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MATRICIAL DECOMPOSITION OF SYSTEMS OVER RINGS*

ANDRÉS SÁEZ-SCHWEDT†

Abstract. This paper extends to non-controllable linear systems over rings the property FC^s ($s > 0$), which means "feedback cyclization with s inputs": given a controllable system (A, B), there exist a matrix K and a matrix U with s columns such that $(A + BK, BU)$ is controllable. Clearly, FC^1 is the usual FC property. The main technique used in this work is the obtention of block decompositions for systems, with controllable subsystems of a certain size. Each of the studied decompositions is associated to a class of commutative rings for which all systems can be decomposed accordingly. Finally, examples are shown of FC^s rings (for $s > 1$) which are not FC rings.

Key words. Systems over commutative rings, Pole assignability.

AMS subject classifications. 93B52, 93B55, 13C99.

1. Introduction.

Let R be a commutative ring with 1. An m-input, n-dimensional system (or a system of size (n, m)) over R will be a pair of matrices (A, B), with $A \in R^{n\times n}$ and $B \in R^{n\times m}$. The pair (A, B) is associated to the state-space description of the discrete-time linear system with constant coefficients, with state and input vectors $x(t), u(t)$, and evolution equation $x(t) = Ax(t-1) + Bu(t)$. See the motivation for studying linear systems over commutative rings in [14].

A system (A, B) is reachable or controllable if any vector x of R^n is reachable at some finite time t, starting from the initial condition $x(0) = 0$ and choosing appropriate input vectors $u(1), \dots, u(t)$. This is equivalent to the condition that R^n is spanned by the columns of the reachability matrix $A^*B = [B|AB|\cdots|A^{n-1}B]$. We say that R is an FC ring or satisfies the FC property (FC stands for feedback cyclization) if, given a reachable system (A, B) over R, there exist a feedback matrix K and a vector u such that the single-input system $(A + BK, Bu)$ is also reachable. See Heymann's Lemma [9] for an elementary solution of the FC problem and of the related pole assignability problem when the ring of scalars is a field, like for example $\mathbb{R}$ or $\mathbb{C}$.

The purpose of this work is to obtain feedback cyclization by means of more than one input, and to generalize the notion of FC to non-reachable systems. The

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following concept was introduced in [5]: The residual rank of the system (A, B), denoted by $\text{res.rk}(A, B)$, is defined as the residual rank of the reachability matrix A^*B, i.e., $\text{res.rk}(A, B) = \max\{i : \mathcal{U}_i(A^*B) = R\}$, where $\mathcal{U}_i(A^*B)$ denotes the ideal of R generated by the $i \times i$ minors of the matrix A^*B, with the convention $\mathcal{U}_0(A^*B) = R$. The case of reachable systems corresponds with maximal residual rank. In particular, $\mathcal{U}_1(B) = R$ for any reachable system (A, B). Also, if two matrices M, M' satisfy $im(M) \subseteq im(M')$, then it is clear from the properties of the ideals of minors that $\text{res.rk}(M) \leq \text{res.rk}(M')$, and the equality holds if the image modules are isomorphic. In particular, given a system (A, B), one has that $\text{res.rk}(A + BK, BU) \leq \text{res.rk}(A, B)$ for all matrices K, U. Finally, $\text{res.rk}(M) \leq \text{rank}(M)$, where *rank* denotes the usual rank of a matrix. For matrices with coefficients in a field, both notions of rank coincide, but not when scalars are taken from an arbitrary commutative ring.

The paper is organized as follows. In Section 2, we define the s-cyclization property for not necessarily reachable systems in this way: if (A, B) is a system with $\text{res.rk}(A, B) = r$, then there exists a feedback matrix K and an input matrix U with s columns such that $\text{res.rk}(A + BK, BU) = r$. A ring R is called a strong FC^s ring if all systems over R are s-cyclizable. This unifies in some sense two previous concepts: for $s = 1$, the strong FC^1 property is the strong feedback cyclization property studied in [12], while for arbitrary s, the strong FC^s property extends to non-reachable systems the FC^s property introduced in [10] for reachable systems.

In Section 3, we present the main technique used to extend results from reachable to non-reachable systems, which consists of various possible decompositions of systems, with a reachable subsystem of a certain size. The first decomposition, K_r, studied in [12], is similar to the classical Kalman controllability decomposition for systems over fields [15, Lemma 3.3.3] and allows to extract a reachable part of dimension r. The second decomposition, K^s, was introduced in [10], and roughly speaking, gives the possibility of concentrating the residual rank of a system (A, B) among A and the first s columns of B. In addition, we introduce a third decomposition, K_r^s, which is in some sense a superposition of the two previous decompositions.

Section 4 is devoted to giving characterizations of those rings for which all systems satisfy the given decompositions. The characterizations of rings obtained in terms of the K_r property involve the existence of unimodular vectors in the image of certain matrices, and a necessary condition is that all unimodular vectors must be completable to invertible matrices. For the K^s and K_r^s properties, the principal obstruction for systems to satisfy such decompositions is the stable range of the ring of scalars (precise definitions will be given later).

In the last section, we present all known examples of strong FC rings, and we study the FC^s rings given in [10], to see if they also satisfy the strong form of FC^s. It is an open question whether $k[x, y]$, for k a field, can be an FC^s ring for some $s > 1$

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(possibly $s = 2$). Finally, some concluding remarks are given.

