3$, we have from (2.4) that
\[
r_j^{S_3}(C) = \sum_{j \neq k, k \in S_1} |c_{jk}| = 0.
\]
Therefore the condition 2) of (3.4) becomes
\[
(|c_{ii}| - r_i^{S_1}(C)) (|c_{jj}| - r_j^{S_2 \cup S_3}(C)) > 0, \quad \forall i \in S_1, \forall j \in S_3,
\]
and it is easy to show that this inequality is fulfilled. The first term is positive since, as before, we have that $|c_{ii}| - r_i^{S_1}(C) = |a_{ii}| - r_i^{S_1}(A) > 0$. The second term of (3.15) is also positive since we have that $c_{jj} = b_{j-t,j-t}$ for all $j \in S_3$ and
\[
r_j^{S_2 \cup S_3}(C) = \sum_{j \neq k, k \in S_2 \cup S_3} |c_{jk}| = \sum_{j \neq k, k \in S_2 \cup S_3} |b_{j-t,k-t}| = r_j^{S_2 \cup S_3}(B), \forall j \in S_3,
\]
and since $B$ is strictly diagonally dominant we have
\[
|b_{j-t,j-t}| - r_j^{S_2 \cup S_3}(B) > 0, \forall j \in S_3.
\]

Therefore equation (3.15) is fulfilled and the proof for the case $S = S_1$ is completed.

When $card(S) < card(S_1)$ the proof is analogous. We only indicate that the key point in this case is the subspace $j \in S_1 \setminus S$ for which it is easy to show that a condition similar to 2) for $C$ in (3.4) still holds.

When $card(S) > card(S_1)$ the preceding theorem is not valid as we show in the following example.

**Example 3.5.** In this example we show a matrix $A$ that is an $S$-SDD matrix with $card(S) > card(S_1)$ and a matrix $B$ that is an SDD matrix but the subdirect sum $C$ is not an $S$-SDD matrix. Let the following matrices $A$ and $B$ be partitioned as
\[
A = \begin{bmatrix}
1.0 & -0.3 & -0.4 & -0.5 \\
-0.9 & 1.0 & -0.4 & -0.7 \\
-0.1 & -0.4 & 1.3 & -0.4 \\
-0.1 & -0.9 & -0.1 & 2.0
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
2.0 & 0.2 & -0.3 & -0.1 \\
0.8 & 2.9 & -0.2 & -0.5 \\
-0.5 & -0.1 & 2.4 & -0.9 \\
-0.6 & -0.8 & -0.8 & 2.3
\end{bmatrix}.
\]

We have from (2.3) that $S_1 = \{1\}$, $S_2 = \{2, 3, 4\}$ and $S_3 = \{5\}$. It is easy to show that $A$ is $\{1, 2\}$-SDD, $A$ is not SDD, and $B$ is SDD. The 3-subdirect sum $C = A \oplus_3 B$
\[
C = \begin{bmatrix}
1.0 & -0.3 & -0.4 & -0.5 & 0 \\
-0.9 & 3.6 & -0.2 & -1.0 & -0.1 \\
-0.1 & 0.4 & 4.2 & -0.6 & -0.5 \\
-0.1 & -1.4 & -0.2 & 4.4 & -0.9 \\
0 & -0.6 & -0.8 & -0.8 & 2.3
\end{bmatrix}
\]

is not a $\{1, 2\}$-SDD: the corresponding condition ii) for $C$ in equation (3.1) fails for $i = 1, j = 5$. 
Remark 3.6. An analogous result to Theorem 3.4 can be obtained when the matrix $B$ is $S$-strictly diagonally dominant with $S = \{n_1 + 1, n_1 + 2, \ldots\}$, $\text{card}(S) \leq \text{card}(S_3)$, and the matrix $A$ is strictly diagonally dominant. The proof is completely analogous, and thus we omit the details.

