Volume 17 *ELA Volume 17 (2008)* Article 7

2008

Quadratic convergence bounds of scaled iterates by the serial Jacobi methods for indefinite Hermitian matrices

Josip Mateja
josip.matejas@kr.htnet.hr

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

QUADRATIC CONVERGENCE BOUNDS OF SCALED ITERATES BY THE SERIAL JACOBI METHODS FOR INDEFINITE HERMITIAN MATRICES*

J. MATEJAŠ†

Abstract. Using the technique from [12], sharp quadratic convergence bounds for scaled Jacobi iterates are derived. The iterates are generated by any serial Jacobi method when applied to a general complex nonsingular Hermitian matrix. The scaled iterates are defined relatively to the diagonal. The estimates depend on the relative separation between the eigenvalues. The assumptions are general, since no monotonic ordering of the diagonal elements within any diagonal block which converges to a multiple eigenvalue is presumed.

Key words. Jacobi method, Scaled matrices, Quadratic convergence.

AMS subject classifications. 65F15, 65G05.

1. Introduction. In [11, 12], we have derived quadratic convergence bounds of scaled iterates $H_S^{(k)} = D_k^{-1/2} H^{(k)} D_k^{-1/2}$, $k \geq 0$, where $D_k = \mathrm{diag}(H^{(k)})$, and $H^{(k)}$ are obtained by the serial Jacobi method applied to the positive definite Hermitian matrix $H = H^{(0)}$. Here $\mathrm{diag}(H^{(k)})$ denotes the diagonal part of $H^{(k)}$. Similar results are obtained for the Kogbetliantz method [7, 14] and for the J-symmetric Jacobi method of Veselić [15]. In this paper, we prove that the results of this kind also hold for the serial Jacobi method when applied to indefinite nonsingular Hermitian matrices. This last result completes our survey of scaled iterates.

The motivation for all these endeavors came from several origins. First, we wanted to generalize the classical quadratic convergence results for the symmetric Jacobi method [4, 9, 10, 17] in order to comply with the new theory of relative perturbations for the eigenvalues and singular values. The norms of scaled matrices and the relative gaps in the spectrum appear naturally in the relative accuracy results (see [1] and many other references from [8]). We note that the pioneering work of Demmel and Veselić [1] has promoted Jacobi method as an accurate eigensolver for the full eigendecomposition of positive definite matrices and recent results of Drmač and Veselić [2, 3] have shown that (one-sided versions of) Jacobi methods can be made very efficient.

*Received by the editors December 28, 2006. Accepted for publication February 7, 2008. Handling Editor: Angelika Bunse-Gerstner.

†Faculty of Economics, University of Zagreb, Kennedyjev trg 6, 10000 Zagreb, Croatia (josip.matejas@kr.htnet.hr).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Second, since Jacobi methods are generally accurate, the termination criterions should carefully be chosen. The one, most often recommended, has been introduced in [1] (although it had already been used in disguised form by Rutishauser [19]): $|h_{ij}| \leq \text{tol} \cdot \sqrt{h_{ii}h_{jj}},\ i \neq j$. This criterion has been additionally theoretically justified in [6], where the structure of scaled almost diagonal Hermitian matrices is revealed. Some results from [6] (here described in Theorem 1) show that the diagonal elements of a Hermitian matrix H are relatively close to the corresponding eigenvalues provided that the off-diagonal part of $H_S = |\operatorname{diag}(H)|^{-1/2}H|\operatorname{diag}(H)|^{-1/2}$ has sufficiently small norm, smaller than the minimum relative gap. Similar results hold for the singular value problem [13]. So, convergence of scaled iterates should be monitored.

Third, note that Jacobi method is not relatively accurate for a general initial indefinite Hermitian matrix H. However, numerical tests and the recent theoretical investigation [16] indicate that Jacobi method is relatively accurate for the scaled diagonally dominant (s.d.d.) indefinite Hermitian matrices. This, together with the results presented here, indicates that standard Jacobi method can be safely used as an accurate eigensolver for general s.d.d. Hermitian matrices.

This paper is closely related to [12] although it presents its counterpart which deals with complex s.d.d. indefinite Hermitian matrices. There are many similarities with the content of [12]. We use the same notation (without introducing it) and the same technique of the proof which is based on induction over the set $\{1, 2, \ldots, p\}$, where p is number of distinct eigenvalues of H. Since this technique is well described and discussed (with figures and all details) in [12], we avoid to explain it. The readers are forewarned of it at several places in the paper. To keep the exposition short, we assume the reader is acquainted with [12]. This paper is sort of continuation of [12].

However, the proofs of several auxiliary results had to be modified because of the differences between the positive definite and the indefinite case. In particular:

- For indefinite s.d.d. Hermitian matrices, the diagonal elements are not all positive. This fact is reflected in the proofs of Lemmas 2 and 5.
- For indefinite s.d.d. Hermitian matrices, the bounds from [6], appearing in the estimates for the structure of indefinite s.d.d. Hermitian matrices are larger than those for the definite case. They are given in Theorem 1 and are used in Lemmas 5, 11 and 14.
- In contrast to [12], we abandon the assumption that the eigenvalues associated with the diagonal elements are monotonically ordered. We shall only require that the diagonal elements affiliated with multiple eigenvalues occupy successive positions on the diagonal (see the asymptotic assumption (A2)), as in the classical result of Hari [4]. It resulted in modifying the proof of Lemma 7.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

Because of these differences, the asymptotic assumption (A1) had to be made more stringent than the one in [12] (the constant 1/6 appearing in [12, p. 25] is replaced by 1/10). Consequently, the constants in the auxiliary lemmas, as well as in Theorem 6, are different from those in [12].

The paper is organized as follows (cf. [12]). In Section 2, we derive some preparatory results. In Section 3, we prove the main result (Theorem 6) and in Section 4, we present some numerical examples.

2. Preparatory results.

2.1. Scaled diagonally dominant matrices. Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with the eigenvalues

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{s_1}, \;\; \lambda_{s_1+1} = \cdots = \lambda_{s_2}, \;\; \ldots, \;\; \lambda_{s_{p-1}+1} = \cdots = \lambda_{s_p}, \tag{2.1}$$

where $s_p = n$. Then p is the number of distinct eigenvalues of H and for each i, $1 \le i \le p$, $n_i = s_i - s_{i-1}$ ($s_0 = 0$) is the multiplicity of λ_{s_i}. We define the appropriate sets of indices

$$\mathcal{N}_r \;=\; \{t \in \mathbb{N} : \; s_{r-1} + 1 \le t \le s_r\}, \quad 1 \le r \le p\,, \tag{2.2}$$

and if $H = (h_{ij})$ is nearly diagonal, we assume

$$\text{for } t \in \mathcal{N}_r\,, \qquad h_{tt} \text{ is affiliated with } \lambda_{s_r}, \quad 1 \le r \le p\,. \tag{2.3}$$

The assumption (2.3) ensures that the diagonal elements of H which correspond to the same multiple eigenvalue occupy successive positions on the diagonal. It means that for each $1 \le r \le p$, the diagonal elements h_{tt}, $t \in \mathcal{N}_r$, belong to the Gerschgorin disc around λ_{s_r}.

For each $i \in \{1, \ldots, p\}$, we define the *relative gap* of λ_{s_i} in the spectrum of H by,

$$\gamma_i = \min_{\substack{1\le j\le p\\ j\ne i}} \left\{ \frac{|\lambda_{s_i} - \lambda_{s_j}|}{|\lambda_{s_i}| + |\lambda_{s_j}|}\,,\; 1 \le i \le p \right\}. \tag{2.4}$$

The *minimum relative gap* is then

$$\gamma = \min_{1\le i\le p} \gamma_i\,. \tag{2.5}$$

For H with nonzero diagonal, the (symmetrically) *scaled* H is defined by

$$H_S = |\mathrm{diag}(H)|^{-1/2} H |\mathrm{diag}(H)|^{-1/2}. \tag{2.6}$$

The spectral matrix norm is denoted by $\|\cdot\|_2$. Since we frequently use the Frobenius matrix norm, it is denoted simply by $\|\cdot\|$. By $\Omega(X)$ we denote the off-diagonal part of

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

X, $\Omega(X) = X - \mathrm{diag}(X)$. Using the notation from [12, pp. 19–20], we now state the result [6, Corollary 3.1], which reveals the structure of an α-s.d.d. Hermitian matrix[1].

Theorem 1. *Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian matrix satisfying the condition (2.1),*

$$\|\Omega(H_S)\|_2 < \frac{\gamma}{\gamma+3}\,,$$

and (2.3). Here γ and H_S are defined by the relations (2.5) and (2.6), respectively. Then the following assertions hold

$$(i)\quad \sum_{j\in\mathcal{N}_r}\left|1-\frac{\lambda_{s_r}}{h_{jj}}\right|^2 + \|\Omega(\pi_r(H_S))\|^2 \le \frac{16}{\gamma_r^2}\|\tau_r(H_S)\|^4,\ \ 1\le r\le p\,,$$

$$(ii)\quad \sum_{j=1}^{n}\left|1-\frac{\lambda_j}{h_{jj}}\right|^2 + \|\Omega(\pi(H_S))\|^2 \le \frac{8}{\gamma^2}\|\tau(H_S)\|^4.$$

2.2. Hermitian Jacobi method. A brief description of Jacobi method for computing the spectral decomposition of Hermitian matrices is given in [12, pp. 20–22]. We consider here the column-cyclic pivot strategy. The final result then holds for any equivalent pivot strategy (in [18], they are called wave front strategies). The total number of rotations in each cycle is

$$N = \frac{n(n-1)}{2}\,. \tag{2.7}$$

The *scaled iterates* are defined by

$$H_S^{(k)} = |\mathrm{diag}(H^{(k)})|^{-1/2} H^{(k)} |\mathrm{diag}(H^{(k)})|^{-1/2}, \quad k\ge 0, \tag{2.8}$$

where $H^{(0)} = H,\ H^{(1)},\ \ldots$ are generated by the method.