2. Strong feedback cyclization with s columns. The following definition is the natural way to extend to arbitrary systems the FC^s property introduced in [10].

Definition 2.1. In [10, Definition 2.1], a reachable system (A, B) over a ring R is said to be s-cyclizable if there exist a matrix K and a matrix U with s columns such that $(A + BK, BU)$ is reachable, and the ring R is called an FC^s ring if any reachable system over R is s-cyclizable. We will say that a system (A, B) over a ring R is s-cyclizable if there exist matrices K, U (U with s columns) such that $\mathrm{res.rk}(A + BK, BU) = \mathrm{res.rk}(A, B)$, and R will be called a strong FC^s ring if all systems over R are s-cyclizable.

Since $\mathrm{res.rk}(A+BK, BU) \le \mathrm{res.rk}(A, B)$ for any matrices K, U, a system (A, B) is s-cyclizable iff there exist matrices K, U as above such that $\mathrm{res.rk}(A+BK, BU) \ge \mathrm{res.rk}(A, B)$. For $s = 1$, one has the strong FC property studied in [11]. It is clear that strong FC^s rings are FC^s rings for all s, since the case of reachable systems corresponds with maximal residual rank. Also, by adding zero columns to the matrix U, it is clear that s-cyclization implies s'-cyclization for all $s' > s$. Hence, FC^s implies $\mathrm{FC}^{s'}$ for all $s' > s$.

Some properties of systems are preserved if the system is affected by certain operations which simplify the structure of the matrices. We recall that two systems (A, B) and (A', B') over a ring R are *feedback equivalent* if there exist invertible matrices $P \in Gl_n(R), Q \in Gl_m(R)$ and a matrix $K \in R^{m\times n}$ such that $(A', B') = (PAP^{-1} + PBK, PBQ)$. The matrices P, Q correspond to changes of basis, while K gives a feedback action that transforms the system (A, B) into the 'closed-loop' system $(A+BK, B)$. We will use repeatedly the fact that the residual rank is invariant under feedback [5, Proposition 2.2]. Also, the s-cyclization property is invariant under feedback, which was already proved for reachable systems in [10, Lemma 2.3].

Proposition 2.2. *Let (A, B) and (A', B') be two feedback equivalent systems over a ring R. If (A', B') is s-cyclizable, then (A, B) is s-cyclizable.*

Proof. Denote by r the common residual rank of (A, B) and (A', B'), and let P, Q, K_1 be matrices such that $(A', B') = (PAP^{-1} + PBK_1, PBQ)$. By hypothesis, there exists a system $\Sigma = (A' + B'K', B'U')$ with residual rank r for some matrix U' with s columns. But Σ is equivalent to $(P^{-1}(A' + B'K')P, P^{-1}B'U')$, which has also residual rank r and is of the form $(A + BK, BU)$, where $K = K_1P + QK'P$ and $U = QU'$ with s columns. This proves that (A, B) is s-cyclizable. □

Next, we prove that the strong FC^s property behaves well under the usual constructions such as products, quotients, lifting modulo the Jacobson radical and forming power series. The next proposition is an immediate generalization of the results

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given in [1, Theorem 1], [11, Proposition 2.5] and [10, Proposition 2.4].

Proposition 2.3. *Let R be a commutative ring with Jacobson radical J, I be an ideal of R, and s be a positive integer. Then the following hold:*

(i) If R is a strong FC^s ring, then R/I is a strong FC^s ring.
(ii) If R/J is a strong FC^s ring, then R is a strong FC^s ring.
(iii) A product $R = \prod_i R_i$ is a strong FC^s ring iff each R_i is a strong FC^s ring.
(iv) R is a strong FC^s ring if and only if $R[[x]]$ is a strong FC^s ring.

Proof. We will illustrate only the proof of (i). The remaining proofs are exactly as in [11, Proposition 2.5].

(i) Let (A, B) be a system over R such that its reduction $(\bar{A}, \bar{B})$ modulo I has residual rank r. One has to find matrices $\bar{K}, \bar{U}$ over R/I, where $\bar{U}$ has s columns, such that $\text{res.rk}(\bar{A} + \bar{B}\bar{K}, \bar{B}\bar{U}) = r$. Like in the proof of [5, Lemma 2.9], one can construct a matrix $B_1 = [B|B']$ such that B' has all entries in I and $\mathcal{U}_r(A^*B_1) = R$.

Since R is a strong FC^s ring, there exist matrices $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ with s columns such that $\text{res.rk}(A + B_1K, B_1U) \geq r$. Reducing modulo I and using that $\bar{B}' = 0$ we get that $\text{res.rk}(\bar{A} + \bar{B}\bar{K}_1, \bar{B}\bar{U}_1) \geq r$, and we are finished. □

We say that a ring R is UCU (Unit-content Contains Unimodular) if, whenever $\mathcal{U}_1(B) = R$, there exists a vector u with Bu unimodular, see [1]. A ring R is called GCU (Good Contains Unimodular) if given a reachable system (A, B), there exists a vector u with Bu unimodular [1, p. 267]. It is clear that UCU rings are GCU [1], and GCU rings are Hermite in the sense of Lam: unimodular vectors can be completed to invertible matrices [2, Lemma 1].

The next theorem proves that for UCU rings, the FC^s and the strong FC^s properties are equivalent.

