2008

On the Brualdi-Liu conjecture for the even permanent

Ian M. Wanless
ian.wanless@sci.monash.edu.au

Follow this and additional works at: https://repository.uwyo.edu/ela

Recommended Citation

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
ON THE BRUALDI-LIU CONJECTURE FOR THE EVEN PERMANENT*  

IAN M. WANLESS†  

Abstract. Counterexamples are given to Brualdi and Liu’s conjectured even permanent analogue of the van der Waerden-Egorychev-Falikman Theorem.  

Key words. Even permanent, Doubly stochastic, Permutation matrix.  

AMS subject classifications. 15A15.  

For an \( n \times n \) matrix \( M = [m_{ij}] \) consider the sum  

\[
\sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)}.
\]

If the sum is taken over all permutations \( \sigma \) of \( [n] = \{1, 2, \ldots, n\} \) then we get \( \text{per}(M) \), the permanent of \( M \). If, however, we only take the sum over all even permutations \( \sigma \) of \( [n] \) then we get \( \text{per}^{ev}(M) \), the even permanent of \( M \).  

Let \( \Omega_n \) denote the set of doubly stochastic matrices (non-negative matrices with row and column sums 1). It is well known that \( \Omega_n \) consists of all matrices which can be written as a convex combination of permutation matrices of order \( n \). By analogy we define \( \Omega_n^{ev} \) to be the set of all matrices which can be written as a convex combination of even permutation matrices of order \( n \).  

The famous van der Waerden-Egorychev-Falikman Theorem states that \( \text{per}(M) \geq n!/n^n \) for all \( M \in \Omega_n \) with equality iff every entry of \( M \) equals \( 1/n \). Similarly, Brualdi and Liu [2] conjectured \( \text{per}^{ev}(M) \geq \frac{1}{2}n!/n^n \) for all \( M \in \Omega_n^{ev} \) with equality iff every entry of \( M \) equals \( 1/n \). They claimed their conjecture was true for \( n \leq 3 \). We show below that their conjecture is false for \( n \in \{4, 5\} \), although we leave open the possibility that it is true for larger \( n \). For background on all of the above, see Brualdi’s new book [1].  

Let  

\[
C_4 = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}.
\]

*Received by the editors 1 January 2007. Accepted for publication 5 June 2008. Handling Editor: Richard A. Brualdi.  
†School of Mathematical Sciences, Monash University, Vic 3800 Australia (ian.wanless@sci.monash.edu.au). Research supported by ARC grant DP0662946.
Then $C_4 \in \Omega_n^{ev}$ since

$$
C_4 = \frac{1}{3} \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
$$

To show that $C_4$ is a counterexample we consider the more general problem of finding $per^{ev}(C_n)$ where $C_n$ is the $n \times n$ matrix with zeroes on the main diagonal and every other entry equal to $1/(n-1)$. Clearly $per(C_n) = D_n/(n-1)^n$ where

$$D_n = n! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots - (-1)^n \frac{1}{n!} \right)$$

is the number of derangements (fixed point free permutations) of $[n]$. Using the cards-decks-hands method of Wilf [4] it can be shown that $\frac{n}{k}x^k e^{-x}$ is a generating function in which the coefficient of $\frac{n}{k}x^k$ is the number of derangements of $[n]$ with exactly $k$ cycles. It can then be deduced that the number of even derangements is $\frac{1}{2} (D_n + (-1)^n (1-n))$ (this result is probably well-known, certainly it is obtained in [3]). Hence

$$per^{ev}(C_n) = \frac{\frac{1}{2} (D_n + (-1)^n (1-n))}{n!} \left( \frac{n}{n-1} \right)^n \left( \frac{1}{e} - \frac{1}{n(n-2)!} \right) \exp \left( 1 + \frac{1}{2n} \right) > 1$$

for $n \geq 5$. It follows that $C_n$ is not a counterexample to the Brualdi-Liu conjecture for any $n \geq 5$. However, $per^{ev}(C_4) = 1/27 < 3/64$ so $C_4$ is a counterexample.

Two further counterexamples arise from the following family of matrices. Let $T_n$ denote the mean of the $(n-1)(n-2)$ permutation matrices corresponding to 3-cycles which move the point 1. For example,

$$
T_5 = \begin{bmatrix}
0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12}
\end{bmatrix}.
$$

Then $T_n \in \Omega_n^{ev}$ by construction. Now given that $per^{ev}(T_4) = 5/108 < 3/64$ and $per^{ev}(T_5) = 11/576 < 12/625$, both $T_4$ and $T_5$ are counterexamples to the Brualdi-Liu conjecture. That the family $\{T_n\}$ contains no further counterexamples is easy to show. The permutation matrices corresponding to 3-cycles alone contribute at least

$$(n-1)(n-2) \frac{1}{(n-1)^2} \left( \frac{n-3}{n-1} \right)^{n-3} \frac{1}{(n-1)(n-2)} = \frac{(n-3)^{n-3}}{(n-1)^{n-1}} \sim \frac{1}{(en)^2}$$

to $per^{ev}(T_n)$. 

REFERENCES


