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## A PARAMETERIZED LOWER BOUND FOR THE SMALLEST SINGULAR VALUE\*

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**Abstract.** This paper presents a parameterized lower bound for the smallest singular value of a matrix based on a new Geršgorin-type inclusion region that has been established recently by these authors. The comparison of the new lower bound with known ones is supplemented with a numerical example.

**Key words.** Eigenvalue, Inclusion region, Smallest singular value, Lower bound.

**AMS subject classifications.** 15A12, 65F15.

**1. Introduction.** To estimate matrix singular values is an attractive topic in matrix theory and numerical analysis, especially to give a lower bound for the smallest one. Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$  and let  $A^*$  be the conjugate transpose of  $A$ . Then the singular values of  $A$  are the square roots of the eigenvalues of  $AA^*$ . Throughout the paper we use  $\sigma_n(A)$  to denote the smallest singular value of  $A$ . Denote  $N := \{1, 2, \dots, n\}$ . Define, for all  $k \in N$ ,

$$r_k(A) := \sum_{j \in N \setminus \{k\}} |a_{kj}|, \quad c_k(A) := \sum_{j \in N \setminus \{k\}} |a_{jk}|, \quad h_k(A) := \frac{1}{2} (r_k(A) + c_k(A)).$$

In the past three decades, several useful lower bounds for the smallest singular value of a matrix have been presented in the literature; see Varah [7] and Qi [6]. By using Geršgorin's theorem (see Chapter 6 of [1]), Johnson [4] obtained a lower bound for  $\sigma_n(A)$ :

$$(1.1) \quad \sigma_n(A) \geq \min_{k \in N} \{|a_{kk}| - h_k(A)\}.$$

Recently, Johnson and Szulc [5] provided several further lower bounds for the

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smallest singular value. Two of them are

$$(1.2) \quad \sigma_n(A) \geq \frac{1}{2} \min_{k \in N} \left\{ \left( 4|a_{kk}|^2 + [r_k(A) - c_k(A)]^2 \right)^{\frac{1}{2}} - 2h_k(A) \right\}$$

and

$$(1.3) \quad \sigma_n(A) \geq \frac{1}{2} \min_{i \neq k} \left\{ |a_{ii}| + |a_{kk}| - \left[ (|a_{ii}| - |a_{kk}|)^2 + 4h_i(A)h_k(A) \right]^{\frac{1}{2}} \right\}.$$

In this paper, after introducing a new inclusion region for eigenvalues of a matrix in Section 2, we present a parameterized lower bound for the smallest singular value in Section 3. In Section 4, a numerical example is given to compare our results with the known ones.

**2. Inclusion regions for eigenvalues.** Recently, a new Geršgorin-type inclusion region for eigenvalues of a matrix has been provided by Huang, Zhang, and Shen in [3]. To introduce the result, we first present some notation. Let  $S$  be a nonempty subset of  $N$ . Denote  $\bar{S} := N \setminus S$  and let  $\mathcal{P}(N)$  denote the power set of  $N$ . For  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ , define, for all  $i \in N$

$$r_i^S(A) := \sum_{k \in S \setminus \{i\}} |a_{ik}|, \quad r_i^{\bar{S}}(A) := \sum_{k \in \bar{S} \setminus \{i\}} |a_{ik}|,$$

$$c_i^S(A) := \sum_{k \in S \setminus \{i\}} |a_{ki}|, \quad c_i^{\bar{S}}(A) := \sum_{k \in \bar{S} \setminus \{i\}} |a_{ki}|.$$

If  $S$  contains a single element, say  $S = \{i_0\}$ , then we let  $r_{i_0}^S(A) = 0$ . Similarly  $r_{i_0}^{\bar{S}}(A) = 0$  if  $\bar{S} = \{i_0\}$ . We sometimes use  $r_i^S$  ( $c_i^S$ ,  $r_i^{\bar{S}}$ ,  $c_i^{\bar{S}}$ ) to denote  $r_i^S(A)$  ( $c_i^S(A)$ ,  $r_i^{\bar{S}}(A)$ ,  $c_i^{\bar{S}}(A)$ , respectively). Define, for all  $i \in S$  and  $j \in \bar{S}$ ,

$$G_i^S(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i^S\}, \quad G_j^{\bar{S}}(A) := \{z \in \mathbb{C} : |z - a_{jj}| \leq r_j^{\bar{S}}\}$$

and

$$G_{i,j}^S(A) := \left\{ z \in \mathbb{C} : z \notin G_i^S(A) \cup G_j^{\bar{S}}(A), (|z - a_{ii}| - r_i^S) (|z - a_{jj}| - r_j^{\bar{S}}) \leq r_i^{\bar{S}} r_j^S \right\}.$$