Theorem 2.4. *If R is UCU, then R is FC^s iff R is strong FC^s.*

Proof. The 'if' part is immediate, regardless of the UCU hypothesis. Conversely, let R be an FC^s ring and consider a system (A, B) over R with $\text{res.rk}(A, B) \geq r$. As the s-cyclization property is invariant under feedback and R is a UCU ring, by [12, Proposition 2.7], we can assume that (A, B) is decomposed in the block form given in Definition 3.1(i), with reachable part (A_1, B_1) of dimension r:

$$\left(A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right).$$

By the FC^s property, there exist matrices K_1, U_1, where U_1 has s columns, such that $(A_1 + B_1K_1, B_1U_1)$ is a reachable pair. Now, define $K = [K_1\, 0]$, $U = U_1$ and consider

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the system $\Sigma = (A + BK, BU)$, which has the form

$$\left(\begin{bmatrix} A_1 + B_1K_1 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} B_1U_1 \\ * \end{bmatrix} \right).$$

Since Σ has a reachable part of dimension r, it must have residual rank $\geq r$ (this is proved in Lemma 3.3 below). Thus, (A, B) is s-cyclizable, and R is a strong FC^s ring. □

The case $s = 1$ of the above result is already proved in [12, Proposition 3.6]. In particular, if the famous conjecture "$\mathbb{C}[x]$ is an FC ring" is true [4, p. 124], then $\mathbb{C}[x]$ will be a strong FC ring.

The following two examples show why it is difficult to obtain counterexamples to the FC^s property for some $s > 1$.

Example 2.5. One dimensional systems are always 1-cyclizable. Such a system $\Sigma = (A, B) = ([a], [b_1 \cdots b_m])$ with residual rank zero is trivially 1-cyclizable, so we may assume that $\text{res.rk}(\Sigma) = 1$, i.e., the ideal generated by $(b_1, \ldots, b_m)$ is R. Thus, there exist scalars $(u_i)_{i=1}^m$ such that $\sum_{i=1}^m b_iu_i = 1$. Taking $U = [u_1 \cdots u_m]'$ (where $'$ denotes transpose), it is clear that $(A, BU) = ([a], [1])$ is reachable, and hence, Σ is 1-cyclizable.

Example 2.6. The following two-dimensional system is 2-cyclizable:

$$\left(A = \begin{bmatrix} 0 & 0 \\ b_1 & a_1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & b_m \end{bmatrix} \right). \tag{2.1}$$

If the ideal generated by $b_1, \ldots, b_m$ is not R, then $\text{res.rk}(A, B) = 1$ and the first column of B already gives a 1-cyclization for the system, i.e., the system is 2-cyclizable. If $(b_1, \ldots, b_m) = R$, there exists scalars $(u_i)_{i=1}^m$ such that $\sum_{i=1}^m b_iu_i = 1$. Now, denoting by u the column vector $[u_2 \cdots u_m]'$, we see that the 2-input system

$$\left(A, B \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \right) = \left(\begin{bmatrix} 0 & 0 \\ b_1 & a_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 - b_1u_1 \end{bmatrix} \right)$$

is reachable, and hence, (A, B) is 2-cyclizable. In particular, if R is a UCU ring, then any two-dimensional system (A, B) is 2-cyclizable: indeed, if $\text{res.rk}(A, B) = 0$, then there is nothing to prove, and if $\text{res.rk}(A, B) \geq 1$, then B has unit content and hence B has a unimodular vector in its image, and therefore, following the proof of [2, Lemma 2], we see that the system (A, B) is feedback equivalent to some system in the form of (2.1), and thus it is 2-cyclizable.

After seeing these two examples, we know that the smallest possible counterexample to the FC^2 property must have sizes $n, m \geq 3$ when the ring R is UCU, or dimension $n = 2$ and $m \geq 3$ inputs when R is not UCU. We remark that in the latter

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case, the system (A, B) must not be equivalent to one in the form (2.1), i.e., the image of B must not have a unimodular vector that can be completed to an invertible matrix.

3. The block decompositions. We begin this section with a precise definition of the studied matricial decompositions.

DEFINITION 3.1. Let (A, B) be a system of size (n, m) over R. We say that:

(i) (A, B) satisfies the property K_r if it is feedback equivalent to

$$\left(PAP^{-1} + PBK = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, PBQ = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right),$$

with (A_1, B_1) reachable of size (r, m), and the remaining blocks are of appropriate sizes. The pair (A_1, B_1) is called the reachable part of the decomposition. By convention, any system satisfies K_0.

(ii) (A, B) satisfies K^s, for $s < m$, if it is equivalent to some system $(A', [B'_1|B'_2])$, with $B'_1 \in R^{n\times s}$, $\text{res.rk}(A', B'_1) = \text{res.rk}(A, B)$, and where the equivalence is given via P, K, Q, for $Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$, with $Q_1 \in R^{s\times s}$. It is 'forbidden' to add multiples of the first s columns of B to the remaining columns of B, any other feedback transformation is allowed. By convention, K^s holds if $m \le s$.

(iii) (A, B) satisfies K_r^s if it is equivalent to

$$\left(PAP^{-1} + PBK = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\right),$$

with (A_{11}, B_{11}) reachable of size (r, s), and where Q has the same structure as in (ii). If $m \le s$, K_r^s is defined as K_r.