It is easy to show that if $A$ is a strictly diagonally dominant matrix, then $A$ is also an $S_1$-strictly diagonally dominant matrix. Therefore we have the following corollary.

**Corollary 3.7.** Let $A$ and $B$ be matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned as in (2.1). If $A$ and $B$ are strictly diagonally dominant and all diagonal entries of $A_{22}$ and $B_{11}$ are positive, then the $k$-subdirect sum $C = A \oplus_k B$ is strictly diagonally dominant, and therefore nonsingular.

Remark 3.8. In the general case of successive $k$-subdirect sums of the form

$$(A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 \oplus \cdots$$

when $A_1$ is S-SDD with $\text{card}(S) \leq n_1 - k_1$ and $A_2, A_3, \ldots$, are SDD matrices, we have that all the subdirect sums are S-SDD matrices, provided that in each particular subdirect sum the quantity $\text{card}(S)$ is no larger than the corresponding overlap, in accordance with Theorem 3.4.

4. Overlapping SDD matrices. In this section we consider the case of square matrices $A$ and $B$ of order $n_1$ and $n_2$, respectively, which are principal submatrices of a given SDD matrix, and such that they have a common block with positive diagonals. This situation, as well as a more general case outlined in Theorem 4.1 later in this section, appears in many variants of additive Schwarz preconditioning; see, e.g., [2], [6], [7], [8]. Specifically, let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

be an SDD matrix of order $n$, with $n = n_1 + n_2 - k$, and with $M_{22}$ a square matrix of order $k$, such that its diagonal is positive. Let us consider two principal submatrices of $M$, namely

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}.$$ 

Therefore the $k$-subdirect sum of $A$ and $B$ is given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}.$$ (4.1)

Since $A$ and $B$ are SDD matrices, according to Corollary 3.7 the subdirect sum given by equation (4.1) is also an SDD matrix. This result can clearly be extended to the sum of $p$ overlapping submatrices of a given SDD matrix with positive diagonal entries. We summarize this result formally as follows; cf. a similar result for...
$M$-matrices in [1]. Here, we consider consecutive principal submatrices defined by consecutive indices of the form $\{i, i+1, i+2, \ldots\}$.

**Theorem 4.1.** Let $M$ be an SDD matrix with positive diagonal entries. Let $A_i$, $i = 1, \ldots, p$, be consecutive principal submatrices of $M$ of order $n_i$, and consider the $p-1$ $k_i$-subdirect sums given by

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1$$

in which $C_0 = A_1$, and $k_i < \min(n_i, n_{i+1})$. Then each of the $k_i$-subdirect sums $C_i$ is an SDD matrix, and in particular

$$C_{p-1} = A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_p} A_p \quad (4.2)$$

is an SDD matrix.

**5. SDD matrices with the structure of a subdirect sum.** We address the following question. Let $C$ be square of order $n$, an SDD matrix with positive diagonal entries, and having the structure of a $k$-subdirect sum. Can we find matrices $A$ and $B$ with the same properties such that $C = A \oplus_k B$? We answer this in the affirmative in the following result.

**Proposition 5.1.** Let

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix},$$

with the matrices $C_{ii}$ of order $n_1-k$, $k$, $n_2-k$, for $i = 1, 2, 3$, respectively, and $C$ an SDD matrix with positive diagonal entries. Then, we can find square matrices $A$ and $B$ of order $n_1$ and $n_2$ such that they are SDD matrices with positive diagonal entries, and such that $C = A \oplus_k B$. In other words, we have

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}$$

such that $C_{22} = A_{22} + B_{22}$.

The proof of this proposition resembles that of [5, Proposition 4.1], where a similar question was studied for $M$-matrices, and we do not repeat it here. We mention that it is immediate to generalize Proposition 5.1 to a matrix $C$ with the structure of a subdirect sum of several matrices such as that of (4.2) of Theorem 4.1.

**REFERENCES**