We shall use $A^{(k)}$, the off-diagonal part of $H_S^{(k)}$, and its norm α_k,

$$A^{(k)} = \Omega(H_S^{(k)}) = H_S^{(k)} - \mathrm{diag}(H_S^{(k)}), \quad \alpha_k = \|\Omega(H_S^{(k)})\|, \quad k\ge 0. \tag{2.9}$$

Thus, the diagonal elements of $A^{(k)} = (a_{lm}^{(k)}),\ k\ge 0$ are zeros and the off-diagonal ones are given by

$$a_{lm}^{(k)} = \frac{h_{lm}^{(k)}}{\sqrt{|h_{ll}^{(k)} h_{mm}^{(k)}|}}\,, \quad l\ne m, \quad k\ge 0. \tag{2.10}$$

[1] The original assumption $h_{11} \ge h_{22} \ge \cdots \ge h_{nn}$ is replaced here with the weaker one (2.3). This is the weakest form of the assumption under which the result holds.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

2.3. Auxiliary lemmas. We proceed in the same way as in [12, pp. 22–23].

LEMMA 2. *Let $H = (h_{lm})$ be a Hermitian matrix of order n. Suppose $\widetilde{H}$ is obtained from H by applying a single Jacobi step which annihilates the element h_{ij}. Let $A = (a_{lm})$ and $\widetilde{A} = (\widetilde{a}_{lm})$ be defined by*

$$A = \Omega(H_S), \quad H_S = \Delta^{-1/2} H \Delta^{-1/2}, \quad \Delta = |diag(H)|,$$
$$\widetilde{A} = \Omega(\widetilde{H}_S), \quad \widetilde{H}_S = \widetilde{\Delta}^{-1/2} \widetilde{H} \widetilde{\Delta}^{-1/2}, \quad \widetilde{\Delta} = |diag(\widetilde{H})|.$$

If $|a_{ij}| < 1$, then

$$(i) \qquad |\widetilde{a}_{il}|^2 + |\widetilde{a}_{jl}|^2 \le \frac{|a_{il}|^2 + |a_{jl}|^2}{1 - |a_{ij}|}, \quad l \neq i, j,$$

$$(ii) \qquad \|\widetilde{A}\|^2 - \|A\|^2 \le |a_{ij}| \frac{\|A\|^2 - 2|a_{ij}|}{1 - |a_{ij}|}\,.$$

If in addition $\|\widetilde{A}\| > \|A\|$, then

$$(iii) \qquad |a_{ij}| \ \le \ \frac{1}{2}\, \|A\|^2.$$

Proof. (i) In the considered Jacobi step only the ith and the jth row and column change. If $h_{ii}h_{jj} > 0$, using the relations [12, rel. (15) and (16)], we obtain

$$(|\widetilde{a}_{il}|^2 + |\widetilde{a}_{jl}|^2) - (|a_{il}|^2 + |a_{jl}|^2) = \kappa_{il} \left[\left(\frac{|\widetilde{h}_{il}|^2}{\widetilde{h}_{ii} h_{ll}} + \frac{|\widetilde{h}_{jl}|^2}{\widetilde{h}_{jj} h_{ll}} \right) - \left(\frac{|h_{il}|^2}{h_{ii} h_{ll}} + \frac{|h_{jl}|^2}{h_{jj} h_{ll}} \right) \right],$$

where

$$\kappa_{il} = \operatorname{sgn}(h_{ii} h_{ll}),$$

and the proof follows the lines of the proof of [12, Lemma 2(i), pp. 38–39].

If $h_{ii}h_{jj} < 0$, using the relations [12, rel. (15) and (16)], we obtain

$$\begin{aligned}
&(|\widetilde{a}_{il}|^2 + |\widetilde{a}_{jl}|^2) - (|a_{il}|^2 + |a_{jl}|^2) \\
&= \kappa_{il} \left[\left(\frac{|\widetilde{h}_{il}|^2}{\widetilde{h}_{ii} h_{ll}} - \frac{|\widetilde{h}_{jl}|^2}{\widetilde{h}_{jj} h_{ll}} \right) - \left(\frac{|h_{il}|^2}{h_{ii} h_{ll}} - \frac{|h_{jl}|^2}{h_{jj} h_{ll}} \right) \right] \\
&= \kappa_{il} \left[\left(\frac{c^2}{\widetilde{h}_{ii}} - \frac{s^2}{\widetilde{h}_{jj}} - \frac{1}{h_{ii}} \right) \frac{|h_{il}|^2}{h_{ll}} + \left(-\frac{c^2}{\widetilde{h}_{jj}} + \frac{s^2}{\widetilde{h}_{ii}} + \frac{1}{h_{jj}} \right) \frac{|h_{jl}|^2}{h_{ll}} \right. \\
&\qquad \left. -2sc \left(\frac{1}{\widetilde{h}_{jj}} + \frac{1}{\widetilde{h}_{ii}} \right) \frac{\Re(e^{\imath\omega} \overline{h_{il}} h_{jl})}{h_{ll}} \right],
\end{aligned} \tag{2.11}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

where s and c denote sine and cosine of the rotation angle and $e^{\imath\omega} = h_{ij}/|h_{ij}|$. Let $t = s/c$. Since $\widetilde{h}_{ii}\widetilde{h}_{jj} = h_{ii}h_{jj} - |h_{ij}|^2$ (see the proof of [12, Lemma 2(i)]), using the relations [12, rel. (13) and (14)], one easily obtains

$$\begin{aligned}
\frac{c^2}{\widetilde{h}_{ii}} - \frac{s^2}{\widetilde{h}_{jj}} - \frac{1}{h_{ii}} &= \frac{c^2(h_{jj} + |h_{ij}|t) - s^2(h_{ii} - |h_{ij}|t)}{\widetilde{h}_{ii}\widetilde{h}_{jj}} - \frac{1}{h_{ii}} \\
&= \frac{c^2 h_{jj} - s^2 h_{ii} + |h_{ij}|t}{h_{ii}h_{jj} - |h_{ij}|^2} - \frac{1}{h_{ii}} \ = \ \frac{h_{jj} - s^2(h_{jj} + h_{ii}) + |h_{ij}|t}{h_{ii}h_{jj} - |h_{ij}|^2} - \frac{1}{h_{ii}} \\
&= \frac{|h_{ij}|^2 + \omega_1 h_{ii}}{h_{ii}(h_{ii}h_{jj} - |h_{ij}|^2)}, \qquad (2.12)
\end{aligned}$$

where $\omega_1 = |h_{ij}|t - s^2(h_{jj} + h_{ii})$. Using again the relations [12, rel. (13) and (14)], we estimate ω_1:

$$\begin{aligned}
\omega_1 &= -s^2(h_{ii} + h_{jj}) + |h_{ij}|t \\
&= -2s^2 h_{ii} - s^2(h_{jj} - h_{ii}) + |h_{ij}|t \ = \ -2s^2 h_{ii} - s^2\, 2|h_{ij}| \cot 2\varphi + |h_{ij}|t \\
&= -2s^2 h_{ii} + |h_{ij}|t(1 - 2c^2 t \cot 2\varphi) \ = \ -2s^2 h_{ii} + 2s^2|h_{ij}|t \\
&= -\frac{1}{2}\tan^2 2\varphi\, \frac{\cos^2 2\varphi}{c^2}\, h_{ii} + \tan 2\varphi\, \cos 2\varphi\, t^2|h_{ij}| \\
&= -\frac{2|h_{ij}|^2}{(h_{jj} - h_{ii})^2}\, \frac{\cos^2 2\varphi}{c^2}\, h_{ii} + \frac{2|h_{ij}|}{h_{jj} - h_{ii}}\, \cos 2\varphi\, t^2|h_{ij}| \\
&= \frac{2|h_{ij}|^2}{h_{jj} - h_{ii}}\, \cos 2\varphi \left(-\frac{h_{ii}}{h_{jj} - h_{ii}}\, \frac{\cos 2\varphi}{c^2} + t^2 \right) \\
&= \frac{2|h_{ij}|^2}{h_{jj} - h_{ii}}\, \cos 2\varphi \left[-\frac{h_{ii}}{h_{jj} - h_{ii}}\, (1 - t^2) + t^2 \right] \\
&= \frac{2|h_{ij}|^2}{h_{jj} - h_{ii}}\, \cos 2\varphi \left[-\frac{h_{ii}}{h_{jj} - h_{ii}} + t^2 \left(1 + \frac{h_{ii}}{h_{jj} - h_{ii}} \right) \right] \\
&= \frac{2|h_{ij}|^2}{h_{jj} - h_{ii}}\, \underbrace{\cos 2\varphi}_{+}\, \underbrace{\frac{t^2 h_{jj} - h_{ii}}{h_{jj} - h_{ii}}}_{+}.
\end{aligned}$$

So, in both cases, $h_{ii} > 0 > h_{jj}$ and $h_{jj} > 0 > h_{ii}$), ω_1/h_{jj} is positive and

$$\frac{\omega_1}{h_{jj}} \ \le \ \frac{2|h_{ij}|^2}{|h_{ii}h_{jj}|}\, \underbrace{\frac{|h_{ii}|}{|h_{jj} - h_{ii}|}}_{<1}\, \underbrace{\cos 2\varphi}_{<1}\, \underbrace{\frac{t^2 h_{jj} - h_{ii}}{h_{jj} - h_{ii}}}_{<1} \ < \ 2|a_{ij}|^2.$$

Combining the above relation with (2.12) we have

$$\left| \frac{c^2}{\widetilde{h}_{ii}} - \frac{s^2}{\widetilde{h}_{jj}} - \frac{1}{h_{ii}} \right| = \left| \frac{1}{h_{ii}}\, \frac{-|a_{ij}|^2 + \frac{\omega_1}{h_{jj}}}{1 + |a_{ij}|^2} \right|$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

$$
(2.13) \qquad < \frac{1}{|h_{ii}|}\,\frac{-|a_{ij}|^2+2|a_{ij}|^2}{1+|a_{ij}|^2} \;=\; \frac{1}{|h_{ii}|}\,\frac{|a_{ij}|^2}{1+|a_{ij}|^2}\,.
$$

In a similar way, one obtains

$$
-\frac{c^2}{\widetilde{h}_{jj}}+\frac{s^2}{\widetilde{h}_{ii}}+\frac{1}{h_{jj}}=\frac{-|h_{ij}|^2+\omega_2 h_{jj}}{h_{jj}(h_{ii}h_{jj}-|h_{ij}|^2)}\,,
$$

where $\omega_2=|h_{ij}|t+s^2(h_{jj}+h_{ii})$, and also

$$
\omega_2=\frac{2|h_{ij}|^2}{h_{jj}-h_{ii}}\,\underbrace{\cos 2\varphi}_{+}\,\underbrace{\frac{h_{jj}-t^2h_{ii}}{h_{jj}-h_{ii}}}_{+}\,.
$$

¿From the above expression, one can see that ω_2/h_{ii} is negative and

$$
\left|\frac{\omega_2}{h_{ii}}\right| \;\le\; \frac{2|h_{ij}|^2}{|h_{ii}h_{jj}|}\,\underbrace{\frac{|h_{jj}|}{|h_{jj}-h_{ii}|}}_{<1}\,\underbrace{\cos 2\varphi}_{<1}\,\underbrace{\frac{h_{jj}-t^2h_{ii}}{h_{jj}-h_{ii}}}_{<1} \;<\; 2|a_{ij}|^2.
$$

Combining the obtained relations, we have

$$
(2.14) \qquad \left|-\frac{c^2}{\widetilde{h}_{jj}}+\frac{s^2}{\widetilde{h}_{ii}}+\frac{1}{h_{jj}}\right|=\left|\frac{1}{h_{jj}}\,\frac{|a_{ij}|^2+\frac{\omega_2}{h_{ii}}}{1+|a_{ij}|^2}\right|<\frac{1}{|h_{jj}|}\,\frac{|a_{ij}|^2}{1+|a_{ij}|^2}\,.
$$