REMARK 3.2. The property K_r was studied in [12], and K^s will be proved to be the natural extension of the property also called K^s in [10], and studied for reachable systems. Note that K^s implies s-cyclization: Proposition 2.2 assures that the s-cyclization property is preserved by feedback, and it is easy to see that a decomposition valid for K^s yields s-cyclization with $K = 0$, $U = \begin{bmatrix} 1_s \\ 0 \end{bmatrix}$. Also, note the following interpretation of K^s: after a suitable transformation, the residual rank of (A, B) can be concentrated in A and in the first s columns of B.

Before obtaining the characterizations of commutative rings, we need a few lemmas, which collect some technical stuff needed to perform our work.

LEMMA 3.3. *Let (A_1, B_1) be a system of size (t, m), and consider any system of size (n, m) given by $A = \begin{bmatrix} A_1 & 0 \\ * & * \end{bmatrix}, B = \begin{bmatrix} B_1 \\ * \end{bmatrix}$. Then we have:*

(i) $\text{res.rk}(A, B) \ge \text{res.rk}(A_1, B_1)$ *(cf. [12, Proposition 2.2(i)]).*

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(ii) *If* (A,B) *is reachable, then so is* (A_1,B_1).

(iii) *If* (A_1,B_1) *is equivalent to* (A_1',B_1') *via* P_1,K_1,Q_1, *then* (A,B) *is equivalent to* $\left(\begin{bmatrix} A_1' & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} B_1' \\ * \end{bmatrix}\right)$ *via* $P = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix}, K = [\, K_1 \quad 0\,]$ *and* $Q = Q_1$.

Proof. (i),(ii) The key fact is that the block formed by the first t rows of the reachability matrix A^*B consists of $[B_1|A_1B_1|\cdots|A_1^{n-1}B_1]$, which by Cayley-Hamilton has the same image as ${A_1}^*B_1$. If for some i, the $i\times i$ minors of ${A_1}^*B_1$ generate R, the same holds for A^*B, and statement (i) follows by definition of residual rank. Also, if (A,B) is reachable, then A^*B is right-invertible, which forces ${A_1}^*B_1$ to be also right-invertible, i.e., (A_1,B_1) is reachable, which proves (ii).

(iii) It is a straightforward verification. ☐

Lemma 3.4. *Let* $A\in R^{n\times n}$, $B_1\in R^{n\times s}$ *and* $B=[B_1|B_2]\in R^{n\times m}$. *Then the following hold:*

(i) $\text{res.rk}(A,B)\geq \text{res.rk}(A,B_1)$, *and if* (A,B_1) *is reachable, then so is* (A,B).

(ii) *If* (A,B_1) *is equivalent to* (A',B_1') *via* P,K_1,Q_1, *then* (A,B) *is equivalent to* $(A',[B_1'|*])$ *via* $P, K = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}$ *and* $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix}$.

Proof. Both (i) and (ii) are immediate. ☐

Lemma 3.5. *Let* (A,B) *be a system of size* (n,m) *and* $s>0$ *a fixed integer.*

(i) *If* (A,B) *satisfies* K_r^s, *then it satisfies* K_r.

(ii) *If* $\text{res.rk}(A,B)=r$ *and* (A,B) *satisfies* K_r^s, *then* K^s *holds.*

(iii) *If* $\text{res.rk}(A,B)=r$ *and* (A,B) *satisfies* K_r *with reachable part* (A_1,B_1) *satisfying* K^s, *then* K^s *and* K_r^s *hold for* (A,B).

Proof. (i) If K_r^s yields a decomposition with reachable part (A_{11},B_{11}) of size (r,s), then by Lemma 3.4(i), $(A_{11},[B_{11}|B_{12}])$ is reachable of size (r,m), which shows that one has a decomposition valid for K_r.

(ii) With respect to the notation of Definition 3.1(iii), observe that

$$r=\text{res.rk}(A,B)\geq \text{res.rk}\left(\begin{bmatrix} A_{11} & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} B_{11} \\ * \end{bmatrix}\right)\geq \text{res.rk}(A_{11},B_{11})=r,$$

where the inequalities hold by Lemma 3.4(i) and Lemma 3.3(i), and the last equality holds by K_r^s. Therefore, the middle system has residual rank r, and K^s holds.

(iii) First, by the K_r property, there exists a feedback equivalence $(A,B)\sim(\bar{A},\bar{B})$, and we know from [12, Lemma 2.2] that this equivalence can be obtained without a matrix Q. Then the K^s property yields an equivalence $(A_1,B_1)\sim(A_1',[B_{11}'|B_{12}'])$, with $\text{res.rk}(A_1',B_{11}')=r$, and the equivalence is given by certain matrices P_1,K_1,Q_1, where Q_1 has the structure required by K^s. By Lemma 3.3(iii),

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we can replace (A_1, B_1) in $(\bar{A}, \bar{B})$ by $(A'_1, [B'_{11}|B'_{12}])$, and thus,

$$(A, B) \approx \left(\begin{bmatrix} A'_1 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} B'_{11} & * \\ * & * \end{bmatrix} \right),$$

where the feedback equivalence consists of some matrices P, K, and $Q = Q_1$ with the required structure. Then (A, B) satisfies K_r^s, and by (ii), K^s also holds. □

As promised in Remark 3.2, we will prove that the property K^s extends the one defined in [10], and that both definitions coincide for reachable systems.

Lemma 3.6. *Let (A, B) be a system of size (n, m), with $B = [B_1|B_2]$ and $B_1 \in R^{n\times s}$. Then (A, B) satisfies K^s if and only if there exist matrices X, Y such that* $\text{res.rk}(A + B_2X, B_1 + B_2Y) = \text{res.rk}(A, B)$.