**PROPOSITION 2.1.** ([3]) *Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ . Then all the eigenvalues of  $A$  are located in*

$$G^S(A) := \left( \bigcup_{i \in S} G_i^S(A) \right) \cup \left( \bigcup_{j \in \bar{S}} G_j^{\bar{S}}(A) \right) \cup \left( \bigcup_{i \in S, j \in \bar{S}} G_{i,j}^S(A) \right).$$

This result is interesting, but it can not be applied directly to estimate  $\sigma_n(A)$ . So we must deduce other inclusion regions. Let  $\alpha \in [0, 1]$  be given. For  $i \in S$ ,  $j \in \bar{S}$ , define the following regions in the complex plane:

$$U_{i,j}^S(A) := \left\{ z \in \mathbb{C} : |z - a_{ii}| - r_i^S \leq \left( r_i^{\bar{S}} r_j^S \right)^\alpha \right\},$$

$$V_{i,j}^S(A) := \left\{ z \in \mathbb{C} : |z - a_{jj}| - r_j^{\bar{S}} \leq \left( r_i^{\bar{S}} r_j^S \right)^{1-\alpha} \right\},$$

PROPOSITION 2.2. *Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ . Then all the eigenvalues of  $A$  are located in*

$$K^S(A) := \left( \bigcup_{i \in S, j \in \bar{S}} U_{i,j}^S(A) \right) \cup \left( \bigcup_{i \in S, j \in \bar{S}} V_{i,j}^S(A) \right).$$

*Proof.* It is sufficient to show that  $G^S(A) \subset K^S(A)$ . Note that  $G_i^S(A) \subseteq U_{i,j}^S(A)$  and  $G_j^{\bar{S}}(A) \subseteq V_{i,j}^S(A)$ . Therefore, for any  $z \in G^S(A)$ , if

$$z \in \bigcup_{i \in S} G_i^S(A) \quad \text{or} \quad z \in \bigcup_{j \in \bar{S}} G_j^{\bar{S}}(A),$$

then  $z \in K^S(A)$ . Otherwise, there exist  $i_0 \in S$  and  $j_0 \in \bar{S}$ , such that

$$\left( |z - a_{i_0 i_0}| - r_{i_0}^S \right) \left( |z - a_{j_0 j_0}| - r_{j_0}^{\bar{S}} \right) \leq r_{i_0}^{\bar{S}} r_{j_0}^S = \left( r_{i_0}^{\bar{S}} r_{j_0}^S \right)^\alpha \left( r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{1-\alpha}$$

which leads to

$$|z - a_{i_0 i_0}| - r_{i_0}^S \leq \left( r_{i_0}^{\bar{S}} r_{j_0}^S \right)^\alpha \quad \text{or} \quad |z - a_{j_0 j_0}| - r_{j_0}^{\bar{S}} \leq \left( r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{1-\alpha}.$$

Hence

$$z \in U_{i_0, j_0}^S(A) \cup V_{i_0, j_0}^S(A).$$

And then  $z \in K^S(A)$ .  $\square$

**3. Main results.** In this section we use the inclusions derived in Section 2 to estimate  $\sigma_n(A)$ . Denote the Hermitian part of  $A$  by

$$H(A) := \frac{1}{2}(A + A^*).$$

Let  $\lambda_{\min}(H(A))$  be the smallest eigenvalue of  $H(A)$ . It is known that  $\lambda_{\min}(H(A))$  is a lower bound for  $\sigma_n(A)$  [2, p.227]. Moreover, define for all  $i \in S, j \in \bar{S}$ , and  $\alpha \in [0, 1]$ ,

$$P_{i,j}^S(A) := |a_{ii}| - \frac{1}{2} [r_i^S(A) + c_i^S(A)] - \left[ \frac{1}{4} (r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)) (r_j^S(A) + c_j^S(A)) \right]^\alpha,$$

$$Q_{i,j}^S(A) := |a_{jj}| - \frac{1}{2} [r_j^{\bar{S}}(A) + c_j^{\bar{S}}(A)] - \left[ \frac{1}{4} (r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)) (r_j^S(A) + c_j^S(A)) \right]^{1-\alpha}.$$

**THEOREM 3.1.** *Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ . Then*

$$(3.1) \quad \sigma_n(A) \geq \min_{i \in S, j \in \bar{S}} \{P_{i,j}^S(A), Q_{i,j}^S(A)\}.$$