Using again the relations [12, rel. (13) and (14)], we have

$$
\begin{aligned}
&2sc\left(\frac{1}{\widetilde{h}_{jj}}+\frac{1}{\widetilde{h}_{ii}}\right) \;=\; 2sc\,\frac{\widetilde{h}_{ii}+\widetilde{h}_{jj}}{\widetilde{h}_{ii}\widetilde{h}_{jj}} \;=\; 2sc\,\frac{h_{jj}+h_{ii}}{h_{ii}h_{jj}-|h_{ij}|^2}\\
&=\;\tan 2\varphi\,\cos 2\varphi\,\frac{h_{jj}+h_{ii}}{h_{ii}h_{jj}-|h_{ij}|^2} \;=\; \frac{2|h_{ij}|}{h_{ii}h_{jj}-|h_{ij}|^2}\,\frac{h_{jj}+h_{ii}}{h_{jj}-h_{ii}}\,\cos 2\varphi\\
&=\;-\frac{1}{\sqrt{|h_{ii}h_{jj}|}}\,\frac{2a_{ij}}{1+|a_{ij}|^2}\,\frac{h_{jj}+h_{ii}}{h_{jj}-h_{ii}}\,\cos 2\varphi\,.
\end{aligned}
$$

Since $\left|\frac{h_{jj}+h_{ii}}{h_{jj}-h_{ii}}\,\cos 2\varphi\right|<1,$ we have

$$
(2.15) \qquad \left|2sc\left(\frac{1}{\widetilde{h}_{jj}}+\frac{1}{\widetilde{h}_{ii}}\right)\right| \;<\; \frac{1}{\sqrt{|h_{ii}h_{jj}|}}\,\frac{2|a_{ij}|}{1+|a_{ij}|^2}\,.
$$

Finally, using the relations (2.11), (2.13), (2.14) and (2.15), we obtain

$$
\begin{aligned}
&(|\widetilde{a}_{il}|^2+|\widetilde{a}_{jl}|^2) \;-\; (|a_{il}|^2+|a_{jl}|^2)\\
&<\frac{|a_{ij}|^2}{1+|a_{ij}|^2}|a_{il}|^2+\frac{|a_{ij}|^2}{1+|a_{ij}|^2}|a_{jl}|^2+\frac{2|a_{ij}|}{1+|a_{ij}|^2}|a_{il}|\,|a_{jl}|
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

$$\le \frac{|a_{ij}|^2}{1+|a_{ij}|^2}(|a_{il}|^2+|a_{jl}|^2) + \frac{|a_{ij}|}{1+|a_{ij}|^2}(|a_{il}|^2+|a_{jl}|^2)$$

$$= \frac{|a_{ij}|^2+|a_{ij}|}{1+|a_{ij}|^2}(|a_{il}|^2+|a_{jl}|^2). \tag{2.16}$$

Since

$$\frac{|a_{ij}|+|a_{ij}|^2}{1+|a_{ij}|^2} < \frac{|a_{ij}|(1+|a_{ij}|)}{1-|a_{ij}|^2} = \frac{|a_{ij}|}{1-|a_{ij}|},$$

the obtained bound $(|a_{ij}|+|a_{ij}|^2)/(1+|a_{ij}|^2)$, is better in the case $h_{ii}h_{jj} < 0$, than the bound $|a_{ij}|/(1-|a_{ij}|)$ which is obtained in the case $h_{ii}h_{jj} > 0$. Using the weaker bound in the relation (2.16), the assertion (i) immediately follows.

The proofs of (ii) and (iii) are exactly the same as those of [12, Lemma 2(ii), p. 40] and [12, Lemma 2(iii), p. 40], respectively. ▯

Lemma 3. *Let H be Hermitian matrix of order $n \ge 3$ and let N be as in the relation (2.7). Let $H^{(0)} = H$, $H^{(1)}$, $\dots$, $H^{(N)}$ be obtained by applying N Jacobi steps to H under any ordering. Let α_k be defined by the relation (2.9). If*

$$\alpha_0 \le \frac{1}{10n},$$

then

$$\alpha_k^2 \le c_k \alpha_0^2, \quad 0 \le k \le N,$$

where

$$c_k = \left(1+\frac{0.00126}{n^2}\right)^k < 1.0007, \quad 0 \le k \le N.$$

Proof. The proof goes in the same way as the proof of [12, Lemma 3] or [11, Lemma 3, pp. 178–179] with suitable modifications of constants. ▯

Lemma 4. *Let H be as in Lemma 3. Let $H^{(0)} = H$, $H^{(1)}$, $\dots$, $H^{(N)}$ be obtained by applying N Jacobi steps to H under the column-cyclic strategy. Let*

$$\eta_{sr}^{(k)} = \left[a_{1r}^{(k)}, \dots, a_{sr}^{(k)}\right]^T, \quad 1 \le s < r \le n, \ 0 \le k \le N$$

and

$$Q_s = 1+2+\cdots+(s-1). \tag{2.17}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

If

$$\alpha_0 \le \frac{1}{10n},$$

then

$$\|\eta_{sr}^{(Q_s)}\|^2 \le K_s \|\eta_{sr}^{(0)}\|^2,$$

where

$$K_s = (1 - \alpha/\sqrt{2})^{-(2s-3)} < 1.1581$$

and

$$\alpha = \sqrt{1.0007}\,\alpha_0\,. \tag{2.18}$$

Proof. The proof follows the lines in the proof of [12, Lemma 4, p. 40] and [11, Lemma 9, pp. 190–193], where we use Lemma 3 to estimate K_s. □

LEMMA 5. *Let $H^{(0)} = H$, $H^{(1)}, \ldots, H^{(N)}$ be as in Lemma 4. In addition, let $H^{(0)}$ satisfy (2.3) and*

$$\alpha_0 \le \frac{1}{10} \min\left\{\frac{1}{n}, \gamma\right\}, \quad n \ge 3, \tag{2.19}$$

where α_0 and γ are defined by the relations (2.9) and (2.5), respectively. Then the following relations hold for $0 \le k \le N$,

(*i*) $(1 - 0.0201\gamma)\lambda_{s_r} < h_{tt}^{(k)} < (1 + 0.0202\gamma)\lambda_{s_r}, \quad t \in \mathcal{N}_r,\ 1 \le r \le p\,,$

(*ii*) $rg(h_{tt}^{(k)}, h_{qq}^{(k)}) \ge 0.9605\gamma, \quad t \in \mathcal{N}_l,\ q \in \mathcal{N}_r,\ l \ne r,$

(*iii*) $|\tan\varphi^{(k)}| \le \dfrac{|a_{ij}^{(k)}|}{2\,0.9605\gamma} \le 0.5206\dfrac{|a_{ij}^{(k)}|}{\gamma}, \quad i \in \mathcal{N}_l,\ j \in \mathcal{N}_r,\ l \ne r,$
where $(i,j) = (i(k), j(k))$ is the pivot pair.

If $h_{tt}^{(k)} h_{qq}^{(k)} < 0$ in (ii) or $h_{ii}^{(k)} h_{jj}^{(k)} < 0$ in (iii), then the constant 0.9605γ can be replaced by 1.

Proof. (*i*) The proof follows the lines in the proof of [12, Lemma 5(*i*), pp. 41–43]. The difference appears in using Theorem 1(*i*) for Hermitian α_k-s.d.d. matrix $\tilde{H}^{(k)}$, instead of using the corresponding result for positive definite matrices. Thus, we obtain for $1 \le r \le p$,

$$1 - \frac{2\alpha_k^2}{\gamma_r} < 1 - \frac{4\|\tau_r(\tilde{H}_S^{(k)})\|^2}{\gamma_r} \le \frac{\lambda_{s_r}}{\tilde{h}_{tt}^{(k)}} \le 1 + \frac{4\|\tau_r(\tilde{H}_S^{(k)})\|^2}{\gamma_r} < 1 + \frac{2\alpha_k^2}{\gamma_r}, \quad t \in \mathcal{N}_r\,.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Hence, in the same way as in [12, Lemma 5(i)], we obtain

$$\lambda_{s_r}\left(1-\frac{2\,1.0007}{10^2}\,\gamma\right)\;\leq\;\widetilde{h}_{tt}^{(k)}\;\leq\;\lambda_{s_r}\left(1+\frac{\frac{2\,1.0007}{10^2}}{1-\frac{2\,1.0007}{10\,30}}\,\gamma\right).$$

According to this relation, we modify the constants in the rest of the proof, and obtain (i).

(ii) From the definition of the function rg(,), we have $\mathrm{rg}(h_{tt}^{(k)},h_{qq}^{(k)})=1$ provided that $h_{tt}^{(k)}h_{qq}^{(k)}<0$. If $h_{tt}^{(k)}h_{qq}^{(k)}>0$, we use ($i$) and follow the lines in the proof of [12, Lemma 5(ii)] or [11, Lemma 5(i), pp. 181–182]. We also use $\kappa=(1-0.0201\gamma)/(1+0.0202\gamma)$.

(iii) As above in the case of opposite sign, we have

$$|\tan\varphi^{(k)}|\;\leq\;\frac{|h_{ij}^{(k)}|}{|h_{jj}^{(k)}-h_{ii}^{(k)}|}\;=\;\frac{|h_{ij}^{(k)}|}{\sqrt{|h_{ii}^{(k)}h_{jj}^{(k)}|}}\,\frac{\sqrt{|h_{ii}^{(k)}h_{jj}^{(k)}|}}{|h_{jj}^{(k)}-h_{ii}^{(k)}|}\;\leq\;\frac{1}{2}|a_{ij}^{(k)}|\,.$$

In the case of same sign, we use the assertion (ii) and follow the lines in the proof of [12, Lemma 5(iii), p. 43] or [11, Lemma 5(ii), p. 182]. ◻

3. Quadratic convergence of scaled iterates.

3.1. Asymptotic assumptions.

According to the conditions used in Lemmas 4 and 5, and the assumptions used in [12, p. 25], we formulate the following asymptotic assumptions:

(A1) H is a complex or real Hermitian matrix of order $n\geq 3$, satisfying

$$\alpha_0\;\leq\;\frac{1}{10}\,\min\left\{\frac{1}{n},\,\gamma\right\},$$

where α_0 and γ are defined by the relations (2.9) and (2.5), respectively.

(A2) The diagonal elements of H satisfy the relation (2.3) i.e.

$$\text{for all } t\in\mathcal{N}_r,\qquad h_{tt}\text{ is affiliated with }\lambda_{s_r},\quad 1\leq r\leq p\,,$$

where the sets $\mathcal{N}_r$, $1\leq r\leq p$, are defined by the relation (2.2).

3.2. The Main Theorem.

Here we state and prove the main result.

THEOREM 6. *Let H satisfy the asymptotic assumptions* (A1) *and* (A2). *Let the sequence $H^{(0)}=H$, $H^{(1)}$, …, $H^{(N)}$ be generated by the column-cyclic Jacobi method. Then*

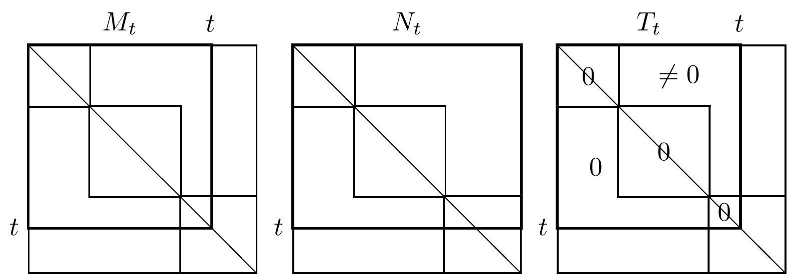
$$\alpha_N\;\leq\;2.8\,\frac{\alpha_0^2}{\gamma}\,,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

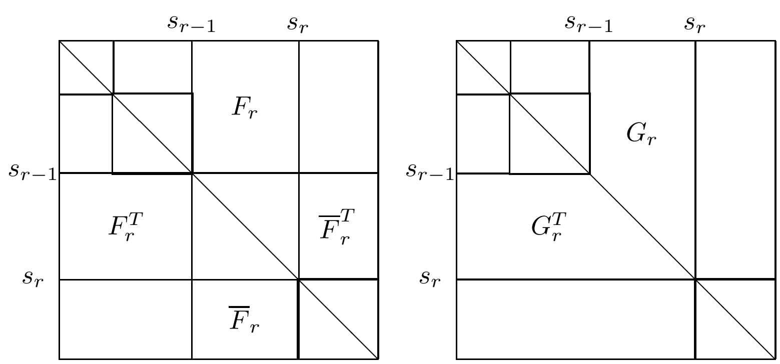
where α_0, α_N and γ are defined by the relations (2.9) and (2.5), respectively.