Proof. Suppose that (A, B) satisfies K^s via an equivalence $(A, B) \sim (A', B')$, with $B' = [B'_1|B'_2]$ and $\text{res.rk}(A', B'_1) = \text{res.rk}(A, B) = r$. Further, suppose that (A', B') is

$$\left(PAP^{-1} + P[B_1\ B_2] \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, P[B_1\ B_2] \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \right).$$

Operating, we see that $A' = PAP^{-1} + PB_1K_1 + PB_2K_2$ and $B'_1 = PB_1Q_1 + PB_2Q_2$. Therefore, (A', B'_1) is equivalent to $(P^{-1}A'P - P^{-1}B'_1Q_1^{-1}K_1P, P^{-1}B'_1Q_1^{-1})$, which has residual rank r and is of the form $(A+B_2X, B_1+B_2Y)$, for $X = K_2P - Q_2Q_1^{-1}K_1P$ and $Y = Q_2Q_1^{-1}$.

Conversely, if $\text{res.rk}(A + B_2X, B_1 + B_2Y) = \text{res.rk}(A, B)$, then the matrices $K = \begin{bmatrix} 0 \\ X \end{bmatrix}$ and $Q = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$ yield an equivalence $(A, B) \sim (A + BK, BQ)$ valid for K^s. □

4. The characterizations. The aim of this section is to characterize those rings for which any system satisfies the studied decompositions. Let us begin recalling some results from [12].

Proposition 4.1. *For a commutative ring R, we have:*

(i) The ring R is Hermite iff K_r holds for any single-input system Σ over R with $r \leq \text{res.rk}(\Sigma)$.

(ii) R is a UCU ring iff K_r holds for all systems Σ over R with $r \leq \text{res.rk}(\Sigma)$.

Proof. See [12, Propositions 2.5 and 2.7]. □

Next, we use the previous decomposition to characterize GCU rings.

Proposition 4.2. *For a ring R, the following statements are equivalent:*

(i) Any reachable system (A, B) over R is equivalent (without Q) to a system $(A', B') = (PAP^{-1} + PBK, PB)$ in which A' is strictly lower triangular.

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(ii) Any reachable n-dimensional system over R satisfies K_r for all $r \leq n$.
(iii) R is a GCU ring.

Proof. (i)⇒(ii) Let (A, B) be a reachable system over R of dimension n, and let (A', B') be an equivalent system as in (i). For any $r \leq n$, consider the system (A_r, B_r) formed by the first $r \times r$ block of A', and the first r rows of B'. Due to the triangular form of A', it follows

$$\left(A' = \begin{bmatrix} A_r & 0 \\ * & * \end{bmatrix}, B' = \begin{bmatrix} B_r \\ * \end{bmatrix} \right).$$

As (A', B') is reachable, by Lemma 3.3(ii), (A_r, B_r) must be also reachable, and K_r holds.

(ii)⇒(iii) Let (A, B) be a reachable system. By K_1, there exists a decomposition $(PAP^{-1} + PBK, PBQ)$ with a reachable part (A_1, B_1), so that B_1 -the first row of PBQ- is unimodular. Then for some column vector u, one has $B_1 u = 1$. Hence, $PBQu$ is unimodular (its first entry is 1), and multiplying by the invertible matrix P^{-1} it follows that $BQu \in im(B)$ is unimodular.

(iii)⇒(i) We proceed by induction on n. The case $n = 1$ being trivial: if a one-dimensional system (a, b) is reachable, then b is a unimodular row and (a, b) is feedback equivalent to $(a + bk = 0, b)$ for some k.

If $n > 1$, then by [2, Lemma 2], we have that (A, B) is equivalent to some system

$$\overline{\Sigma} = \left(\begin{bmatrix} 0 & 0 \\ b' & A' \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & B' \end{bmatrix} \right),$$

where $\Sigma' = (A', [b'|B'])$ is reachable by Eising's Lemma [6, Lemma 1]. By induction, Σ' can be transformed into a normal form $(A'', [b''|B''])$ via a feedback equivalence without Q. Therefore, we can apply [12, Lemma 2.4] to "copy & paste" the blocks A'', b'', B'' into $\overline{\Sigma}$, obtaining a new system

$$\overline{\overline{\Sigma}} = \left(\overline{\overline{A}} = \left[\begin{array}{c|cccc} 0 & 0 & \cdots & \cdots & 0 \\ \hline * & 0 & \cdots & \cdots & 0 \\ \vdots & * & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & * & 0 \end{array} \right], \overline{\overline{B}} = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & * \end{array} \right] \right).$$

If, after the sequence of equivalences, we have $\overline{\overline{A}} = PAP^{-1} + PBK$ and $\overline{\overline{B}} = PBQ$ for some invertible matrices P and Q, then we can remove Q by right-multiplication by Q^{-1}, obtaining an equivalent system in the required form. ∎

REMARK 4.3. If a statement on a system (A, B) involves only feedback reduction of the matrix A, then we are allowed to apply any transformation $(A, B) \mapsto (PAP^{-1}+$

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PBK, PBQ) and remove Q at the end, as was done in the previous proposition. However, for statements involving both matrices (A, B), we may apply only those transformations that preserve the studied property. For example, when working with the K^s property, there are some restrictions on the matrix Q (see Definition 3.1).