*Proof.* We first define a diagonal matrix  $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ , where  $e^{i\theta_k} a_{kk} = |a_{kk}|$  if  $a_{kk} \neq 0$  and  $\theta_k = 0$  if  $a_{kk} = 0, k \in N$ . Since  $D$  is unitary, the singular values of  $DA$  are the same as those of  $A$ . Consequently, we have

$$(3.2) \quad \sigma_n(A) = \sigma_n(DA) \geq \lambda_{\min}(H(DA)).$$

Denote  $B = (b_{kl}) := H(DA) = \frac{1}{2}(DA + A^*D^*)$ . Thus,  $b_{kk} = |a_{kk}|$ , for  $k \in N$ , and

$$b_{kl} = \frac{1}{2} (e^{k\theta_k} a_{kl} + \bar{a}_{lk} e^{-k\theta_k}), \quad \text{for all } k \neq l, \quad k, l \in N.$$

Since  $B$  is a Hermitian matrix, its eigenvalues are all real. Let  $\lambda_{\min}(B)$  denote the smallest eigenvalue of  $B$ . Then, by using Proposition 2.2,  $\lambda_{\min}(B)$  must satisfy at least one of the following conditions

$$\lambda_{\min}(B) \geq |b_{ii}| - r_i^S(B) - [r_i^{\bar{S}}(B)r_j^S(B)]^\alpha, \quad i \in S, j \in \bar{S},$$

$$\lambda_{\min}(B) \geq |b_{jj}| - r_j^{\bar{S}}(B) - [r_i^{\bar{S}}(B)r_j^S(B)]^{1-\alpha}, \quad i \in S, j \in \bar{S}.$$

It follows that

$$\lambda_{\min}(B) \geq \min_{i \in S, j \in \bar{S}} \left\{ |b_{ii}| - r_i^S(B) - [r_i^{\bar{S}}(B)r_j^S(B)]^\alpha, \right. \\ \left. |b_{jj}| - r_j^{\bar{S}}(B) - [r_i^{\bar{S}}(B)r_j^S(B)]^{1-\alpha} \right\}.$$

By applying the triangle inequality, we have

$$\begin{aligned} & |b_{ii}| - r_i^S(B) - [r_i^{\bar{S}}(B)r_j^S(B)]^\alpha \\ &= |a_{ii}| - r_i^S(H(DA)) - [r_i^{\bar{S}}(H(DA))r_j^S(H(DA))]^\alpha \\ &\geq |a_{ii}| - \frac{1}{2} (r_i^S(A) + c_i^S(A)) - \left[ \frac{1}{4} (r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)) (r_j^S(A) + c_j^S(A)) \right]^\alpha \\ &= P_{i,j}^S(A). \end{aligned}$$

Similarly, one can obtain

$$|b_{ii}| - r_i^{\bar{S}}(B) - \left[ r_i^{\bar{S}}(B) r_j^S(B) \right]^{1-\alpha} \geq Q_{i,j}^S(A).$$

Then from (3.2), we have

$$\sigma_n(A) \geq \min_{i \in S, j \in \bar{S}} \{P_{i,j}^S(A), Q_{i,j}^S(A)\}. \quad \square$$

Since the bound (3.1) holds for any nonempty  $S \in \mathcal{P}(N)$  and any  $\alpha \in [0, 1]$ , we have obtain the following corollaries:

COROLLARY 3.2.  $\sigma_n(A) \geq \max_{S \in \mathcal{P}(N)} \max_{\alpha \in [0,1]} \min_{i \in S, j \in \bar{S}} \{P_{i,j}^S(A), Q_{i,j}^S(A)\}.$

COROLLARY 3.3. ([4])  $\sigma_n(A) \geq \min_{i \in N} \{ |a_{ii}| - \frac{1}{2} (r_i(A) + c_i(A)) \}.$

**4. Numerical example.** In this section, we give a numerical example to compare our bound (3.1) with known ones.

EXAMPLE 4.1. Consider the following matrices

$$A_1 = \begin{bmatrix} 11 & 5 & 6 \\ 4 & 12 & -5 \\ 3 & 4 & 13 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 18 & 2 & -5 \\ 6 & 15 & 8 \\ -6 & -3 & 17 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 9 & 1 \\ 2 & -2 & -13 \end{bmatrix}.$$

Table 1. Comparison of lower bounds for  $\sigma_n(A)$

Matrix	$\sigma_n(A_i)$	$S$	$\alpha$	(1.1)	(1.2)	(1.3)	(3.1)
$A_1$	5.8446	{1}	0.57	2.0000	2.1803	2.4861	2.5886
$A_2$	9.6861	{2}	0.56	5.5000	6.1172	5.7827	5.7989
$A_3$	4.5433	{3}	0.10	2.5000	3.6921	2.3028	2.8377

From Table 1, we can see that bounds (1.2), (1.3), (3.1) are not comparable. Note that the tightness of our bounds depend on the choice of  $S$  and  $\alpha$  in which, unfortunately, we do not find a method such that the derived bounds are optimal. However, these parameters offer the possibility to optimize the estimation. We hope that future research will propose a method to determine the parameters  $S$  and  $\alpha$  that can give tighter lower bounds.

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