Proof. We provide here just an outline of the proof. One has to follow the proof of [12, Theorem 6].

The proof of Theorem 6 uses induction over the set $\{1, 2, \ldots, p\}$. We use the notation from [12, Section 5] (relations [12, rel. (28)–(31), p. 26] and figures [12, Fig. 1,2, p. 27]). The matrices and matrix blocks appearing in the proof are sketched below.



The matrices M_t, N_t and T_t.



The blocks F_r, $\overline{F}_r$ and G_r.

In the proof of Theorem 1, we shall use the inequality

$$\|T_{s_r}^{(Q_{s_r})}\| \,\leq\, C_r\,\frac{\|N_{s_r}\|^2}{\gamma}\,, \quad 1 \leq r \leq p\,, \tag{3.1}$$

where Q_{s_r} is given by (2.17) and

$$C_r \,=\, 1.8\,\xi^{s_r} \prod_{i=1}^{r} \left(1 + 30\frac{\|G_i\|^2}{\gamma^2}\right)^{1/2}, \quad 1 \leq r \leq p\,. \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

Here

$$\xi \;=\; \left(1 - 0.521\,\frac{\alpha_0^2}{\gamma}\right)^{-1}. \tag{3.3}$$

Using the assumption (A1), we have $\alpha_0/\gamma \le 1/10$ and $\alpha_0 \le 1/(10n) \le 1/30$, so that

$$\xi \;\le\; \left(1 - 0.521\,\frac{\alpha_0}{\gamma}\alpha_0\right)^{-1} \;\le\; 1.0018\,. \tag{3.4}$$

Using the inequality

$$\prod_l (1 + x_l) \;\le\; \left(1 - \sum_l x_l\right)^{-1}, \quad x_l \ge 0,\;\; \sum_l x_l < 1, \tag{3.5}$$

with $x = \frac{0.521\alpha_0^2/\gamma}{1-0.521\alpha_0^2/\gamma}$ and the assumption (A1), we obtain

$$\xi^n \;=\; (1+x)^n \;\le\; (1-nx)^{-1} \;\le\; \left(1 - \frac{0.521\,n\alpha_0\,\alpha_0/\gamma}{1-0.521\,\alpha_0\,\alpha_0/\gamma}\right)^{-1} \;<\; 1.0053\,. \tag{3.6}$$

Similarly, using the relations (3.5) and (3.6), and the assumption (A1), we obtain from the relation (3.2)

$$C_r^2 \;\le\; \xi^{2s_r}\,\frac{1.8^2}{1-30\sum_{i=1}^r \frac{\|G_i\|^2}{\gamma^2}} \;\le\; \xi^{2n}\,\frac{1.8^2}{1-30\frac{\alpha_0^2}{2\gamma^2}} \;\le\; 1.0053^2\,\frac{1.8^2}{1-15\,(1/10)^2} \;<\; 3.853\,.$$

Therefore, C_r is uniformly bounded from above and from below,

$$1.8 \;\le\; C_r \;<\; \sqrt{3.853}\,, \quad 1 \le r \le p\,. \tag{3.7}$$

The proof of the inequality (3.1) uses induction with respect to r. As in [12, p. 28], we divide it into three parts: the induction base is checked in PART I and the induction step is proved in PART II and PART III.

PART I

We assume that

$$\|T_{s_{r-1}}^{(I)}\| \;\le\; C_{r-1}\,\frac{\|N_{s_{r-1}}\|^2}{\gamma} \tag{3.8}$$

holds for some $2 \le r \le p$, where $I := Q_{s_{r-1}} = 1 + 2 + \cdots + (s_{r-1}-1)$. The induction base (for $r = 2$) is now trivial because $T_{s_1}^{(I)} = O$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

PART II

Let $II := Q_{s_{r-1}} + s_{r-1}(s_r - s_{r-1}) = I + s_{r-1}n_r$. We prove in this part

$$\|T^{(II)}_{s_{r-1}}\| \le \xi^{s_r - s_{r-1}}\, C_{r-1}\, \frac{\|N_{s_r}\|^2}{\gamma}, \tag{3.9}$$

$$\|F^{(II)}_r\| \le 8.983 \frac{\|N_{s_r}\|^2}{\gamma^2} \|G_r\|\,. \tag{3.10}$$

PART III

Let $III := Q_{s_r} = 1 + 2 + \cdots + (s_r - 1) = II + n_r(n_r - 1)/2$. We prove

$$\|F^{(III)}_r\| \le 1.015\, \|F^{(II)}_r\|, \tag{3.11}$$

$$T^{(III)}_{s_{r-1}} = T^{(II)}_{s_{r-1}}\,. \tag{3.12}$$

This is illustrated in [12, Fig. 3, p. 29].

To complete the induction step, we use the relations (3.12), (3.11), (3.9), (3.10), (3.2) and (3.7). We have

$$\begin{aligned}
\|T^{(III)}_{s_r}\|^2 &= \|T^{(III)}_{s_{r-1}}\|^2 + \|F^{(III)}_r\|^2 \;\le\; \|T^{(II)}_{s_{r-1}}\|^2 + 1.015^2 \|F^{(II)}_r\|^2 \\
&\le \xi^{2(s_r - s_{r-1})} C^2_{r-1} \frac{\|N_{s_r}\|^4}{\gamma^2} + 1.015^2\, 8.983^2\, \frac{\|N_{s_r}\|^4}{\gamma^4} \|G_r\|^2 \\
&\le \xi^{2(s_r - s_{r-1})} C^2_{r-1} \left[1 + \left(\frac{1.015\, 8.983}{1.8}\right)^2 \frac{\|G_r\|^2}{\gamma^2}\right] \frac{\|N_{s_r}\|^4}{\gamma^2} \;\le\; C^2_r \frac{\|N_{s_r}\|^4}{\gamma^2}\,,
\end{aligned}$$

and the relation (3.1) is now proved. Now, we can complete the proof of the main theorem,

$$\begin{aligned}
\alpha^2_N &= 2\|T^{(N)}_{s_p}\|^2 + \sum_{i=1}^{p} \|A^{(N)}_{ii}\|^2 \\
&\le 2\|T^{(N)}_{s_p}\|^2 + \frac{8}{\gamma^2} \left(2\|T^{(N)}_{s_p}\|^2\right)^2 \;\le\; 2\|T^{(N)}_{s_p}\|^2 \left(1 + \frac{16}{\gamma^2}\|T^{(N)}_{s_p}\|^2\right) \\
&\le 2C^2_p \frac{\|N_{s_p}\|^4}{\gamma^2} \left(1 + 16C^2_p \frac{\|N_{s_p}\|^4}{\gamma^4}\right) \;\le\; 2C^2_p \frac{\alpha_0^4}{\gamma^2} \left(1 + 16C^2_p \frac{\alpha_0^4}{\gamma^4}\right) \\
&\le 2\, 3.853\, (1 + 16\, 3.853\, (1/10)^4)\, \frac{\alpha_0^4}{\gamma^2} \;\le\; 2.8^2\, \frac{\alpha_0^4}{\gamma^2}\,.
\end{aligned} \tag{3.13}$$

Here, we have used Theorem 1(ii) to bound $\sum_{i=1}^p \|A^{(N)}_{ii}\|^2$. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

3.3. Details of the Proof. Here we present the proof of the relation (3.1). The proof is quite similar to the proof in [12, Section 5.3, pp. 30–35], so we avoid here full explanation of all details.

PART II

In the following lemma, we use

$$w_m = Q_{s_{r-1}} + (m-1-s_{r-1})s_{r-1}, \quad s_{r-1}+1 \le m \le s_r\,. \tag{3.14}$$

Lemma 7. *Let H satisfy the assumptions of Theorem 6. Let w_m be defined by the relation (3.14). Then*

(i) $a_{km}^{(w_m+k)} = 0, \quad 1 \le k \le s_{r-1}\,,$

(ii) $|a_{lm}^{(w_m+k)}| \le 1.0027\left(|a_{lm}^{(w_m)}| + \dfrac{1.0412}{\gamma}\displaystyle\sum_{t=1}^{k}|a_{lt}^{(w_m)}a_{tm}^{(w_m+t-1)}|\right),$
$1 \le k \le s_{r-1}, \quad k < l \le m-1$

(iii) $|a_{lm}^{(w_m+k)}| \le \dfrac{1.045}{\gamma}\displaystyle\sum_{t=l+1}^{k}|a_{tm}^{(w_m+t-1)}|\left(|a_{lt}^{(w_m)}| + \dfrac{1.0412}{\gamma}|a_{lm}^{(w_m+l-1)}a_{tm}^{(w_m+l-1)}|\right),$
$1 \le l < k \le s_{r-1}$

(iv) $|a_{lt}^{(w_m+s_{r-1})}| \le \xi\,|a_{lt}^{(w_m)}| + \dfrac{1.043}{\gamma}\left(|a_{lm}^{(w_m+l-1)}a_{tm}^{(w_m+l-1)}| + |a_{lm}^{(w_m+t-1)}a_{tm}^{(w_m+t-1)}|\right),$
$1 \le l \ne t \le s_{r-1}$

(v) $|a_{lt}^{(w_m+s_{r-1})}| \le \xi\,|a_{lt}^{(w_m)}| + \dfrac{1.043}{\gamma}\,|a_{lm}^{(w_m+l-1)}a_{tm}^{(w_m+l-1)}|, \quad 1 \le l \le s_{r-1} < t < m,$

where ξ in (iv) and (v) is given by the relation (3.3).

Proof. In the proof we shall omit the index w_m.

(i) This obvious since $a_{km}^{(k)}$ is the annihilated pivot element.