Next step is to characterize those rings satisfying K^s for all systems. We need to introduce the stable range of a ring, a useful tool for studying stability properties of matrices. Following [8], recall that a ring R has s in its stable range, or is s-stable, if given a unimodular row $(a_1, \ldots, a_s, b)$, there exist elements $k_1, \ldots, k_s$ in R such that $(a_1 + k_1 b, \ldots, a_s + k_s b) = R$. It is easy to see that if R is s-stable, then it is also s'-stable for $s' \geq s$. The stable range of a ring is the smallest s for which R is s-stable. The following matricial characterization is immediate: R is s-stable iff given a unimodular row $[a|b]$ with $a \in R^{1\times s}$ and $b \in R^{1\times k}$, there exists a matrix X with $a + bX$ unimodular.

Proposition 4.4. *For a commutative ring R, we have:*

(i) The ring R is s-stable iff K^s holds for one-dimensional reachable systems.
(ii) If R is GCU, then it is s-stable iff K^s holds for all reachable systems over R. In particular, s-stable GCU rings are FC^s rings.
(iii) If R is UCU, then it is s-stable iff K^s holds for all systems over R. In particular, s-stable UCU rings are strong FC^s rings.

Proof. (i) It suffices to consider one-dimensional systems with more than s inputs, i.e., systems of the form $(a, [c|d])$, where $c \in R^{1\times s}$ and with $[c|d]$ unimodular. Requiring that all such systems satisfy K^s is equivalent to the existence of K such that $c + dK$ is unimodular, i.e., R is an s-stable ring.

(ii) It is proved in [10, Theorem 3.3].

(iii) From (i), it is clear that if K^s holds for all systems, then R is s-stable, without using the UCU hypothesis. Conversely, suppose that R is an s-stable UCU ring, let (A, B) be a system over R of size (n, m) with $m \geq s$ (otherwise K^s holds by convention), and assume that $\text{res.rk}(A, B) \geq r$. By Proposition 4.1(ii), (A, B) satisfies K_r, and we know from [12, Proposition 2.2] that the equivalence yielding the decomposition can be obtained without Q, thus preserving the presence or absence of property K^s. Hence, we can assume that (A, B) in decomposed in the form of K_r, with a reachable part (A_1, B_1) of size (r, m). Since UCU implies GCU, the reachable pair (A_1, B_1) satisfies the property K^s, and we use Lemma 3.5(iii) to conclude that K^s holds for (A, B). ☐

The next step will be to characterize rings satisfying simultaneously the conditions s-stable and GCU, as well as s-stable and UCU.

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PROPOSITION 4.5. *For a ring R, the following statements are equivalent:*

(i) Any reachable system (A, B) of size (n, m) over R satisfies K_r^s for all $r \leq n$.
(ii) Any reachable system (A, B) of size (n, m) satisfies K^s and K_r for all $r \leq n$.
(iii) R is an s-stable GCU ring.

Proof. (i)⇒(ii) As was seen in Lemma 3.5, K_r^s implies K_r, and also K^s when applied to a reachable part of maximal size (in this case $r = n$).

(ii)⇒(i) Let (A, B) be a reachable system over R. First, decompose $(A, B) \sim (A', [B'_1|B'_2])$ according to K^s, i.e., with (A', B'_1) reachable of size (n, s). Then for any $r \leq n$, since (A', B'_1) satisfies K_r by hypothesis, one can apply Lemma 3.4(i) to replace (A', B'_1) by its decomposition, whose reachable part is of size (r, s). This gives a decomposition valid for K_r^s.

(ii)⇔(iii) By Proposition 4.2, the GCU condition is equivalent to the second part of (ii). But Proposition 4.4(ii) implies that, in the presence of the GCU property, R is s stable if and only if K^s holds for all reachable systems. Therefore, statements (ii) and (iii) are clearly equivalent. □

With an analogous formulation and almost with the same proof, we have:

PROPOSITION 4.6. *For a ring R, the following statements are equivalent:*

(i) Any system (A, B) over R satisfies K_r^s for all $r \leq \text{res.rk}(A, B)$.
(ii) Any system (A, B) over R satisfies K^s and K_r for all $r \leq \text{res.rk}(A, B)$.
(iii) R is an s-stable UCU ring.

Proof. The equivalence (i)⇔(ii) can be proved exactly like in the above theorem. To see that (ii)⇔(iii), we use Proposition 4.1(ii) and Proposition 4.4(iii). □

At this point, let us see what happens if we allow arbitrary feedback transformations in K_s and K_r^s, without the restrictions on the matrix Q imposed by Definition 3.1. We shall call these properties weak K^s and weak K_r^s, respectively. As a result we will obtain weakened versions of Propositions 4.5 and 4.6. The obstruction to obtaining stronger results will be the following: suppose that (A, B) is such that $\text{res.rk}(A + BK, BU) = \text{res.rk}(A, B)$ for some matrix K and some matrix U with s columns. If U can be completed to an invertible matrix Q (i.e., the columns of U span a free rank s direct summand of R^n with a free complement), then the equivalence $(A, B) \sim (A+BK, BQ)$ is adequate for the weak K^s property. Only in the case $s = 1$, we were able to solve this gap.

PROPOSITION 4.7. *Consider the following statements for a commutative ring R:*

(i) Reachable systems of size (n, m) satisfy weak K_r^s for all $r \leq n$.
(ii) Reachable systems of size (n, m) satisfy weak K^s and K_r for all $r \leq n$.

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(iii) R is a GCU ring with the FC^s property.

Then (i)⇔(ii)⇒(iii), and if $s = 1$, then all conditions are equivalent to the FC property.

Proof. The proof of (i)⇔(ii)⇒(iii) offers no difficulty and is only a repetition of the usual techniques. We will prove that (iii) implies (ii) in the case $s = 1$. For this, let R be an FC ring and (A, B) a reachable system over R. The existence of a reachable system $(A + BK, Bu)$ forces u to be a unimodular vector. Since FC rings are GCU rings [1], unimodular vectors can be completed to invertible matrices. Therefore, there exists an $m \times m$ invertible matrix $Q = [u|*]$, which yields the systems equivalence $(A, B) \sim (A + BK, BQ) = (A + BK, [Bu|*])$, valid for the decomposition K^s. By the characterization of GCU rings given in Proposition 4.2, it follows that K_r holds for all $r \leq n$, therefore we have (ii). □

Proposition 4.8. *Consider the following statements for a ring R:*

(i) Any system (A, B) satisfies weak K_r^s for all $r \leq \text{res.rk}(A, B)$.
(ii) Any system (A, B) satisfies weak K^s and K_r for all $r \leq \text{res.rk}(A, B)$.
(iii) R is a UCU ring with the strong FC^s property.

Then (i)⇔(ii)⇒(iii), and if $s = 1$, then all conditions are equivalent to the strong FC property.

Proof. It is a straightforward adaptation of the previous proposition. □

We will end this section by describing how our block decompositions can be used to derive feedback canonical or reduced forms for systems, provided we know how to reduce reachable systems with s inputs.

Let (A, B) be a system of size (n, m) over a principal ideal domain (PID) such that $\text{res.rk}(A, B) = r$. It is known that PIDs are UCU rings [4, p. 119] and also 2-stable [8, Theorem 2.3]. Therefore, by Proposition 4.6, the system (A, B) satisfies the property K_r^2 with a reachable part (A_{11}, B_{11}) of size $(r, 2)$. Combining Lemma 3.3(iii) and Lemma 3.4(ii), we can replace the reachable part by any known canonical or reduced form (see [13]), obtaining

$$\left(A^c = \left[\begin{array}{ccccc|c} 0 & & & & 0 & \\ 1 & \ddots & & & \vdots & \\ \vdots & \ddots & \ddots & & \vdots & \mathbf{0} \\ \vdots & & 1 & 0 & \vdots & \\ 0 & & & a & 0 & \\ \hline & & * & & & * \end{array}\right], B^c = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & c_2 & \\ \vdots & \vdots & * \\ 0 & c_{r-1} & \\ 0 & b & \\ \hline & * & * \end{array}\right] \right),$$

with $gcd(a, b) = 1$. Finally, some of the $*$'s can be removed as follows. Denote

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$A^c = (a_{ij})$ and $B^c = (b_{ij})$, and perform the following algorithm:

- For $i = r+1, \dots, n$, apply the row transformation given by $P : \{row_i - b_{i1} row_1\}$ on A^c and B^c, and apply the column operations $P^{-1} : \{col_1 + b_{i1} col_i\}$ on A^c. This cleans the first column of B^c.
- For $i = n, \dots, r+1$ and $j = r-2, \dots, 1$, apply $P : \{row_i - a_{ij} row_{j+1}\}$ on A^c and B^c, and apply $P^{-1} : \{col_{j+1} + a_{ij} col_i\}$ on A^c. This cleans all the $*$'s below the 1's in the first $r-2$ columns of A^c.
- For $j = 3, \dots, m$, apply $Q : \{col_j - b_{1j} col_1\}$ on B^c, cleaning the first row.

With a similar procedure, we could recover the reduced form obtained in [11, Theorem 2.8] for systems over strong FC rings. By Proposition 4.8 (case $s = 1$), strong FC rings are precisely those rings for which any system of residual rank $\geq r$ satisfies the decomposition weak K_r^1, which simply says that we can extract a reachable subsystem of size $(r, 1)$ in the controller canonical form. Moreover, if R is a UCU ring with stable range 1, Proposition 4.6 assures that the same reduced form can be obtained, but with a matrix Q of a very special form, which can be used in induction arguments.

5. Examples and concluding remarks. First, we recall all known examples of strong FC (FC^1) rings (note that the Krull dimension can be arbitrary).

PROPOSITION 5.1. *The following commutative rings are strong FC rings:*

(i) Fields, local and semilocal rings.
(ii) Local-global rings, including rings with many units.
(iii) Zero-dimensional rings, including von Neumann regular rings and Artin rings.
(iv) 1-stable Bezout domains, in particular the ring of all algebraic integers and the ring $H(\Omega)$ of holomorphic functions on a noncompact Riemann surface.
(v) Any other ring obtained by applying Proposition 2.3 to the above rings.

Proof. For (i)–(iv), see [11], where it is proved that all these rings are UCU and 1-stable. For (v), use Proposition 2.3 to construct more examples. □

When searching for examples of an FC^s ring R that is not a strong FC^s ring, by Theorem 2.4, R cannot be a UCU ring. This was solved in [11, Example 3.3] for $s = 1$: using a construction from [7, p. 149], it is shown the existence of a 1-stable Dedekind domain which is an FC ring but not a strong FC ring.