(ii) We follow the proof of [12, Lemma 8(ii)] or [11, Lemma 7(ii), pp. 184–185], and use Lemmas 5(ii) and 3, and the relations (2.18), (3.3) and (3.6) to obtain

$$\frac{|s^{(t-1)}h_{lt}|}{\sqrt{|h_{ll}h_{mm}^{(k)}|}} \le \frac{1}{0.9605\gamma}\,|a_{tm}^{(t-1)}a_{lt}|\sqrt{\frac{|h_{mm}^{(t-1)}|}{|h_{mm}^{(k)}|}}, \quad 1 \le t \le k\,, \tag{3.15}$$

$$\begin{aligned}|h_{mm}^{(q)}| &\ge h_{mm}^{(q-1)} - |h_{qm}^{(q-1)}\tan\varphi^{(q-1)}| \;\ge\; |h_{mm}^{(q-1)}|\left(1-\frac{\frac{\alpha^2}{2}}{0.9605\gamma}\right)\\ &\ge |h_{mm}^{(q-1)}|\left(1-\frac{1.0007\alpha_0^2}{2\,0.9605\gamma}\right) \;\ge\; |h_{mm}^{(q-1)}|\xi^{-1}, \quad 1 \le q \le s_{r-1}\end{aligned} \tag{3.16}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

and

$$\frac{|h_{mm}^{(t-1)}|}{|h_{mm}^{(k)}|} \leq \xi^{k-t+1} \leq \xi^n < 1.0053, \quad 1 \leq t \leq k \leq s_{r-1}. \tag{3.17}$$

This implies (ii).

Since $h_{tt}^{(k)} = h_{tt}^{(t)} = h_{tt} - |h_{tm}^{(t-1)}| \tan\varphi^{(t-1)}$, $1 \leq t \leq k \leq s_{r-1}$, we have the relation analogous to (3.16), i.e.,

$$\frac{|h_{tt}|}{|h_{tt}^{(k)}|} \leq \left(1 - \frac{\frac{\alpha^2}{2}}{0.9605\gamma}\right)^{-1} \leq \xi \tag{3.18}$$

$$< 1.0018, \quad 1 \leq t \leq k \leq s_{r-1}\,. \tag{3.19}$$

(iii) Here, in the proof of [12, Lemma 8(iii)] or [11, Lemma 7(iii), pp. 185–186], we cannot estimate the term $|h_{tt}|/|h_{ll}|$ for all $t > l$. However, we can easily overcome this problem by using a proper association of the factors as follows. Using the relations (3.15), (3.17) and (3.19), we have

$$\begin{aligned}
\frac{|s^{(t-1)} h_{lt}|}{\sqrt{|h_{ll}^{(k)} h_{mm}^{(k)}|}} &\leq \frac{1}{0.9605\gamma} |a_{tm}^{(t-1)} a_{lt}| \sqrt{\frac{|h_{mm}^{(t-1)}|}{|h_{mm}^{(k)}|}} \sqrt{\frac{|h_{ll}|}{|h_{ll}^{(k)}|}} \\
&\leq \frac{1}{0.9605\gamma} |a_{tm}^{(t-1)} a_{lt}| \sqrt{1.0053\, 1.0018}, \quad 1 \leq t \leq k\,.
\end{aligned}$$

Now, using the relation [12, rel. (13)], Lemma 5(ii) and the relations (3.17) and (3.19), we have

$$\begin{aligned}
\frac{|s^{(t-1)} s^{(l-1)} h_{tm}^{(l-1)}|}{\sqrt{|h_{ll}^{(k)} h_{mm}^{(k)}|}} &\leq \frac{|h_{tm}^{(t-1)}|}{|h_{tt} - h_{mm}^{(t-1)}|} \frac{|h_{lm}^{(l-1)}|}{|h_{ll} - h_{mm}^{(l-1)}|} \frac{|h_{tm}^{(l-1)}|}{\sqrt{|h_{tt} h_{mm}^{(l-1)}|}} \sqrt{\frac{|h_{tt} h_{mm}^{(l-1)}|}{|h_{ll}^{(k)} h_{mm}^{(k)}|}} \\
&= \frac{|a_{tm}^{(t-1)}|}{\mathrm{rg}(h_{tt}, h_{mm}^{(t-1)})} \frac{\sqrt{|h_{tt} h_{mm}^{(t-1)}|}}{|h_{tt}| + |h_{mm}^{(t-1)}|} \frac{|a_{lm}^{(l-1)}|}{\mathrm{rg}(h_{ll}, h_{mm}^{(l-1)})} \frac{\sqrt{|h_{ll} h_{mm}^{(l-1)}|}}{|h_{ll}| + |h_{mm}^{(l-1)}|} \\
&\quad \cdot |a_{tm}^{(l-1)}| \sqrt{\frac{|h_{tt} h_{mm}^{(l-1)}|}{|h_{ll}^{(k)} h_{mm}^{(k)}|}} \\
&\leq \frac{|a_{tm}^{(t-1)} a_{lm}^{(l-1)} a_{tm}^{(l-1)}|}{(0.9605\gamma)^2} \frac{|h_{tt}|}{|h_{tt}| + |h_{mm}^{(t-1)}|} \frac{|h_{mm}^{(l-1)}|}{|h_{ll}| + |h_{mm}^{(l-1)}|} \sqrt{\frac{|h_{ll} h_{mm}^{(t-1)}|}{|h_{ll}^{(k)} h_{mm}^{(k)}|}} \\
&\leq \frac{|a_{tm}^{(t-1)} a_{lm}^{(l-1)} a_{tm}^{(l-1)}|}{(0.9605\gamma)^2} \sqrt{1.0053\, 1.0018}\,.
\end{aligned}$$

The obtained relations yield the assertion (iii).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

(iv) In a similar way as in the proof of (iii) (see the proof of [12, Lemma 8(iv)] or [11, Lemma 7(iv), pp. 186–187]), we obtain

$$\frac{|h_{lt}|}{\sqrt{h_{ll}^{(s_{r-1})}h_{tt}^{(s_{r-1})}}} \le |a_{lt}| \sqrt{\frac{|h_{ll}h_{tt}|}{|h_{ll}^{(s_{r-1})}h_{tt}^{(s_{r-1})}|}} \le \xi\, |a_{lt}|\,,$$

$$\frac{|s^{(l-1)}h_{tm}^{(l-1)}|}{\sqrt{h_{ll}^{(s_{r-1})}h_{tt}^{(s_{r-1})}}} \le \frac{|a_{lm}^{(l-1)}a_{tm}^{(l-1)}|}{0.9605\gamma} \underbrace{\sqrt{\frac{|h_{tt}^{(l-1)}h_{ll}|}{|h_{tt}^{(s_{r-1})}h_{ll}^{(s_{r-1})}|}}}_{\le 1.0018}$$

and

$$\frac{|s^{(t-1)}h_{lm}^{(t-1)}|}{\sqrt{h_{ll}^{(s_{r-1})}h_{tt}^{(s_{r-1})}}} \le \frac{|a_{tm}^{(t-1)}a_{lm}^{(t-1)}|}{0.9605\gamma} \underbrace{\sqrt{\frac{|h_{ll}^{(t-1)}h_{tt}|}{|h_{ll}^{(s_{r-1})}h_{tt}^{(s_{r-1})}|}}}_{\le 1.0018},$$

which together imply (iv).

(v) We use the assertion (iv) and the proof of [12, Lemma 8(v), p. 43]. ◻

Using Lemma 7, we can estimate the elements of the mth column, prior and after annihilations. Let

$$\eta_m^{(k)} = \left[a_{1m}^{(k)}, \ldots, a_{s_{r-1},m}^{(k)}\right], \quad s_{r-1}+1 \le m \le s_r,\ 0 \le k \le N, \tag{3.20}$$

$$\rho_m = \left[a_{1m}, a_{2m}, \ldots, a_{m-1,m}\right], \quad s_{r-1}+1 \le m \le s_r\,. \tag{3.21}$$

LEMMA 8. *Let H satisfy the assumptions of Theorem 6. Let w_m, $\eta_m^{(w_m)}$ and $M_{s_{r-1}}^{(w_m)}$ be defined by the relations (3.14), (3.20) and [12, rel. (28)], respectively. Then*

$$(i) \quad \sum_{l=1}^{s_{r-1}} (a_{lm}^{(w_m+q_l)})^2 \le \mu_m^2 \|\eta_m^{(w_m)}\|^2 \quad \textit{for any} \quad 0 \le q_l < l,\ \ 1 \le l \le s_{r-1},$$

$$(ii) \quad \sum_{l=1}^{s_{r-1}} (a_{lm}^{(w_m+q_l)})^2 \le \frac{0.547\mu_m^2}{\gamma^2}\, \|\eta_m^{(w_m)}\|^2 \left(\|M_{s_{r-1}}^{(w_m)}\| + \frac{1.473\mu_m^2}{\gamma} \|\eta_m^{(w_m)}\|^2 \right)^2$$
for any $l \le q_l \le s_{r-1}$, $1 \le l \le s_{r-1}$,

where

$$\mu_m = \frac{1.0027}{1 - \frac{0.7383}{\gamma}\, \|M_{s_{r-1}}^{(w_m)}\|}\,. \tag{3.22}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Proof. We follow the lines in the proof of [12, Lemma 9] or [11, Lemma 8, pp. 188–190]. We use Lemma 7(ii) to prove (i) and Lemma 7(iii) to prove (ii). □

The next lemma estimates the norms of the matrices $M_{s_{r-1}}$ and $T_{s_{r-1}}$ after the annihilations in the mth column of the block F_r are completed.

LEMMA 9. *Let H satisfy the assumptions of Theorem 6. Then for $s_{r-1}+1 \le m \le s_r$ hold*

$$\begin{aligned}(i)\quad \|M_{s_{r-1}}^{(w_m+1)}\| \;&\le\; \xi\,\|M_{s_{r-1}}^{(w_m)}\| \\ &+ \frac{1.476\mu_m^2\|\eta_m^{(w_m)}\|^2}{\gamma}\left[1+\frac{\sqrt{0.547}}{\gamma}\left(\|M_{s_{r-1}}^{(w_m)}\| + \frac{1.473\mu_m^2}{\gamma}\,\|\eta_m^{(w_m)}\|^2\right)\right],\\ (ii)\quad \|T_{s_{r-1}}^{(w_m+1)}\| \;&\le\; \xi\,\|T_{s_{r-1}}^{(w_m)}\| \\ &+ \frac{1.043\mu_m^2\|\eta_m^{(w_m)}\|^2}{\gamma}\left[1+\frac{\sqrt{0.547}}{\gamma}\left(\|M_{s_{r-1}}^{(w_m)}\| + \frac{1.473\mu_m^2}{\gamma}\,\|\eta_m^{(w_m)}\|^2\right)\right],\end{aligned}$$

where w_m, $\eta_m^{(k)}$, μ_m and ξ are defined by the relations (3.14), (3.20), (3.22) and (3.3), respectively.