The next examples are mainly algebraic. We will only give the references where it is proved that the studied rings are GCU or UCU, indicating the stable range, if known, and then Proposition 4.4 will assure either the FC^s or strong FC^s property.

- An elementary divisor ring R is a UCU ring [4, p. 119]. By [10, p. 339], R is

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an FC^2 ring (hence a strong FC^2 ring) which may not be FC.

• If R is a zero-dimensional ring, then $R[x]$ is a UCU ring with the FC^2 property, thus a strong FC^2 ring (see [10, p. 240]). As commented in [1, p. 269] and [10, p. 240], $R[x]$ is not an elementary divisor ring if R has nonzero nilradical, and $R[x]$ is not an FC ring if some residual field $R/\mathfrak{m}$ has finite characteristic.

• The ring $\mathbb{Z}[[x]]$ inherits from $\mathbb{Z}$ the UCU and FC^2 properties and the absence of the FC property. Therefore, it is a strong FC^2 ring that is not a strong FC ring. Since $\mathbb{Z}[[x]]$ is neither an elementary divisor ring nor is of the form $R[x]$, for R zero-dimensional, this example is not included in the two previous cases.

• Let R be a Dedekind domain. By [8, Theorem 2.3], R is 2-stable. If R has torsion-free class group, R is an FC^2 ring [10, p. 240]. The interesting case is when R is not a UCU ring, otherwise it would be an elementary divisor domain (see [1, Proposition 4]). Such an R is a candidate of an FC^2 but not strong FC^2 ring.

• In [3], several interesting examples are given of polynomial rings with the UCU property, called BCU in that reference. For example, (i) $R = V[x]$, where V is a valuation ring, or (ii) $R = D[x]$, for D a principal ideal domain with countably many maximal ideals and containing an uncountable field. In case (i), since V has stable range 1, the stable s range of $V[x]$ should not be too high, while in case (ii), by Bass stable range theorem [8, Theorem 2.3], R is 2- or 3-stable. We have been unable to classify these examples as strong FC^s rings with the smallest possible value of s. In fact, we know no example of a UCU ring with stable range higher than 2.

We conclude that a solution could be obtained to the feedback cyclization problem for systems over a wide class of rings, by allowing more than one input (but as few as possible). The residual rank and the matricial decompositions have allowed us to treat successfully the case of non-reachable systems. Also, we have reenforced the idea that the main obstruction for a ring R to solve the feedback cyclization problem is the stable range and the existence of certain unimodular vectors, rather than the Krull dimension (see [10]). Although a positive answer is given to the FC^s (resp. strong FC^s) problem for GCU (resp. UCU) rings with finite stable range, it is not known if certain rings which are not GCU, like $k[x, y]$, with k a field, can be FC^s rings (or strong FC^s) for some $s \geq 2$. We close stating two important questions which remain unsolved:

Open problem 5.2. *Do the properties GCU or UCU imply stable range* ≤ 2*?*

Open problem 5.3. *Is* FC^s *(resp., strong* FC^s*) for* $s \geq 2$ *possible without the GCU (resp., UCU) property?*

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REFERENCES

[1] J. Brewer, D. Katz, and W. Ullery. Pole assignability in polynomial rings, power series rings and Prüfer domains. *J. Algebra*, 106:265–286, 1987.

[2] J. Brewer, D. Katz, and W. Ullery. On the pole assignability property over commutative rings. *J. Pure Appl. Algebra*, 48:1–7, 1987.

[3] J. Brewer, L. Klingler, and F. Minaar. Polynomial rings which are BCS-rings. *Comm. Algebra*, 18:209–223, 1990.

[4] R. Bumby, E.D. Sontag, H.J. Sussmann, and W. Vasconcelos. Remarks on the pole-shifting problem over rings. *J. Pure Appl. Algebra*, 20:113–127, 1981.

[5] M. Carriegos, J.A. Hermida-Alonso, and T. Sánchez-Giralda. Pole-shifting for linear systems over commutative rings. *Linear Algebra Appl.*, 346:97–107, 2002.

[6] R. Eising. Pole-assignment for systems over rings. *Systems Control Lett.*, 2:225–229, 1982.

[7] D.R. Estes and R.M. Guralnick. Module equivalences: local to global when primitive polynomials represent units. *J. Algebra*, 77:138–157, 1982.

[8] D. Estes and J. Ohm. Stable range in commutative rings. *J. Algebra*, 7:343–362, 1967.

[9] M. Heymann. Comments "On Pole Assignment in Multi-Input Controllable Linear Systems". *IEEE Trans. Aut. Contr.*, 13:748–749, 1968.

[10] A. Sáez-Schwedt. Feedback cyclization for rings with finite stable range. *Linear Algebra Appl.*, 427:234–241, 2007.

[11] A. Sáez-Schwedt and T. Sánchez-Giralda. Strong feedback cyclization for systems over rings. *Systems Control Lett.*, 57:71–77, 2008.

[12] A. Sáez-Schwedt and T. Sánchez-Giralda. Coefficient assignability and a block decomposition for systems over rings. *Linear Algebra Appl.*, 429:1277–1287, 2008.

[13] W. Schmale. A symmetric approach to cyclization for systems over $\mathbb{C}[y]$. *Linear Algebra Appl.*, 263:221–232, 1997.

[14] E.D. Sontag. Linear systems over commutative rings: a survey. *Ricerche di Automatica*, 7:1–34, 1976.

[15] E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd edition. Springer, New York, 1998.