Proof. Using Lemmas 7(iv) and 8, the proof follows the lines of the proof of [12, Lemma 10, pp. 43–45]. □

LEMMA 10. *Let H satisfy the assumptions of Theorem 6. Let $s_{r-1}+1 \le q < m \le s_r$ and let w_q, w_m, $\eta_q^{(k)}$, $\eta_m^{(k)}$, μ_q and μ_m be defined by the relations (3.14), (3.20) and (3.22). Then*

$$(i)\quad |a_{qm}^{(w_m+l)}| \le 1.0027\left(|a_{qm}^{(w_m)}| + \frac{1.0412\mu_m}{\gamma}\|\eta_q^{(w_m)}\|\,\|\eta_m^{(w_m)}\|\right),\quad 0 \le l \le s_{r-1},$$

$$(ii)\quad |a_{qm}^{(w_m)}| = |a_{qm}^{(w_q+1)}| \le 1.0027\left(|a_{qm}^{(w_q)}| + \frac{1.0412\mu_q}{\gamma}\|\eta_q^{(w_q)}\|\,\|\eta_m^{(w_q)}\|\right).$$

Proof. Using Lemmas 7(ii) and 8(i), the proof is implied by the proof of [12, Lemma 11, pp. 45–46]. □

LEMMA 11. *Let H satisfy the assumptions of Theorem 6. If the hypothesis (3.8) holds, then*

$$\|M_{s_{r-1}}^{(I)}\| \;\le\; 5.668\,\frac{\|N_{s_{r-1}}\|^2}{\gamma}\,,\quad I = 1+2+\cdots+(s_{r-1}-1).$$

Proof. Following the lines in the proof of [12, Lemma 12, p. 46], we apply first

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Theorem 1(i) and the relation (3.8), to obtain

$$\|M^{(I)}_{s_{r-1}}\|^2 \;\le\; 2C^2_{r-1}\frac{\|N_{s_{r-1}}\|^4}{\gamma^2} + \frac{16}{\gamma^2}\left(2\|T^{(I)}_{s_{r-1}}\|^2 + \|N^{(I)}_{s_{r-1}}\|^2 - \|M^{(I)}_{s_{r-1}}\|^2\right)^2 .$$

After that, we apply Lemma 4, the relation (3.7) and the assumption (A1). We obtain

$$\|M^{(I)}_{s_{r-1}}\|^2 \;\le\; \frac{\|N_{s_{r-1}}\|^4}{\gamma^2}\left[2\,3.853 + 16\left(2\,\frac{1}{100}\,3.853 + 1.1581\right)^2\right] \;\le\; 5.668^2\,\frac{\|N_{s_{r-1}}\|^4}{\gamma^2}$$

which completes the proof. □

Lemma 12. *Let H satisfy the assumptions of Theorem 6. Let w_m and μ_m be defined by the relations (3.14) and (3.22), respectively. If the hypothesis (3.8) holds, then for $s_{r-1}+1 \le m \le s_r$, it holds that*

$$(i)\quad \mu_m \;\le\; \mu = 1.05,$$

$$(ii)\quad \|M^{(w_{m+1})}_{s_{r-1}}\| \;\le\; 5.668\,\xi^{m-s_{r-1}}\,\frac{\|N_m\|^2}{\gamma},$$

$$(iii)\quad \|T^{(w_{m+1})}_{s_{r-1}}\| \;\le\; \xi^{m-s_{r-1}}C_{r-1}\frac{\|N_m\|^2}{\gamma}.$$

Proof. We use the same technique as in the proof of [12, Lemma 13, pp. 46–48]. The induction base ($m = s_{r-1}+1$) is proved by using the relation (3.22) and Lemmas 11, 9 and 4. The induction step ($m-1 \to m$) is proved by using the assertions (ii) and (iii) as the induction hypothesis. Thus, using the relations (3.22) and (3.6), the assertion (ii) (for $m-1$), and the assumption (A1), we obtain

$$\mu_m \;\le\; \frac{1.0027}{1 - 0.7383\,5.668\,1.0053\,(1/10)^2} \;<\; 1.05 \;=\; \mu$$

which is (i) (for m). Now, using Lemma 9(i), the assertions (i) (for m) and (ii) (for $m-1$), Lemma 4, the relation (3.6) and the assumption (A1), we have

$$\begin{aligned}
&\|M^{(w_{m+1})}_{s_{r-1}}\| \\
&\le \xi\,\|M^{(w_m)}_{s_{r-1}}\| + \frac{1.476\mu^2\|\eta^{(w_m)}_m\|^2}{\gamma}\left[1 + \frac{\sqrt{0.547}}{\gamma}\left(\|M^{(w_m)}_{s_{r-1}}\| + \frac{1.473\mu^2}{\gamma}\|\eta^{(w_m)}_m\|^2\right)\right] \\
&\le \xi\,5.668\xi^{m-1-s_{r-1}}\,\frac{\|N_{m-1}\|^2}{\gamma} + \frac{1.476\,1.05^2\,1.1581\|\rho_m\|^2}{\gamma} \\
&\quad\cdot\left[1 + \frac{\sqrt{0.547}}{\gamma}\left(5.668\,\xi^{m-1-s_{r-1}}\,\frac{\|N_{m-1}\|^2}{\gamma} + \frac{1.473\,1.05^2 1.1581}{\gamma}\|\rho_m\|^2\right)\right]
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

$$
\begin{aligned}
&\le 5.668\,\xi^{m-s_{r-1}}\,\frac{\|N_{m-1}\|^2}{\gamma} + 1.885\,\frac{\|\rho_m\|^2}{\gamma}\left(1+\frac{\sqrt{0.547}}{\gamma}5.668\,\xi^{m-1-s_{r-1}}\,\frac{\|N_m\|^2}{\gamma}\right)\\
&\le 5.668\,\xi^{m-s_{r-1}}\,\frac{\|N_{m-1}\|^2}{\gamma} + 1.885\,\frac{\|\rho_m\|^2}{\gamma}\left(1+\sqrt{0.547}\;5.668\,1.0053\,\frac{\alpha_0^2}{\gamma^2}\right)\\
&\le 5.668\,\xi^{m-s_{r-1}}\,\frac{\|N_m\|^2}{\gamma}\,,
\end{aligned}
$$

that is (ii) (for m). Similarly, using Lemma 9(ii) and the assertion (iii) (for $m-1$), we obtain (iii) (for m). This makes the induction step, and thus, it completes the proof. ▯

For $m = s_r$, the assertion (iii) is just the relation (3.9).

LEMMA 13. *Let H satisfy the assumptions of Theorem 6. Let $s_{r-1}+1 \le q \le s_r$ and let w_m, $\eta_m^{(k)}$ and ρ_m be defined by the relations (3.14), (3.20) and (3.21), respectively. If the hypothesis (3.8) holds, then*

$$
\|\eta_q^{(w_{s_r+1})}\| \;\le\; 1.013\,\|\eta_q^{(w_{q+1})}\| + \frac{1.198}{\gamma}\sum_{m=q+1}^{s_r} |a_{qm}^{(w_{q+1})}|\,\|\rho_m\|\,,
$$

where for $q = s_r$, the empty sum is assumed to be zero.

Proof. In the same way as in the proof of [12, Lemma 14, pp. 48–49], using Lemmas 7(v), 10(i), 8(i), 12(i) and 4 together with the relation (3.6), we obtain the following estimates

$$
\|\eta_q^{(w_{m+1})}\| \le \underbrace{\left(\xi + \frac{1.043\;1.0412\;1.0027\;\mu^2\;1.1581}{\gamma^2}\;\|\rho_m\|^2\right)}_{z_m}\|\eta_q^{(w_m)}\|
$$

$$
+\underbrace{\frac{1.043\;\mu\,1.0027\,\sqrt{1.1581}}{\gamma}}_{z}\,|a_{qm}^{(w_{q+1})}|\,\|\rho_m\|\,,
$$

$$
\|\eta_q^{(w_{s_r+1})}\| \;\le\; \|\eta_q^{(w_{q+1})}\|\prod_{m=q+1}^{s_r} z_m \;+\; z\sum_{m=q+1}^{s_r} z_{s_r}\,\cdots\,z_{m+1}\,|a_{qm}^{(w_{q+1})}|\,\|\rho_m\|\,,
$$

and

$$
\begin{aligned}
\prod_{m=q+1}^{s_r} z_m &\le \xi^n\prod_{m=2}^{n}\left(1+1.391\,\frac{\|\rho_m\|^2}{\gamma^2}\right) \;\le\; \xi^n\left(1-\frac{1.391}{\gamma^2}\sum_{m=2}^{n}\|\rho_m\|^2\right)^{-1}\\
&\le 1.0053\left(1-1.391\,\frac{1}{2}\,\frac{1}{100}\right)^{-1} \;<\; 1.013\,,
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

which imply the desired assertion. □

Now we are able to prove the relation (3.10).

LEMMA 14. *Let H satisfy the assumptions of Theorem 6. If the hypothesis (3.8) holds, then*

$$\|F_r^{(II)}\| \;\leq\; 8.983\, \frac{\|N_{s_r}\|^2}{\gamma^2}\, \|G_r\|\,.$$

Proof. We follow the lines in the proof of [12, Lemma 15, pp. 49–51]. First, we use Lemmas 8(ii), 12(i), 4 and 12(ii), and the relation (3.6), to obtain

$$\begin{aligned}
&\|\eta_q^{(w_{q+1})}\|^2 \\
&\leq \frac{0.547\mu^2}{\gamma^2}\, 1.1581\, \|\rho_q\|^2 \left(5.668\, \xi^{q-1-s_{r-1}}\, \frac{\|N_{q-1}\|^2}{\gamma} + \frac{1.437\mu^2\, 1.1581}{\gamma}\, \|\rho_q\|^2\right)^2 \\
&\leq \frac{0.547\, 1.05^2}{\gamma^2}\, 1.1581\, \|\rho_q\|^2 \left(5.668\, \xi^n \frac{\|N_q\|^2}{\gamma}\right)^2 \;\leq\; 22.677\, \frac{\|\rho_q\|^2}{\gamma^2}\, \frac{\|N_{s_r}\|^4}{\gamma^2}\,.
\end{aligned}$$

After that, we use Lemmas 10(ii), 12(i) and 4, to obtain

$$\sum_{q=s_{r-1}+1}^{s_r} \sum_{m=q+1}^{s_r} |a_{qm}^{(w_{q+1})}|^2 \;\leq\; 1.0027^2 \left(\frac{\|A_{rr}\|}{\sqrt{2}} + \frac{1.0412\, \mu\, 1.1581}{\gamma} \|G_r\|^2\right)^2.$$

To bound $\|A_{rr}\|$, we use Theorem 1(i),

$$\|A_{rr}\| \;\leq\; \frac{4}{\gamma}\left(\|F_r\|^2 + \|\overline{F}_r\|^2\right) \;\leq\; \frac{4}{\gamma}\|N_{s_r}\|^2.$$

We complete the proof by combining the obtained estimates with Lemma 13, in the same way as in the proof of [12, Lemma 15, p. 51]. □

By proving the relations (3.9) and (3.10), PART II of the proof has been completed.

PART III

For $s_{r-1} + 2 \leq m \leq s_r$, let

$$v_m \;=\; Q_{s_{r-1}} + (s_r - s_{r-1})s_{r-1} + (m - s_{r-1} - 1)(m - s_{r-1} - 2)/2\,. \tag{3.23}$$

LEMMA 15. *Let H satisfy the assumptions of Theorem 6. Let v_m be defined by the relation (3.23). Then*

$$\|F_r^{(v_{s_r+1})}\| \;\leq\; 1.015\, \|F_r^{(v_{s_{r-1}+2})}\|\,.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Proof. In the proof of [12, Lemma 16, pp. 52–53], we use Theorem 1(i) to obtain

$$\|A_{rr}^{(k)}\| \;\leq\; \frac{4}{\gamma}\left(\|F_r^{(k)}\|^2 + \|\overline{F}_r^{(k)}\|^2\right).$$

Thus, we have $\tilde{a} \leq \sqrt{2}\,\alpha^2/\gamma$. The proof is completed by using the relation (2.18) and the assumption (A1). □

The last lemma proves the relation (3.11) and completes PART III of the proof.

4. Numerical examples. We have made several experiments using MATLAB. The main m-file that we have used is displayed below. In the first (second) part of that file, we generate an s.d.d. symmetric matrix of order n with simple (multiply) eigenvalues. We cannot display all the m-functions which are called in this m-file; instead, we briefly describe what they do. The m-function `symd(a,k)` generates an almost diagonal symmetric matrix, from the vector `a`, by the formula `A=Q*diag(a)*Q'`, where `Q` is an orthogonal matrix whose off-diagonal elements are of order 10^{-k} (cf. [5]). Then `A` is symmetrically scaled to be an s.d.d. matrix. After that, three method are applied to `A`: two Jacobi methods (`djacobivpa(A,180)` and `djacobi(A)`), and the QR method which hides within the intrinsic `eig(A)` function. The first two methods are coded almost in the same way, using the row-cyclic pivot strategy.

The first method, within the m-file `djacobivpa(A,k)` is the control one. It uses variable precision arithmetic (vpa) with `k` decimal digits. We have taken `k=180`, large enough to watch the asymptotic convergence during several cycles. Its input is the matrix `A` in double precision and its outputs are also in double precision: `V` (the eigenvector matrix), `Lambda` (the eigenvalue diagonal matrix) and `OFF` (the two-column matrix: the first (second) column contains the off-norms of Jacobi iterates (scaled iterates) obtained after each full cycle). Within `djacobivpa(A,180)` all computations are performed using vpa.

The second and third method are the standard double precision algorithms.

The eigenvalues computed by these three methods are displayed in non-increasing ordering as vectors: `c0` (here are "exact" eigenvalues rounded to double precision), `c1` (here are the eigenvalues computed by `djacobi(A)`) and `c3` (here are the eigenvalues computed by `eig(A)`). The relative errors of the eigenvalues contained in `c1` and `c2` are computed and stored in the vectors `c4` and `c5`, respectively. From the entries of `c1`, the absolute and relative gaps are computed. From the columns of `OFF`, we can watch the off-norm reduction of simple and scaled iterates, that is, we can watch the asymptotic convergence.

In the second part of the main m-file, we use the m-function `dmult3vpa(V1,B,`

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

`V1',128)` and the orthogonal matrix `V` (already available from the first part) to compute the new `A` by the formula `A=V*B*V'`, where `B=diag(b)`, and `b` is the given vector with multiple entries (which will be recomputed as eigenvalues of `A`). Although, the input matrices and the output matrix to `dmult3vpa` are in standard double precision, the computation within `dmult3vpa` uses vpa with 128 decimal digits. We have chosen

```
%%%
 % case of simple eigenvalues of an s.d.d. symmetric matrix
 format short e; n=12; a1=[1 2 3 4]; a2=[a1 a1 -a1 -a1 -a1]; a=a2(1:n);
 A1=symd(a,3);  d=logspace(10,-10,n);  D=diag(d);   A=D*A1*D;    A=A+A'
 [V Lambda OFF] = djacobivpa(A,180);     [V1 Lambda1] = djacobi(A);
 [k1 k2]=size(OFF);   c0=sort(diag(Lambda),'descend');
 c1=sort(diag(Lambda1),'descend');     c2=sort(eig(A),'descend');
 c3=(c1-c0)./c0;       c4=(c2-c0)./c0;       format long e;
 disp('          Exact                    Jacobi                    QR
 rel.err. Jacobi                     rel.err.QR')
 disp([c0 c1 c2 c3 c4]);   gaps=gap(c0);   sort(gaps,'descend');
 disp('       Absolute gaps         Relative gaps');   disp(gaps(1:n,:))
 disp('                    Minima gaps ');   disp(gaps(n+1,:));
 disp('     Initial off-norm       Initial scaled off-norm');
 disp(OFF(1,:));         disp('     Off-norm per cycle
 Scaled off-norm per cycle');   disp(OFF(2:k1,:));
%%%
 % case of multiple eigenvalues of an s.d.d. symmetric matrix
 b1=[1 1 1 1];   b2=[1e16*b1 2e4*b1 -2e-3*b1 -5e-17*b1];    b=b2(1:n);
 B=diag(b);  A2=dmult3vpa(V,B,V',128);  format short e;  A=0.5*(A2+A2')
 [V Lambda OFF] = djacobivpa(A,180);    c0=sort(diag(Lambda),'descend');
 [V1, Lambda1] = djacobi(A);           c1=sort(diag(Lambda1),'descend');
 [V2, Lambda2] = eig(vpa(A));     sort(diag(double(Lambda2)),'descend');
 c2=sort(eig(A),'descend');      c3=(c1-c0)./c0;      c4=(c2-c0)./c0;
 c5=(c6-c0)./c0;    format long e;
 disp('            Exact                    Jacobi                    QR
 rel.err. Jacobi            rel.err.QR             rel err. eig(vpa(A))')
 disp([c0 c1 c2 c3 c4 c5]);   gaps=gap(c0);    sort(gaps,'descend');
 disp('       Absolute gaps         Relative gaps');  disp(gaps(1:n,:));
 disp('                    Minima gaps ');   disp(gaps(n+1,:));
 [k1 k2]=size(OFF);
 disp('      Initial off-norm        Initial scaled off-norm');
 disp(OFF(1,:));
 disp('       Off-norm per cycle      Scaled off-norm per cycle');
 disp(OFF(2:k1,:));
```

Table 1: The m-file used in the experiment.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

the entries of `b` to be of the same sign and of the similar magnitudes as the diagonal elements of `A` from the first part of the main m-file. We hope that this might ensure that an s.d.d. symmetric matrix `A` with multiple eigenvalues (actually, with small clusters of eigenvalues of width not much larger than the machine precision) will be generated. The problem with matrix generation, in this part of the program, lies in the fact that the input matrices to `dmult3vpa` are in double precision and therefore `V` is just within machine precision (double precision) close to an orthogonal matrix.

Then we apply four methods to this new `A` in the same way as described above. The additional method (provided by MATLAB), `eig(vpa(A,180))`, computes the spectral precision of `A` to 180 decimal digits and serves only as a control method for `djacobivpa(A,k)`. The computation in this part is more delicate than in the first part, and we wanted an additional control.

In our experiments, we have been changing `n` (between 6 and 30) and the vectors `a1`, `a2`, `b1`, `b2` in an arbitrary fashion. The results that we display are typical and are obtained as output from the m-file given below.

As can be seen from Table 2, the case of simple eigenvalues delivered an expected behavior of the Jacobi method. Except for the off-norm reduction, which is closer to cubic than to quadratic convergence. This is often the case when the scaling diagonal matrix `D` has decreasingly ordered diagonal elements and for smaller matrix dimension. When we have changed the command `d=logspace(10,-10,n);` into `d=logspace(-10,10,n);`, we got the following sequence of scaled off-norms per cycle (we display just four figures): `1.996e-002`, `1.745e-005`, `2.987e-011`, `4.015e-022`, `7.792e-045`, `5.782e-094`, `1.310e-199`. In all considered cases, the relative accuracy of the Jacobi method has been in average just few ulps, while the intrinsic MATLAB function `eig` delivered quite erroneous small eigenvalues.

In the case of (almost) multiple eigenvalues, Jacobi method proved to be relatively accurate on scaled almost diagonal symmetric matrices, while this is certainly not true for the QR method (Table 3). The asymptotic behavior of Jacobi method is here most instructive. Since the relative gaps are so tiny, in the first cycles Jacobi behaves as if multiple eigenvalues were present (see the analysis of Jacobi method in the presence of clusters [4, last section]). So, quadratic (or faster) asymptotic convergence takes place. As the scaled off-norm, call it α_0 becomes smaller and approaches the magnitude of relative gaps, the asymptotic convergence slows down. And when α_0 becomes smaller than the minimum gap, its reduction per cycle is subjected to the rule described by the main theorem here. Then, actually $\beta_0 = \alpha_0/\gamma$, reduces quadratically per cycle.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

A =

2.0000e+020	8.8742e+015	3.3952e+014	8.5408e+011	-4.8888e+010	-2.3741e+008
8.8742e+015	9.2405e+016	7.5958e+010	4.6505e+008	-2.9021e+005	-3.9117e+005
3.3952e+014	7.5958e+010	3.2020e+013	2.6113e+006	-8.5534e+005	-1.1924e+004
8.5408e+011	4.6505e+008	2.6113e+006	9.8628e+009	-5.0751e+002	-5.8053e+002
-4.8888e+010	-2.9021e+005	-8.5534e+005	-5.0751e+002	-5.6960e+005	-5.3446e-001
-2.3741e+008	-3.9117e+005	-1.1924e+004	-5.8053e+002	-5.3446e-001	-2.6317e+002
-1.1253e+007	-3.9913e+004	-1.0796e+003	-1.6837e+001	-1.3281e-002	-1.0234e-004
-9.2888e+004	-7.6369e+002	-2.5099e+000	-3.8046e-002	-5.0881e-004	-8.5806e-006
2.9505e-001	-4.1079e-001	-1.1690e-003	-1.9748e-003	4.9874e-006	1.0215e-007
1.2615e+001	-5.3873e-004	-2.6700e-004	-2.5249e-005	9.3148e-008	3.7510e-010
1.3057e-001	3.5830e-004	-1.0724e-007	-1.4150e-007	1.9112e-009	5.4289e-011
-5.3137e-003	-1.3299e-005	-1.3834e-007	5.7874e-012	-4.4112e-011	-7.5379e-013

-1.1253e+007	-9.2888e+004	2.9505e-001	1.2615e+001	1.3057e-001	-5.3137e-003
-3.9913e+004	-7.6369e+002	-4.1079e-001	-5.3873e-004	3.5830e-004	-1.3299e-005
-1.0796e+003	-2.5099e+000	-1.1690e-003	-2.6700e-004	-1.0724e-007	-1.3834e-007
-1.6837e+001	-3.8046e-002	-1.9748e-003	-2.5249e-005	-1.4150e-007	5.7874e-012
-1.3281e-002	-5.0881e-004	4.9874e-006	9.3148e-008	1.9112e-009	-4.4112e-011
-1.0234e-004	-8.5806e-006	1.0215e-007	3.7510e-010	5.4289e-011	-7.5379e-013
-9.1194e-002	1.5580e-009	1.4767e-009	2.2776e-011	6.5178e-014	-2.4384e-015
1.5580e-009	-2.8090e-005	3.6650e-011	7.5438e-013	6.2077e-015	-2.7659e-016
1.4767e-009	3.6650e-011	1.6223e-009	1.0839e-015	2.1443e-017	-6.8156e-019
2.2776e-011	7.5438e-013	1.0839e-015	7.4953e-013	5.5574e-019	-9.0855e-021
6.5178e-014	6.2077e-015	2.1443e-017	5.5574e-019	2.5973e-016	-2.2376e-022
-2.4384e-015	-2.7659e-016	-6.8156e-019	-9.0855e-021	-2.2376e-022	8.0000e-020

Exact	Jacobi	QR
2.000046750664956e+020	2.000046750664956e+020	2.000046750664957e+020
9.240438209042706e+016	9.240438209042706e+016	9.240438209042619e+016
3.201932426557301e+013	3.201932426557301e+013	3.201932426557377e+013
9.862765376960863e+009	9.862765376960865e+009	9.862765376847088e+009
1.622261859324245e-009	1.622261859324244e-009	7.849982703729516e-003
7.495254709475820e-013	7.495254709475820e-013	7.999981069902117e-020
2.597255642838921e-016	2.597255642838921e-016	-2.816494199892072e-005
7.999981012923348e-020	7.999981012923348e-020	-9.119541211584167e-002
-2.808957050812652e-005	-2.808957050812652e-005	-4.867318339132567e+000
-9.119517758082542e-002	-9.119517758082543e-002	-2.631734820328961e+002
-2.631734820325797e+002	-2.631734820325797e+002	-1.639945514979879e+004
-5.696131240766835e+005	-5.696131240766834e+005	-5.696131240771967e+005

rel.err. Jacobi	rel.err. QR	Absolute gaps
-1.638361702750479e-016	1.638361702750479e-016	5.693499505946509e+005
0	-9.350205915066759e-015	2.630822868549989e+002
0	2.366734830862110e-014	9.116708801031730e-002
1.933888275663550e-016	-1.153583695316064e-011	2.808957050812661e-005
-2.549466992023072e-016	4.838911200647703e+006	2.596455644737629e-016
0	-9.999998932660547e-001	2.596455644737629e-016
0	-1.084411620274501e+011	7.492657453832981e-013
0	-1.139945356976755e+018	1.621512333853297e-009
-1.206188534650962e-016	1.732774891717876e+005	9.862765376960863e+009
1.521767726754488e-016	2.884826740121735e+003	3.200946150019605e+013
0	6.131423858946626e+001	9.237236276616149e+016
-2.043761228564364e-016	9.008899495511716e-013	1.999122706844052e+020

Relative gaps	Initial off-norm	Initial scaled off-norm
9.990763838287010e-001	8.880711476893470e+015	7.641990047129683e-003
9.990763838287010e-001		
9.993071977711009e-001		
9.993841577618235e-001	Off-norm per cycle	Scaled off-norm per cycle
9.993841562985136e-001	1.033617937736517e+005	2.405416528131962e-014
9.993072000741221e-001	1.685298539763208e-024	7.703180036125967e-042
9.990763768166148e-001	2.755405903042885e-080	5.190144088267827e-094
9.990763768166148e-001	2.672916428515878e-187	5.432172951866892e-203
9.993841388876070e-001	0	0
9.993072139595326e-001		
9.990764044898099e-001	Minima gaps	
9.990764044898099e-001	2.596455644737629e-016	9.990763768166148e-001

Table 2: Simple eigenvalues.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

ELA
http://math.technion.ac.il/iic/ela

A =

1.0000e+016	8.6982e-005	-5.8954e-002	-7.4715e-002	-2.4439e+006	-1.1860e+004
8.6982e-005	1.0000e+016	4.8938e-003	3.7359e-003	2.0350e+005	-4.1187e+004
-5.8954e-002	4.8938e-003	1.0000e+016	-7.5174e+000	-2.4121e+008	-3.5979e+006
-7.4715e-002	3.7359e-003	-7.5174e+000	1.0000e+016	-3.0286e+008	-5.8758e+008
-2.4439e+006	2.0350e+005	-2.4121e+008	-3.0286e+008	-1.9985e+004	1.7882e+001
-1.1860e+004	-4.1187e+004	-3.5979e+006	-5.8758e+008	1.7882e+001	-1.9965e+004
-5.6179e+002	-4.2651e+003	-3.3121e+005	-1.7022e+007	5.2353e-001	1.0003e+000
-4.6392e+000	-8.2200e+001	-7.3447e+002	-3.8169e+004	1.1737e-003	2.2430e-003
2.5895e-005	-4.4448e-002	-3.6509e-001	-2.0023e+003	6.0826e-005	1.2540e-004
6.3102e-004	-1.1870e-004	-9.0075e-002	-2.5655e+001	7.8253e-007	1.5369e-006
6.5272e-006	3.8149e-005	-1.0278e-004	-1.4403e-001	4.4330e-009	1.2600e-008
-2.6555e-007	-1.4137e-006	-4.0384e-005	2.8941e-005	-1.4966e-012	-5.9447e-011

-5.6179e+002	-4.6392e+000	2.5895e-005	6.3102e-004	6.5272e-006	-2.6555e-007
-4.2651e+003	-8.2200e+001	-4.4448e-002	-1.1870e-004	3.8149e-005	-1.4137e-006
-3.3121e+005	-7.3447e+002	-3.6509e-001	-9.0075e-002	-1.0278e-004	-4.0384e-005
-1.7022e+007	-3.8169e+004	-2.0023e+003	-2.5655e+001	-1.4403e-001	2.8941e-005
5.2353e-001	1.1737e-003	6.0826e-005	7.8253e-007	4.4330e-009	-1.4966e-012
1.0003e+000	2.2430e-003	1.2540e-004	1.5369e-006	1.2600e-008	-5.9447e-011
-2.0000e+004	6.4997e-005	3.2644e-004	5.1820e-006	1.6110e-008	-6.0216e-010
6.4997e-005	-2.0000e+004	2.6080e-002	5.4113e-004	4.4622e-006	-1.9873e-007
3.2644e-004	2.6080e-002	2.0000e-003	-7.0058e-010	-5.7901e-012	2.5915e-013
5.1820e-006	5.4113e-004	-7.0058e-010	2.0000e-003	-1.2037e-013	5.3771e-015
1.6110e-008	4.4622e-006	-5.7901e-012	-1.2037e-013	2.0000e-003	4.4338e-017
-6.0216e-010	-1.9873e-007	2.5915e-013	5.3771e-015	4.4338e-017	2.0000e-003

Exact	Jacobi	QR
1.000000000000004e+016	1.000000000000000e+016	1.000000000000000e+016
1.000000000000001e+016	1.000000000000000e+016	1.000000000000000e+016
1.000000000000000e+016	1.000000000000000e+016	1.000000000000000e+016
1.000000000000000e+016	1.000000000000000e+016	9.999999999999996e+015
2.000000000000002e-003	2.000000000000001e-003	1.852542487720570e+000
2.000000000000000e-003	2.000000000000000e-003	1.062114463469422e+000
2.000000000000000e-003	2.000000000000000e-003	3.137526950677154e-001
2.000000000000000e-003	2.000000000000000e-003	2.000000000000000e-003
-2.000000000000000e+004	-2.000000000000000e+004	-1.999805600114654e+004
-2.000000000000000e+004	-2.000000000000000e+004	-1.999903881970658e+004
-2.000000000000000e+004	-2.000000000000000e+004	-1.999999995759706e+004
-2.000000000000000e+004	-2.000000000000000e+004	-2.000020236358738e+004

rel.err. Jacobi	rel.err. QR	Absolute gaps
-4.199999999999982e-015	-3.999999999999982e-015	3.637978807091713e-012
3.637978807091713e-012	-5.999999999999996e-016	0
-3.999999999999998e-016	0	0
0	0	0
0	-4.000000000000000e-016	0
-6.505213034913021e-016	9.252712438602841e+002	0
0	5.300572317347110e+002	0
-2.168404344971009e-016	1.558763475338577e+002	0
-2.168404344971009e-016	0	1.734723475976807e-018
0	-9.719994267325092e-005	0
0	-4.805901467098010e-005	0
0	-2.120147109963000e-009	6.000000000000000e+000
0	1.011817936887382e-005	3.800000000000000e+001

Relative gaps	Initial off-norm	Initial scaled off-norm
9.094947017729283e-017	7.038941602976600e+008	4.998902418448497e-002
0		
0		
0	Off-norm per cycle	Scaled off-norm per cycle
0	2.201009424095137e+001	2.201931286417376e-015
0	5.977604186695108e-001	5.977604187210803e-017
0	3.809484058360602e-006	3.809484058362190e-022
4.336808689942016e-016	1.224016819631911e-015	1.224016819631908e-031
0	1.223732657523249e-047	1.223732657523246e-063
0	2.614185022365451e-162	2.614185022365445e-178
2.999999999999999e-016	0	0
1.899999999999995e-015		
	Minima gaps	
	0	0

Table 3: Multiple eigenvalues.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 62-87, February 2008

Acknowledgment. The author is indebted to Professor V. Hari for providing MATLAB m-files and for conducting numerical experiments, as well as for reading the paper.

REFERENCES

[1] J.W. Demmel and K. Veselić. Jacobi's method is more accurate than QR. *SIAM J. Matrix Anal. Appl.,* 13:1204–1245, 1992.

[2] Z. Drmač and K. Veselić. New fast and accurate Jacobi SVD algorithm I. *LAPACK Working note*, 169, 2005.

[3] Z. Drmač and K. Veselić. New fast and accurate Jacobi SVD algorithm II. *LAPACK Working note*, 170, 2005.

[4] V. Hari. On sharp quadratic convergence bounds for the serial Jacobi methods. *Numer. Math.*, 60:375–406, 1991.

[5] V. Hari. Structure of almost diagonal matrices. *Mathematical Communications*, 4:135–158, 1999.

[6] V. Hari and Z. Drmač. On scaled almost diagonal Hermitian matrix pairs. *SIAM J. Matrix Anal. Appl.*, 18:1000–1012, 1995.

[7] V. Hari and J. Matejaš. Quadratic convergence of scaled iterates by Kogbetliantz method. *Computing*, 16:83–105, 2003.

[8] I.C.F. Ipsen. Relative perturbation results for matrix eigenvalues and singular values. *Acta Numerica*, 7:151-201, 1998.

[9] H.P.M. Van Kempen. On the Quadratic Convergence of the Special Cyclic Jacobi Method. *Num. Math.*, 9:19–22, 1966.

[10] W.F. Mascarenhas. *On the Convergence of the Jacobi Method for Arbitrary Orderings*. University of Minnesota, Minneapolis, 1993.

[11] J. Matejaš. Quadratic convergence of scaled matrices in Jacobi method. *Numer. Math.*, 87:171–199, 2000.

[12] J. Matejaš. Convergence of scaled iterates by the Jacobi method. *Linear Algebra Appl.*, 349:17–53, 2002.

[13] J. Matejaš and V. Hari. Scaled almost diagonal matrices with multiple singular values. *ZAMM*, 78(2):121–131, 1998.

[14] J. Matejaš and V. Hari. Scaled iterates by Kogbetliantz method. *Proceedings of the 1st Conference on Applied Mathematics and Computations*, Dubrovnik, Croatia, September 13–18, 1999. Publisher Department of Mathematics, University of Zagreb, 1–20, 2001.

[15] J. Matejaš and V. Hari. Quadratic convergence estimate of scaled iterates by J-symmetric Jacobi method. *Linear Algebra Appl.*, 417:434–465, 2006.

[16] J. Matejaš. Accuracy of the Jacobi method for indefinite almost diagonal matrices. Preprint, University of Zagreb, 2006.

[17] J.H. Wilkinson. Note on the quadratic convergence of the cyclic Jacobi process. *Num. Math.*, 4:296–300, 1962.

[18] G. Shroff and R. Schreiber. On the convergence of the cyclic Jacobi method for parallel block orderings. *SIAM J. Matrix Anal. Appl.*, 10:326–346, 1989.

[19] J. Wilkinson and C. Reinsch. *Handbook for Automatic Computation*. Springer-Verlag, New York, 1971.